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Properties of a Complete Fuzzy Normed Algebra

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Abstract:

The aim of this paper is to translate the basic properties of the classical complete normed algebra to the complete fuzzy normed algebra at this end a proof of multiplication fuzzy continuous is given. Also a proof of every fuzzy normed algebra V without identity can be embedded into fuzzy normed algebra V_e with identity e and V is an ideal in V_e is given. Moreover the proof of the resolvent set of a non zero element in complete fuzzy normed space is equal to the set of complex numbers is given. Finally basic properties of the resolvent space of a complete fuzzy normed algebra is given.

Key words: Complete fuzzy normed algebra, Fuzzy bounded operator, Fuzzy norm of operator, The $\omega^* -$fuzzy topology, A character.

Introduction:

Sadeqi and Amiripour in 2007 (1) introduced the definition of fuzzy Banach algebra and proved some results of this space. Dinda et al. in 2010 (2) introduced some properties of intuitionistic fuzzy complete fuzzy normed algebra. A condition for continuous product in a fuzzy normed algebra is introduced in 2016 by Binzar et al (3) also they proved some properties of fuzzy Banach algebra. In this research, first we recall the definition of complete fuzzy normed algebra such that the basic properties of the ordinary complete normed algebra is proved for a complete fuzzy normed algebra. The inverse operator $x \to x^{-1}$ is fuzzy continuous mapping and the resolvent set $\rho_V(x)$ is equal to $\mathbb{C} \setminus \{0\}$ were proved. After that basic properties of the resolvent space were proved such as $\rho_V(xy) \cup \{0\} = \rho_V(yx) \cup \{0\}$ for any x and y in a complete fuzzy normed algebra $(V, L_V, \circledast, \odot)$ with identity.

Preliminaries

In this section, we recall basic properties of fuzzy normed space and other concepts.

Definition 2.1.(3) A quadruple $(V, L_V, \circledast, \odot)$ is called a fuzzy normed algebra if:

1. $(\circledast, \odot)$ is continuous t-norm.
2. $(V, +, \cdot)$ is an algebra over the field $\mathbb{F}$.
3. $(V, L_V, \circledast)$ is a fuzzy normed space.
4. $L_V(ab, ts) \geq L_V(a, t) \odot L_V(b, s)$ for all $a, b \in V$ and $t > 0,\ s > 0$.

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Example 2.2.(3) If $(V, \|\cdot\|)$ is a normed algebra then $(V, L_V, \circledast, \odot)$ is a fuzzy normed algebra where

$$L_V(u, t) = \begin{cases} 0 & \text{if } t \leq \| u \| \\ 1 & \text{if } t \geq \| u \| \end{cases}$$

Example 2.3.(3) If $(V, \|\cdot\|)$ is a normed algebra then $(V, L_V, \wedge, \cdot)$ and $(V, L_V, \cdot, \cdot)$ are a fuzzy normed algebra where

$$L_V(u, t) = \begin{cases} \dfrac{t}{t + \| u \|} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Definition 2.4.(3) If $(V, L_V, \circledast)$ is a complete fuzzy normed space then the fuzzy normed algebra $(V, L_V, \circledast, \odot)$ is called complete fuzzy normed algebra.

Definition 2.5.(4) Suppose that U is any non empty set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u): U \to [0,1]$. Then $\widetilde{A}$ is represented by $\tilde{A} = \{(u, \mu_{\widetilde{A}}(u)): u \in U, 0 \leq \mu_{\widetilde{A}}(u) \leq 1\}$.

Definition 2.6.(4) Suppose that $\circledast: [0,1] \times [0,1] \to [0,1]$ is a binary operation then $\circledast$ is called a continuous **t-norm** (or **triangular norm**) if for all $\alpha, \beta, \gamma, \delta \in [0,1]$ it has the following properties**:**

(1) $\alpha \circledast \beta = \beta \circledast \alpha$, (2) $\alpha \circledast 1 = \alpha$, (3) $(\alpha \circledast \beta) \circledast \gamma = \alpha \circledast (\beta \circledast \gamma)$.
(4) If $\alpha \leq \beta$ and $\gamma \leq \delta$ then $\alpha \circledast \gamma \leq \beta \circledast \delta$.

Remark 2.7.(4) For any α, β, γ and σ in $[0,1]$ then:
(1) If $\alpha > \beta$ then there is γ such that $\alpha \circledast \gamma \geq \beta$.
(2) There is δ such that $\delta \circledast \delta \geq \sigma$.

Definition 2.8.(4) A fuzzy normed space is a triple $(V, L_V, \circledast)$ where V is a vector space over the field $\mathbb{F}$, $\circledast$ is a t-norm and $L_V: V \times [0, \infty) \to [0,1]$ is a fuzzy set has the following properties for all $a, b \in V$ and $\alpha, \beta > 0$:

1- $L_V(a, \alpha) > 0$.
2- $L_V(a, \alpha) = 1$ if and only if $a = 0$.
3- $L_V(ca, \alpha) = L_V\left(a, \frac{\alpha}{|c|}\right)$ for all $c \neq 0 \in \mathbb{F}$.
4- $L_V(a, \alpha) \circledast L_V(b, \beta) \leq L_V(a + b, \alpha + \beta)$.
5- $L_V(a, .): [0, \infty) \to [0,1]$ is a continuous function of α.
6- $lim_{\alpha \to \infty} L_V(a, \alpha) = 1$.

Remark 2.9.(4) Assume that $(V, L_V, \circledast)$ is a fuzzy normed space and let $a \in V, t > 0, 0 < q < 1$, if $L_V(a, t) > (1 - q)$ then there is s with $0 < s < t$ such that $L_V(a, s) > (1 - q)$.

Definition 2.10.(4) Suppose that $(V, L_V, \circledast)$ is a fuzzy normed space. Then $FB(a, r, t) = \{ b \in V : L_V(a - b, t) > (1 - r)\}$ is called open fuzzy ball with the center $a \in V$ and radius r, with $r > 0$.

Definitions 2.11. (4) A subset W of a fuzzy normed space $(V, L_V, \circledast)$ is said to be open if for any $w \in W$ we can find $0 < r < 1$ and $t > 0$ with $FB(w, r, t) \subseteq W$.

Lemma 2.12.(4) Suppose that $(V, L_V, \circledast)$ is a fuzzy normed space then $L_V(x - y, t) = L_V(y - x, t)$ for all $x, y \in V$ and $t > 0$.

Definition 2.13.(4) Assume that $(V, L_V, \circledast)$ is a fuzzy normed space. $W \subseteq V$ is called fuzzy bounded if we can find $t > 0$ and $0 < r < 1$ such that $L_V(w, t) > (1 - r)$ for each $w \in W$.

Definition 2.14.(4) A sequence (v_n) in a fuzzy normed space $(V, L_V, \circledast)$ is called **converges to** $v \in V$ if for each $0 < r < 1$ and $t > 0$ we can find positive natural number such that N with $L_V[v_n - v, t] > (1 - r)$ for all $n \geq N$. Or in other word $lim_{n\to\infty} v_n = v$ or simply represented by $v_n \to v$, v is known the limit of (v_n) or equivalently $lim_{n\to\infty} L_V[v_n - v, t] = 1$.

Definition 2.15.(4) A sequence (v_n) in a fuzzy normed space $(V, L_V, \circledast)$ is said to be a Cauchy sequence if for all $0 < q < 1$, $t > 0$, there is a number N with $L_V[v_m - v_n, t] > (1 - q)$ for all $m, n \geq N$.

Definition 2.16.(4) Suppose that $(V, L_V, \circledast)$ is a fuzzy normed space and let W be a subset of V. Then the closure of W is written by $\overline{W}$ or $CL(W)$ and which is $\overline{W} = \{W \subseteq B: B$ is closed in $V\}$, where a subset B is said to be a closed subset of V if it contains all its limit points**.**

Definition 2.17.(4) Suppose that $(V, L_V, \circledast)$ is a fuzzy normed space and $W \subseteq V$. Then W is called dense in V when $\overline{W} = V$.

Definition 2.18.(4) A fuzzy normed space $(V, L_V, \circledast)$ is said to be complete if every Cauchy sequence in V converges to a point in V.

Definition 2.19.(4) Suppose that $(V, L_V, \circledast)$ and $(W, L_W, \odot)$ are two fuzzy normed spaces. The operator $S: V \to W$ is said to be fuzzy continuous at $v_0 \in V$ if for all $t > 0$ and for all $0 < \alpha < 1$ there is s [is depends on t, α and v_0] and there is p [is depends on t, α and v_0] with $L_V[v - v_0, s] > (1 - p)$ we have $L_W[S(v) - S(v_0), t] > (1 - \alpha)$ for all $v \in V$.

Definition 2.20.(4) Let $(V, L_V, \circledast)$ and $(U, L_U, \odot)$ be two fuzzy normed spaces. The operator $T: D(T) \to U$ is said to be fuzzy bounded if there exists $r, 0 < r < 1$ such that $L_U(Tv, t) \geq (1 - r) \circledast L_V(v, t)$ for each $v \in D(T) \subseteq V$ and $t > 0$ where $D(T)$ is the domain of T.

Theorem 2.21.(4) Suppose that $(V, L_V, \circledast)$ and $(U, L_U, \odot)$are two fuzzy normed spaces. The operator $S: D(S) \to U$ is fuzzy bounded if and only if $S(A)$ is fuzzy bounded for every fuzzy bounded subset A of $D(S)$. Put $FB(V, U) = \{S: V \to U, S$ is a fuzzy bounded operator$\}$ when $(V, L_V, \circledast)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces.

Theorem 2.22.(4) Suppose that $(V, L_V, \circledast)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. Define $L(T, t) = inf_{x\in D(T)} L_U(Tx, t)$ for all $T \in FB(V, U)$ and $t > 0$ then $(FB(V, U), L, *)$ is fuzzy normed space.

Theorem 2.23.(4) Suppose that $(V, L_V, \circledast)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces with $S: D(S) \to U$ is a linear operator where $D(S) \subseteq V$. Then S is fuzzy bounded if and only if S is fuzzy continuous.

Theorem 2.24.(5) Let $(V, L_V, \circledast)$ be a complete fuzzy normed space and W be a closed subspace of V, Then $(\frac{V}{W}, Q, \circledast)$ is a complete fuzzy normed space.

Lemma 2.25.(5) Assume that $(V, L_V, \circledast)$ is a fuzzy normed space and suppose that W is a subset of V. Then $w \in \overline{W}$ if and only if there is a sequence (w_n) in W with (w_n) converges to w.

Theorem 2.26.(5) Suppose that $(V, L_V, \circledast)$ is a fuzzy normed space and assume that W is a subset of V. Then W is dense in V if and only if for every $u \in V$ there is $w \in W$ such that $L_V[u - w, t] > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ and $t > 0$.

Definition 2.27.(5) Suppose that $(V, L_V, \circledast)$ and $(U, L_U, \odot)$ are two fuzzy normed spaces. A sequence operators $T_n \in FB(V, U)$ is said to be:

1) Fuzzy Uniform Convergent Operator**:** if there is $T: V \to U$ with $L[T_n - T, t] \to 1$ for any $t > 0$ and $n \geq N$.
2) Fuzzy strong Convergent Operator: if there is $T: V \to U$ with $L_U[T_n v - Tv, t] \to 1$ for every $t > 0, v \in V$ and $n \geq$ N.
3) Fuzzy weak Convergent Operator**:** if there is $T: V \to U$ with $L_R[f(T_n v) - f(Tv), t] \to 1$ for every $t > 0, f \in FB(U, R)$ and $n \geq N$.

Respectively, T is called uniform strong and weak operator limit of (T_n).

Theorem 2.28.(5) Let $(V, L_V, \circledast)$ be a complete fuzzy normed space and W be a closed subspace of V, Then $(\frac{V}{W}, Q, \circledast)$ is a complete fuzzy normed space.

Proposition 2.29.(6) Let $(V, L_V, \circledast)$ be a fuzzy normed space and W be a closed subspace of V. Then the quotient space $\frac{V}{W}$ is fuzzy normed space with respect to the quotient fuzzy norm defined by: $Q[x + W, t] = sup\{L_V(x + a, t): a \in W\} = \frac{V}{W} = \{v + W: v \in V\}$.

Proposition 2.30.(6) Let $(V, L_V, \circledast)$ be a fuzzy normed space and W be a closed subspace of V. Then the quotient space $\frac{V}{W}$ is fuzzy normed space with respect to the quotient fuzzy norm defined by:$Q[x + W, t] = sup\{L_V(x + a, t): a \in W\} = \frac{V}{W} = \{v + W: v \in V\}$.

Remark 2.31.(7) Let $\lambda \in \mathbb{R}$ if the infinite series $1 + \lambda + \lambda^2 + \ldots + \lambda^n + \ldots = \sum_{n=0}^{\infty} \lambda^n$ converge then $\sum_{n=0}^{\infty} \lambda^n = \frac{1}{(1-\lambda)}$ when $|\lambda| < 1$. In $(\mathbb{R}, L_{\mathbb{R}})$ the infinite series $\sum_{n=0}^{\infty} \lambda^n$ always converges to $\frac{1}{(1-\lambda)}$ since $0 < L_{\mathbb{R}}(\lambda, t) < 1$.

Lemma 2.32.(7) Let $(V, L_V, \circledast, \odot)$ be a complete fuzzy normed algebra with identity e. then $a_1 a_2 .. a_n$ is invertible if and only if each a_i is invertible.

Definition 2.33.(8) Let $\mathbb{R}$ is the vector space of real numbers over the field $\mathbb{R}$ and $\circledast, \odot$ be a continuous t-norm. A fuzzy set $L_{\mathbb{R}}$: $\mathbb{R} \times (0, \infty)$ is called a fuzzy absolute value on $\mathbb{R}$ if it satisfies the following conditions for all $a, b \in \mathbb{R}$:

1) $0 \leq L_{\mathbb{R}}(a, t) < 1$ for all $t > 0$.
2) $L_{\mathbb{R}}(a, t) = 1$ for all $t > 0$ if and only if $a = 0$.
3) $L_{\mathbb{R}}(a + b, t + s) \geq L_{\mathbb{R}}(a, t) \odot L_{\mathbb{R}}(b, s)$.
4) $L_{\mathbb{R}}(ab, st) \geq L_{\mathbb{R}}(a, t) \circledast L_{\mathbb{R}}(b, s)$.
5) $L_{\mathbb{R}}(a, \cdot)$: $[0, \infty) \to [0, \infty)$ is a continuous function of t.
6) $lim_{t \to \infty} L_{\mathbb{R}}(a, t) = 1$.

Properties of Fuzzy normed algebra

Lemma 3.1. If $(V, L_V, \circledast, \odot)$ is a fuzzy normed algebra then multiplication is a fuzzy continuous function.

Proof.

Suppose that (a_n) and (b_n) are two sequences in V. If $a_n \to a$ and $b_n \to b$ as $n \to \infty$ then for any given $0 < \gamma < 1$ and $0 < \alpha < 1$ there is N such that $L_V(a_n - a, t) > (1 - \gamma)$ for all $n \geq N$ and $L_V(b_n - b, t) > (1 - \alpha)$, for all $n \geq N$. Put $L_V(a_n, t) \geq (1 - \beta)$ and $L_V(b, s) = (1 - \theta)$ for some $0 < \beta, \theta < 1$, let $(1 - \beta) \odot (1 - \alpha) = (1 - \varepsilon)$ and $(1 - \gamma) \odot (1 - \theta) = (1 - \delta)$ for some $0 < \delta,\ \varepsilon < 1$.

Now

$$L_V(a_n b_n - ab, t^2 + s^2) = L_V(a_n b_n - a_n b + a_n b - ab, t^2 + s^2)$$
$$\geq L_V(a_n(b_n - b), t^2) \circledast L_V((a_n - a)b, s^2))$$
$$\geq L_V(a_n, t) \odot L_V(b_n - b, t) \circledast L_V(a_n - a, s) \odot L_V(b, s)$$
$$= \big((1 - \beta) \odot (1 - \alpha)\big) \circledast \big((1 - \gamma) \odot (1 - \theta)\big)$$
$$= (1 - \varepsilon) \circledast (1 - \delta)$$

Let $(1 - \varepsilon) \circledast (1 - \delta) > (1 - r)$ for some $0 < r < 1$. Then $L_V(a_n b_n - ab, t^2 + s^2) > (1 - r)$ for all $n \geq$ N, that is $a_n b_n \to ab$.

Hence multiplication is a continuous function.

Theorem 3.2. A fuzzy normed algebra $(V, L_V, \circledast, \odot)$ such that V is a vector space over $\mathbb{C}$ without identity can be embedded into fuzzy normed algebra V_e with identity e and V is an ideal in V_e.

Proof.

Put $V_e = V \times \mathbb{C}$ and define a multiplication in V_e by $(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta) \ldots (*)$, where $\alpha, \beta \in \mathbb{C}$. Then $(V_e, +, \cdot)$ is an algebra with

identity $\hat{e} = (0,1)$ since $(a, \alpha) \cdot (0,1) = (a, \alpha)$ and $(V_e, L_{V_e}, \circledast, \odot)$ is a fuzzy normed space with fuzzy norm $L_{V_e}: V_e \times [0, \infty) \to [0,1]$ defined by: $L_{V_e}((a, \alpha), t) = L_V(a, t) \circledast L_{\mathbb{C}}(\alpha, t)$.
Now
$L_{V_e}((a, \alpha) \cdot (b, \beta), st)$
$= L_{V_e}((ab + \beta a + \alpha b, \alpha\beta), st)$
$\geq L_V(ab + \beta a + \alpha b, st) \circledast L_{\mathbb{C}}(\alpha\beta, st)$
$\geq L_V(ab, st) \circledast L_{\mathbb{C}}(\alpha\beta, st)$
$\geq [L_V(a, s) \odot L_V(b, t)] \circledast [L_{\mathbb{C}}(\alpha, s) \odot L_{\mathbb{C}}(\beta, t)]$
$\geq [L_V(a, s) \circledast L_{\mathbb{C}}(\alpha, s)] \odot [L_V(b, t) \circledast L_{\mathbb{C}}(\beta, t)]$
$= L_{V_e}[(a, \alpha), s] \odot L_{V_e}[(b, \beta), t]$
Hence $(V_e, L_{V_e}, \circledast, \odot)$ is a fuzzy normed algebra with identity $\hat{e} = (0,1)$.
Now we can define a mapping $T: V \to V_e$ by $T(a) = (a, 0)$ it is clear that T is one to one so V can be embedded into V_e and $V \cong V \times \{0\}$. Also it is clear that V is an ideal of V_e.

Proposition 3.3. $(V_e, L_{V_e}, \circledast, \odot)$ is a complete if and only if $(V, L_V, \circledast, \odot)$ is complete.
Proof.
Suppose that $V_e = V \times \mathbb{C}$ is complete and let $(x_n), (\lambda_n)$ be two Cauchy sequence in V and $\mathbb{C}$ respectively that is for any given $0 < \varepsilon < 1$, $0 < \alpha < 1$, $t > 0$ there is N_1 and N_2 such that $L_V[x_n - x_m, t] > (1 - \varepsilon)$ for all $n, m > N_1$ also, $L_{\mathbb{C}}[\lambda_n - \lambda_m, t] > (1 - \alpha)$ for all $n, m > N_2$. Let $N = N_1 \wedge N_2$.
Now
$L_{V_e}[(x_n, \lambda_n) - (x_m, \lambda_m), t]$
$= L_{V_e}[(x_n - x_m, \lambda_n - \lambda_m), t]$
$= L_V[x_n - x_m, t] \circledast$
$L_{\mathbb{C}}[\lambda_n - \lambda_m, t] > (1 - \varepsilon) \circledast (1 - \alpha)$.

Choose $0 < r < 1$ with $(1 - \varepsilon) \circledast (1 - \alpha) > (1 - r)$.
Then $L_{V_e}[(x_n, \lambda_n) - (x_m, \lambda_m), t] > (1 - r)$ for all $m, n > N$. Thus,$[(x_n, \lambda_n)]$ is Cauchy sequence in V_e but V_e is complete so there is $(x, \lambda) \in V_e$ such that $(x_n, \lambda_n) \to (x, \lambda)$. That is

$$1 = \lim_{n\to\infty} L_{V_e}[(x_n, \lambda_n) - (x, \lambda), t]$$
$$= \lim_{n\to\infty} L_V(x_n - x, t)$$
$$\circledast \lim_{n\to\infty} L_{\mathbb{C}}(\lambda_n - \lambda, t)$$

Therefore, $lim_{n\to\infty} L_V(x_n - x, t) = 1$and $lim_{n\to\infty} L_{\mathbb{C}}(\lambda_n - \lambda, t) = 1$. Hence $V$is complete.
Conversely, assume that V is complete and let (x_n, λ_n) be a Cauchy sequence in V_e then for any given $t > 0$ $L_{V_e}[(x_n, \lambda_n) - (x_m, \lambda_m), t]$ converges to 1 as $n\to \infty$ and $m\to \infty$ or $[L_V(x_n - x_m, t) \circledast L_{\mathbb{C}}(\lambda_n - \lambda_m, t)]$ converges to 1 as $n\to \infty$ and $m\to \infty$. Hence $L_V(x_n - x_m, t)$ converges to 1 and $L_{\mathbb{C}}(\lambda_n - \lambda_m, t)$ converges to 1 as $n\to \infty$ and $m\to \infty$. This implies (x_n) and (λ_n) are Cauchy sequences in V and $\mathbb{C}$ respectively, but V and $\mathbb{C}$are complete so there is $x \in V$ and $\lambda \in \mathbb{C}$ such that $lim_{n\to\infty} L_V(x_n - x, t) = 1$ and $lim_{n\to\infty} L_{\mathbb{C}}(\lambda_n - \lambda, t) = 1$.
Now
$lim_{n\to\infty} L_{V_e}[(x_n, \lambda_n) - (x, \lambda), t] = lim_{n\to\infty} L_V(x_n - x, t) \circledast lim_{n\to\infty} L_{\mathbb{C}}(\lambda_n - \lambda, t)$
$= 1 \circledast 1 = 1.$
Hence $(x_n, \lambda_n) \to (x, \lambda)$, thus, V_e is complete.

Theorem 3.4. Let $(V, L_V, \circledast, \odot)$ be a fuzzy normed algebra with identity e. Then there is a fuzzy norm H on V such that L_V is equivalent to H and $(V, H, \circledast)$ is a fuzzy normed algebra with $H(e, t) = 1$ for $t > 0$.
Proof.
For each $x \in V$ let N_x be a linear operator defined by $N_x(v) = xv$ for all $v \in V$. Now if $N_x = N_y$ it follows that $N_x(e) = N_y(e)$. And so $x = y$ hence $x \longmapsto N_x$ is an injective map from V into the set of all linear operator on V.
Now since $L_V(N_x(u), t^2) = L_V(xu, t^2) \geq L_V(x, t) \odot L_V(u, t)$for $u \in V$ which implies that N_x is a fuzzy bounded and $L_V(N_x, t^2) \geq kL_V(x, t)$.

Put $H(a, t) = L_V(N_a, t^2)$ so $H(x, t) \geq kL_V(x, t)$ …(1), for some $k, 0 \leq k \leq 1$.
On the other hand:
$H(a, t) = L(N_a, t^2) = inf_{b\in D(N_a)} L_V(N_a(b), t^2)$
$= inf_{b\in D(N_a)} L_V(ab, t^2) \leq L_V(ae, t^2)$
$= L_V(a, t) \odot L_V(e, t) \leq \frac{1}{k} L_V(a, t)$ $so\, H(a, t) \leq \frac{1}{k} L_V(a, t)$ ... (2).
From equations (1) and (2) we get $kL_V(a, t) \leq H(a, t) \leq \frac{1}{k} L_V(a, t)$ for all $a \in V$ and $t > 0$. Hence $L_V(a, t)$ is equivalent to $H(a, t)$.
Now $H(ab, t^2) = L_V(N_{ab}, t^2) = L_V(N_a . N_b, t^2)$
$\geq L_V(N_a, t) \odot L_V(N_b, t) = H(a, t) \odot H(b, t)$
Therefore, $(V, H, \circledast, \odot)$ is a fuzzy normed algebra. We now have $H(e, t) = L_V(N_e, t) = L_V(I_V, t) = 1$ where I_V is the identity operator on V.

Theorem 3.5. Every fuzzy normed algebra $(V, L_V, \circledast, \odot)$ can be embedded in $FB(V)$ as a closed subalgebra,
Proof.
We know that $N_x: V \to V$ is defined by $N_x(a) = xa$ for all $a \in V$. Then $N_x \in FB(V)$ since $N_x(a_1 + a_2) = x(a_1 + a_2) = xa_1 + xa_2 = N_x(a_1) + N_x(a_2)$ and $N_x(\alpha a) = x(\alpha a) = \alpha(xa) = \alpha N_x(a)$. Also N_x is fuzzy bounded since $L_V(N_x(y), t^2) = L_V(xy, t^2) \geq L_V(x, t) \odot L_V(y, t)$. Put $L_V(x, t) =$

$(1-r)$ for some $0 < r < 1$. That is $L_V(N_x(y), t^2) \geq (1-r) \odot L_V(y,t)$.
Therefore, N_x is fuzzy bounded. Now put $FN = \{N_x\colon x \in V\}$ we will show that FN is a subalgebra of $FB(V)$.
Now we show that $N_{a+b} = N_a + N_b$and $N_{ab} = N_a \,.\, N_b$ also $N_{\alpha a} = \alpha N_a$ and $N_e = I_V$.
$N_{a+b}(x) = (a+b)x = ax + bx = N_a(x) + N_b(x)$
$N_{\alpha a}(x) = (\alpha a)x = \alpha(ax) = \alpha N_a(x)$
$N_{ab}(x) = (ab)x = a(bx) = N_a . N_a(x)$
$N_e(x) = e.x = x = I_V(x)$
Now since $L_V(N_x(u), t^2) = L_V(xu, t^2) \geq L_V(x,t) \odot L_V(u,t)$ for $u \in V$which implies that N_x is a fuzzy bounded so FN is a subalgebra of $FB(V)$ and the mapping $N\colon V \to FB(V)$ is defined by $N(a) = N_a$ is an isometric so it is injective. Moreover, the image of the map N i.e $N(V) = FN$ is closed subalgebra of $FB(V)$. Now suppose that (N_{a_n}) be a sequence in FN such that $N_{a_n} \to T$ in $FB(V)$ then $N_{a_n}(x) = xa_n = N_e(a_n)x$ and so as $n \to \infty$, $T(a)\, x = N_e(a)x$ i.e $T = N_e$. Thus, $N(V)$ is closed.

Proposition 3.6. If $(V, L_V, ⊛, ⊙)$ is a complete fuzzy normed algebra then the inverse operator $x \to x^{-1}$ is fuzzy continuous mapping.
Proof.
First we show that the inverse map is fuzzy continuous at e, if $0 < \varepsilon < 1$ be given and for all $t > 0$ we want to find $0 < \delta < 1$ and s > 0 such that $L_V(a-e,t) > (1-\delta)$ implies $L_V(a^{-1} - e, s) > (1-\varepsilon)$. Now since $L_V(a,t) < 1$ implies:
$a^{-1} = \sum_{n=0}^{\infty}(e-a)^n$ so for any $s < t$
$L_V(a^{-1} - e, s) = L_V(\sum_{n=0}^{\infty}(e-a)^n - e, s)$
$= L_V(\sum_{n=1}^{\infty}(e-a)^n, s)$
$\geq (1-\delta) ⊛ (1-\delta) \odot (1-\delta) ⊛ (1-\delta)$.
Put $(1-\delta) ⊛ (1-\delta) \odot (1-\delta) ⊛ (1-\delta) \odot ... > (1-\varepsilon)$.
We get $L_V(a^{-1} - e, s) > (1-\varepsilon)$. Now $x_n \to x$ implies $x_n x^{-1} \to xx^{-1} = e$ implies $(x_n x^{-1})^{-1} \to e$ implies $x{x_n}^{-1} \to e$ implies $x_n^{-1} \to x^{-1}$.

Definition 3.7. Let $(V, L_V, ⊛, ⊙)$ be a complete fuzzy normed algebra with identity e and let $0 \neq x \in V$. Then the resolvent set of x is denoted $\rho_V(x)$ and defined by $\rho_V(x) = \{\lambda \in \mathbb{C}\colon (x - \lambda e)^{-1}$ exists$\}$. The spectrum of x is denoted by $\sigma_V(x)$ and is defined to be $\sigma_V(x) = (\rho_V(x))^c$.

Theorem 3.8. Suppose that $(V, L_V, ⊛, ⊙)$ is a complete fuzzy normed algebra with identity e. Then the resolvent $\rho_V(x) = \mathbb{C} \setminus \{0\}$.
Proof.
For any given $0 \neq x \in V$ and for any $0 \neq \lambda \in \mathbb{C}$ then $x - \lambda e = -\lambda\left(e - \frac{x}{y}\right)$ has inverse given by a convergent series expansion in powers of $\frac{x}{\lambda}$ since $0 < L_V(\frac{x}{\lambda}, t) < 1$. Hence $\rho_V(x) = \mathbb{C}\setminus\{0\}$, so $\sigma_V(x) = \{0\}$.

Lemma 3.9. Let $(V, L_V, ⊛, ⊙)$ be a complete fuzzy normed algebra with identity e. If x and y are invertible elements of V then xy and yx are invertible.
Proof.
Since $L_V(xy, ts) \geq L_V(x,t) \odot L_V(y,s)$ that is $L_V(xy, ts) < 1$. Similarly $L_V(yx, ts) < 1$. This implies that $(e - xy)$ and $(e - yx)$ are both invertible with inverse given by $a = (e-xy)^{-1} = \sum_{n=0}^{\infty}(xy)^n$ and $b = (e-yx)^{-1} = \sum_{n=0}^{\infty}(yx)^n$, respectively.

Lemma 3.10. Let $(V, L_V, ⊛, ⊙)$ be a complete fuzzy normed algebra with identity e. Suppose x and y element of V such that $(e - xy)$ is invertible. Let $a = (e-xy)^{-1}$ then $b = e + yax$ is the inverse of $(e - yx)$.
Proof.
$b(e-yx) = (e + yax)(e-yx) = e - yx + yax - yaxyx = e - yx + ya(e-xy)x = e - yx + y[a(e-xy) = e]x = e$.

Properties of the resolvent space

Theorem 4.1. If $(V, L_V, ⊛, ⊙)$ is a complete fuzzy normed algebra with identity e then $V \cong C$.
Proof.
For any $x \in V$ there is $\lambda \in \mathbb{C}$ with $(x - \lambda e)$ is not invertible then $x - \lambda e = 0$ that is $x = \lambda e$ for some $\lambda \in \mathbb{C}$. Hence $V \cong \mathbb{C}$.

Theorem 4.2. Let $(V, L_V, ⊛, ⊙)$ be a complete fuzzy normed algebra and suppose W is a closed ideal in V. Then $\left(\frac{V}{W}, Q, ⊛, ⊙\right)$ is a complete fuzzy normed algebra. If V has identity then $\frac{V}{W}$ has identity. Moreover, the identity of $\frac{V}{W}$ has fuzzy norm equal to 1.
Proof.
We know that $\frac{V}{W}$ is complete fuzzy normed space by Theorem 2.24 Since W is an ideal it is easy to see that $\frac{V}{W}$ is an algebra with multiplication given by $(x+W)(y+W) = xy + W$. Now $Q[(x+W)(y+W), ts] = Q[(xy+W), ts]$
$= sup_{a\in W} L_V(xy + a, ts)$
$\geq sup_{a\in W} L_V[(x+a)(y+b), st]$

$\geq \quad sup_{a\in W} L_V(x+a,t) \odot sup_{b\in W} L_V(y+b,s)$

$= Q(x + W, t) \odot Q(y + W)$

Thus, $(\frac{V}{W}, Q, \circledast, \odot)$ is a complete fuzzy normed algebra. Furthermore, if e is the identity of V with $L_V(e,t) = 1$ then $e + W$ is the identity of $\frac{V}{W}$. Also $Q[e+W,t] = sup_{a\in W} L_V[e+a : t] \geq L_V(e,t) = 1$ where, $[a = 0]$ so $Q[e + W , t] = 1$.

Proposition 4.3. Let $(V, L_V, \circledast, \odot)$ be a complete fuzzy normed algebra with identity e and let $a \in V$, then:

(i) $\rho_V(P(a)) = P(\rho_V(a))$ for any complex polynomial P.

(ii) If a is invertible then $\rho_V(a^{-1}) = \rho_V(a)^{-1}$.

Proof (i).

Suppose that P has degree $n \geq 1$ for any $\mu \in \mathbb{C}$, let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the n complex roots of the polynomial $P(\cdot) - \mu$. Then for any $z \in \mathbb{C}$ we hav $P(z) - \mu = \alpha(z - \lambda_1) \dots .(z - \lambda_n)$for some $\alpha \neq 0 \in \mathbb{C}$ and so $P(a) - \mu e = \alpha(a - \lambda_1 e) \dots .(a - \lambda_n e)$. Now if $a_1, a_2, .., a_n$ with $a_i a_j = a_j a_i$ for any $1 \leq i, j \leq n$ then the product $a_1, a_2, .., a_n$ is invertible if and only if each a_j is invertible by Lemma 2.32 suppose that $\mu \in \rho_V(P(a))$ then $P(a) - \mu e$ is invertible and so there is $a - \lambda_i e$ for some $1 \leq i \leq n$ that is $\lambda_i \in \rho_V(a)$. But $P(\lambda_i) = \mu$ which shows that $\mu \in \rho_V(P(a))$.

Conversely, Suppose that $\lambda \in \rho_V(a)$ and let $\mu = P(\lambda)$. Then by Lemma 2.32 it follows that $\lambda = \lambda_i$ for some $1 \leq i \leq n$ and that is $P(a) - \mu e$ is invertible. Thus, $P(\lambda) \in \rho_V(P(a))$.

Proof (ii).

Suppose that $a \in V$ is invertible i.e. $0 \notin \rho_V(a)$. For any $\lambda \in \mathbb{C}$ we have $(a - \lambda e) = a(e - \lambda a^{-1}) = a\lambda(\lambda^{-1}e - a^{-1})$ which implies that $(a - \lambda e)$ is invertible if and only if $(a^{-1} - \lambda^{-1}e)$ is invertible.

Proposition 4.4. Let $(V, L_V, \circledast, \odot)$ be a complete fuzzy normed algebra and $T: V \to \mathbb{C}$ then T is fuzzy continuous.

Proof.

By Theorem 4.1 $V \cong \mathbb{C}$ and for any $0 \neq x \in V$ there is $\lambda \in \mathbb{C}$ such that $x = \lambda e$. Now let (x_n) be a sequence in V converge to $x \in V$ that is for any $t > 0$, we have $lim_{n\to\infty} L_V(x_n - x, t) = 1$or $\lim_{n\to\infty} x_n = x$ or $\lim_{n\to\infty} \lambda_n e = \lambda e$. Now $\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(\lambda_n e) = \lim_{n\to\infty} \lambda_n T(e) = \lim_{n\to\infty} \lambda_n . 1 = \lambda$.

Hence $Tx_n \to T(x)$. Therefore, T is fuzzy continuous.

Definition 4.5. A nonzero complex−valued homomorphism on a complete fuzzy normed algebra is called a character. By Proposition 4.5 character are necessarily fuzzy continuous.

Proposition 4.6. Any commutative complete fuzzy normed algebra $(V, L_V, \circledast, \odot)$ with identity e has at least one character.

Proof.

By Theorem 4.1 $V \cong \mathbb{C}$ and the effecting isomorphism is a character.

Definition 4.8. Let $(V, L_V, \circledast, \odot)$ be a complete fuzzy normed algebra with identity e. Then the set of characters of V is called structure of V and is denoted by $St(V)$. We define the ω^*-fuzzy topology on $FB(V, \mathbb{K})$ the dual of the complete fuzzy normed space $(V, L_V, \circledast, \odot)$.

Definition 4.9. The ω^*-fuzzy topology on $FB(V, \mathbb{K})$ is generated by the neighborhoods $N(p, S, \varepsilon, t) = \{w \in FB(V, \mathbb{K}) : L_{\mathbb{K}}(w(a) - p(a), t) > (1 - \varepsilon)\}$ for all $a \in V$ and $s, t > 0$ where $p \in FB(V, \mathbb{K})$, $0 < \varepsilon < 1$, and S is any finite subset of V. Thus, a set E in $FB(V, \mathbb{K})$ is open in ω^*-fuzzy topology if and only if for each $\Psi \in E$ there is some $N(\Psi, s, \varepsilon, t)$ with $N(\psi, s, \varepsilon, t) \subseteq E$.

Proposition 4.10. The ω^*-fuzzy topology on $FB(V, \mathbb{K})$ where $(V, L_V, \circledast, \odot)$ is a complete fuzzy normed algebra with identity e is a Hausdorff space.

Proof.

Let $f_1, f_2 \in FB(V, \mathbb{K})$ with $f_1 \neq f_2$ then there exists $a \in V$ such that $f_1(a) \neq f_2(a)$. Let $L_{\mathbb{K}}(f_1(a) - f_2(a), t) = r$ for some $0 < r < 1$ then for any $r < r_o < 1$ we can find r_1 such that $r_1 \circledast r_1 \geq r_o$ by Remark 2.7.

Now consider $N(f_1, \{a\}, 1 - r_1, \frac{t}{2})$ and $N(f_2, \{a\}, 1 - r_1, \frac{t}{2})$. Clearly $N(f_1, \{a\}, 1 - r_1, \frac{t}{2}) \cap N(f_2, \{a\}, 1 - r_1, \frac{t}{2}) = \emptyset$.

If there exists $y \in N(f_1, \{a\}, 1 - r_1, \frac{t}{2}) \cap N(f_2, \{a\}, 1 - r_1, \frac{t}{2})$ then $= L_{\mathbb{K}}(f_1(a) - f_2(a), t) \geq L_{\mathbb{K}}\left(f_1(a) - y(a), \frac{t}{2}\right) \circledast L_{\mathbb{K}}\left(f_2(a) - y(a), \frac{t}{2}\right) \geq r_1 * r_1 > r$, which is contradiction. Therefore, ω^*-fuzzy topology is Hausdorff space.

Proposition 4.11. The structure $St(V)$ of a commutative complete fuzzy normed algebra $(V, L_V, \circledast, \odot)$ with identity e, is ω^*-closed subset of $FB(V, \mathbb{K})$.

Proof.

Let (f_n) be a sequence is $St(V)$ converging to $T \in FB(V, \mathbb{K})$ then by the definition of the ω^*- fuzzy topology $f_n(x) \to T(x)$ for each $x \in V$. Now for any $x, y \in V$ we have $T(xy) = lim_{n\to\infty} f_n(xy) = lim_{n\to\infty} f_n(x) . lim_{n\to\infty} f_n(y) =$

$T(x)T(y)$. It follows that $T \in St(V)$. Hence $St(V)$ is closed.

Theorem 4.12. Let $(V, L_V, \circledast, \odot)$ be a commutative complete fuzzy normed algebra with identity e. Then for each $x \in V$ and $f \in St(V)$ define $\Psi_x: St(V) \to \mathbb{C}$ by $\Psi_x(f) = f(x)$. Then Ψ is homomorphism from V into $FC(St(V))$.

Proof.

Ψ is a homomorphism since $\Psi_{xy}(f) = f(xy) = f(x)f(y) = \Psi_x(f)\Psi_y(f)$ for any $x, y \in V$ and $f \in St(V)$. Similarly we can show that Ψ is linear. To show that $\Psi_x \in FC(St(V))$ suppose that $f_n \to f$ in $St(V)$ then $\Psi_x(f_n) = f_n(x) \to f(x) = \Psi_x(f)$ by the definition of the ω^* −fuzzy topology. That is Ψ_x is fuzzy continuous.

Conclusion:

In this paper the multiplication as a continuous function was proved. Also every fuzzy normed algebra $(V, L_V, \circledast, \odot)$ without identity can be embedded into fuzzy normed algebra V_e with identity e as well as V is an ideal in V_e. The resolvent set $\rho_V(x)$ is equal to $\mathbb{C} \setminus\{0\}$. And the ω^* −fuzzy topology on fuzzy bounded operator $FB(V, \mathbb{K})$ which is the dual of the complete fuzzy normed space $(V, L_V, \circledast, \odot)$ it is a Hausdorff space is defined. Likewise, the structure $St(V)$ of a commutative complete fuzzy normed algebra $(V, L_V, \circledast, \odot)$ with identity e is ω^* −closed subset of $FB(V, \mathbb{K})$.

Conflicts of Interest: None.

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خواص الجبر القياسي الضبابي التام

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الخلاصة:

الهدف من هذا البحث هو استنباط الخواص الاساسية للجبر القياسي التام الاعتيادي لبرهان خواص مشابه لها للجبر القياسي الضبابي التام. في هذا الاتجاه برهن ان عملية الضرب مستمرة ضبابيا. كذلك برهن أن كل عنصر غير صفري في الجبر القياسي الضبابي التام يكون قابل للعكس. لوحظ ان مجموعة اعادة الحل لأي عنصر غير صفري تساوي مجموعة الاعداد العقدية .

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