The Existence of a Polynomial Inverse Integrating Factors and Studies About the Limit Cycles for Cubic, Quartic and Quintic Polynomial Systems

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Abstract:  
The main aims of our study are the presence of reverse polynomial integration factors and studies about the limit cycles for real polynomial in two-dimensional autonomous system of the form

\[
\begin{align*}
\dot{\alpha} &= \sum_{j=0}^{m} \sum_{i=0}^{j} a_{i,j-i} \alpha^i \beta^{j-i} = P_m(\alpha, \beta) \\
\dot{\beta} &= \sum_{j=0}^{m} \sum_{i=0}^{j} b_{i,j-i} \alpha^i \beta^{j-i} = Q_m(\alpha, \beta)
\end{align*}
\]

(1)

Where \(P_m(\alpha, \beta)\) and \(Q_m(\alpha, \beta)\) are polynomials of degree \(m\), \(m = 3, 4\) and 5.

Key words: Limit cycles, Polynomials Inverse integrating factors, Polynomial systems.

Introduction:  
Some results of polynomial inverse integrating factor and study about limit cycle of system (1) are obtained, there are many works achieved on the two-dimensional autonomous system, According to polynomial inverse integrating factor and limit cycles, readers can judge Andronov(1), Ferragut (2), García et al.(3), Giné & Llibre (4, 5), García & Grau (6), Coll et al. (7), Perko(8), Poincaré (9), Van der Pol (10) and Yan-Qian(11).

Our project is structured as follows and consists of two sections:

The first section, is an introduction to planar differential systems. The most basic definitions and results of general type are also given, including the main topics discussed in this paper. The method must be lead to find polynomials inverse integrating factor, and some results about the existence of inverse polynomial integrating factors of system (1) are given in section two.

Introduction to planar differential systems (3-6):  
Firstly, the following systems of form are concerned with independent polynomial planar differential

\[
\begin{align*}
\dot{\alpha} &= P(\alpha, \beta), \\
\dot{\beta} &= Q(\alpha, \beta)
\end{align*}
\]

where \(P(\alpha, \beta)\) and \(Q(\alpha, \beta)\) are continuously partial first order derivative functions of class \(U\) and \(U \subseteq \mathbb{R}^2\), secondly, restrict \(P(\alpha, \beta)\) and \(Q(\alpha, \beta)\) to be elements in a real polynomial loop in two variables \(\mathbb{R}[\alpha, \beta]\). The point indicates the derivation of the independent variable \(t\), which is usually called time, that is \(\dot{\alpha} = \frac{d\alpha}{dt}\). The vector field linked to system (1.1) will be denoted as

\[
\chi(\alpha, \beta) = P(\alpha, \beta) \frac{\partial}{\partial \alpha} + Q(\alpha, \beta) \frac{\partial}{\partial \beta}
\]

Definition 1 (3):

Define the divergence of a vector field \(\chi\), written \(\text{div} \chi\) or \(\nabla \cdot \chi\), as the dot product of del with \(\chi\). So if the vector \(\chi = (P, Q)\), then

\[
\text{div} \chi = \nabla \cdot \chi = \left( \frac{\partial}{\partial \alpha} \frac{\partial}{\partial \beta} \right) (P, Q) = \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta}.
\]

Notice that \(\text{div} \chi\) is a scalar.

Definition 2 (12):

The nonzero function \(V: U \rightarrow \mathbb{R}\) is said to be the inverse integral factor of (1.1) for category
\[ \Sigma (U, R), \text{ and is not local empty and fulfill the} \]
\[ \text{following linear PDEs:} \]
\[ P \frac{\partial V}{\partial \alpha} + Q \frac{\partial V}{\partial \beta} = \left( \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta} \right) V. \quad (1.2) \]
\[ \text{In short, the reverse integration factor of } V(\alpha, \beta) \]
\[ \text{system (1.1) fulfill} \]
\[ \chi V = V \, d\nu \chi. \]
\[ \text{The inverse integration factor is a significant} \]
\[ \text{material in our study.} \]

**Theorem 1 (3), (13):**

Let \( V_1, \ldots, V_p \) be inverse integration factors of (1.1) and \( a_1, \ldots, a_p \in \mathbb{R} \). Then, the function \( V = \sum_{i=1}^{p} a_i V_i \) is inverse integration factor of (1.1).

**Definition 3 (4), (15):**

The singular point \( (\alpha_0, \beta_0) \) of (1.1) is said to be center if there is a neighborhood \( N \) of \( (\alpha_0, \beta_0) \) such that all the trajectories of \( N \setminus \{ (\alpha_0, \beta_0) \} \) are periodic.

**Definition 4 (5), (16-17):**

If C is an isolated closed curve, then C is called a limit cycle. The important relationship between the limit cycle and the inverse integrating factor, verified in (1) and (14).

**Theorem 2 (2), (6):**

Let \( V : U \to R \) be the reverse merge factor of (1.1). If \( \rho \subset U \) is a boundary cycle of (1.1), then \( \rho \) is included in group.

\[ \sum = \{ (\alpha, \beta) \in U : V(\alpha, \beta) = 0 \}. \]

**Inverse Integrating Factors for polynomial systems (4), (6):**

This portion is concentrated on some polynomial results of the inverse integral factors of cubic, quartic and quintic polynomial systems.

Firstly, systems are considered real polynomial differential systems (1.1) of the degree \( m = 3, 4, 5 \). Our aims are to obtain the polynomial inverse integral factor \( V(\alpha, \beta) \).

**Method for calculating inverse polynomial integrating factors for systems (2), (6), (14):**

Consider the real planar systems (1), suppose that \( P_m \) and \( Q_m \) have no intersection factors; in general to find the inverse integrating factors use equation (1.2).

**Method 1.** Look for real inverse polynomial integrals factor of degree \( k \in N \), and write \( V(\alpha, \beta) \) as

\[ V(\alpha, \beta) = \sum_{i+j=0}^{k} V_{i,j} \alpha^i \beta^j, \]

Where \( V_{i,j} \in R \). Equation (1.2) is a polynomial equation, because \( P_m(\alpha, \beta), Q_m(\alpha, \beta) \) and \( V \) are Polynomial functions which can be written as aligner system with an unknown \( V_{i, j}, i + j = 0, \ldots, k \).

**Theorem 3 (12):**

Let \( V(\alpha, \beta) \) be the inverse polynomial inverse factor to incorporate the degree of \( k \) into the system (1). Then equation (1.2) is equal in value to a linear homo system:

\[ A_k V^k = 0, \quad (2.1) \]

Were \( A_k \) is matrix and

\[ V^k = (V_{0,0}, V_{0,1}, V_{0,1}, V_{2,0}, V_{2,1}, V_{2,0}, V_{3,1}, V_{4,0}, V_{k-1,1}, \ldots, V_{k,1}, V_{k,0})^T \]

\( V^k \) is vector of coefficient for \( V(\alpha, \beta) \).

**The presence of an inverse integrals factor for cubic polynomial systems**

**Theorem 4**

The nonlinear systems

\[ \dot{\alpha} = -\beta - 3\alpha^2 \beta \]
\[ \dot{\beta} = \alpha - 9\alpha \beta^2 \]

have an inverse integral factor

\[ V(\alpha, \beta) = v_{0,0}(1 + 12 \alpha^2 + 54 \alpha^4 + 108 \alpha^6 + 81 \alpha^8). \]

**Proof:** let \( V(\alpha, \beta) \) be the inverse integral factor so by (Definition 2) and equation (1.2), it must be the following system

\[ P \frac{\partial V}{\partial \alpha} + Q \frac{\partial V}{\partial \beta} = \left( \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta} \right) V \]
\[ (-\beta - 3\alpha^2 \beta) \frac{\partial V}{\partial \alpha} + (\alpha - 9\alpha \beta^2) \frac{\partial V}{\partial \beta} = (-24 \alpha \beta) V \]

Solve our semi-linear equation (2.3), the result would be

\[ d\alpha = \frac{\partial V}{\partial \beta} V \]
\[ d\beta = \frac{\partial V}{\partial \alpha} V \]

in general \( V(\alpha, \beta) \) is extremely difficult to solve. Thus, the method is ignored Eq. 2.3. So by the Method, \( V(\alpha, \beta) = \sum_{i+j=0}^{k} V_{i,j} \alpha^i \beta^j \), apply for \( k = 8 \) it gives us

\[ V(\alpha, \beta) = v_{0,0} + v_{2,0} \alpha^2 + v_{4,0} \alpha^4 + v_{6,0} \alpha^6 + v_{0,8} \alpha^8 \]
\[ (-\beta - 3\alpha^2 \beta) \frac{\partial V}{\partial \alpha} + (\alpha - 9\alpha \beta^2) \frac{\partial V}{\partial \beta} = (-24 \alpha \beta) V \]
\[ (-\beta - 3\alpha^2 \beta) \left( 2v_{2,0} \alpha + 4v_{4,0} \alpha^3 + 6v_{6,0} \alpha^5 \right) + 8v_{8,0} \alpha^7 \]
\[ = (-24 \alpha \beta) \left( v_{0,0} + v_{2,0} \alpha^2 + v_{4,0} \alpha^4 + v_{6,0} \alpha^6 + v_{8,0} \alpha^8 \right) \]

The coefficients of \( \alpha^5 \beta, \alpha^3 \beta, \alpha^2 \beta, \alpha \beta \) and \( \alpha \beta \) are:

\[ -12v_{2,0} + v_{2,0} = 0 \]
\[ -9v_{2,0} + 2v_{2,0} = 0 \]
\[ -2v_{4,0} + v_{6,0} = 0 \]
\[ -3v_{6,0} + 4v_{8,0} = 0 \]

The above system will give
Theorem 4

Let \( \mathbf{A} = \begin{pmatrix} -12 & 1 & 0 & 0 & 0 \\ 0 & -9 & 2 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \) and \( \mathbf{V} = \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{2,0} \\ v_{3,0} \\ v_{4,0} \end{pmatrix} \).

\[
\mathbf{A} \mathbf{V}^5 = 0,
\]

After solving the system \( v_{4,0} = 54v_{0,0}, \ v_{2,0} = 12v_{0,0}, \ v_{6,0} = 108v_{0,0} \) and \( v_{8,0} = 81v_{0,0} \), \( v_{0,0} \) is an independent constant.

Substitute \( v_{2,0}, v_{4,0}, v_{6,0} \) and \( v_{8,0} \) in \( V(\alpha, \beta) \):

\[
V(\alpha, \beta) = v_{0,0}(1 + 12 \alpha^2 + 54 \alpha^4 + 108 \alpha^6 + 81 \alpha^8).
\]

Corollary 1

The nonlinear system

\[
\dot{\alpha} = -\beta + \alpha^2 \beta \\
\dot{\beta} = \alpha - 2a^3 + 9\alpha b^2
\]

has an inverse integrating factor \( V(\alpha, \beta) = v_{0,0}(1 - 12 \alpha^2 + 54 \alpha^4 - 108 \alpha^6 + 81 \alpha^8) \).

Proof: similar to (Theorem 4) after substitute in \( V(\alpha, \beta) \), get the system

\[
\begin{pmatrix} 4 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,1} \\ v_{2,2} \end{pmatrix} = 0.
\]

After solving the system \( v_{1,1} = 4v_{0,0} \) and \( v_{2,2} = 4v_{0,0} \), \( v_{0,0} \) is an independent constant.

Substitute \( v_{1,1} \) and \( v_{2,2} \) in \( V(\alpha, \beta) \):

\[
V(\alpha, \beta) = v_{0,0}(1 + 4\alpha \beta + 4\alpha^2 \beta^2).
\]

Theorem 5

The cubic systems in all the previous corollaries cannot contain limit cycles.

Proof: To prove that, it is followed straight by (Theorem 2) since the illustration of reverse integrals factors is not associated with convex functions.

The presence of inverse integrals factors for Quadratic polynomial systems

Theorem 6

The nonlinear system

\[
\begin{align*}
\dot{\alpha} &= a_1 \alpha + a_0 \beta + a_2 \alpha^2 + a_1 \alpha \beta + a_0 \beta^2 \\
\dot{\beta} &= b_1 \beta + b_0 \beta^2 + b_1 \alpha \beta + b_0 \beta^2 \\
\end{align*}
\]

has an inverse integrating factor \( V(\alpha, \beta) = v_{0,0}(1 - 12 \alpha^2 + 54 \alpha^4 - 108 \alpha^6 + 81 \alpha^8) \).

Corollary 2

The nonlinear system

\[
\dot{\alpha} = -\beta + \alpha^3 - \alpha \beta^2 \\
\dot{\beta} = \alpha + \alpha^2 \beta - \beta^3
\]

has an inverse integrating factor \( V(\alpha, \beta) = v_{0,0}(1 + 4\alpha \beta + 4\alpha^2 \beta^2) \).

\[
V(\alpha, \beta) = v_{2,0} \alpha^2 + v_{0,2} \beta^2 + v_{3,0} \alpha^3 + v_{1,2} \alpha \beta^2 + v_{4,0} \alpha^4 + v_{2,2} \alpha^2 \beta^2 + v_{0,4} \beta^4 + v_{5,0} \alpha^5 + v_{1,4} \alpha \beta^4
\]

find the optimal solutions of the inverse integrating factor.

Proof: Let \( V(\alpha, \beta) \) be the inverse integrating factor so by (Definition 2) and equation (1.2), the result must be given as below

\[
P \frac{\partial V}{\partial \alpha} + Q \frac{\partial V}{\partial \beta} = \left( \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta} \right) V
\]

After substituting \( P \) and \( Q \), quasi linear equation is given.

In general, it is extremely difficult to solve quasi linear equation. Thus, the method is ignored.

\[
V(\alpha, \beta) = \sum_{n+j=0}^{k} V_{i,j} \alpha^i \beta^j
\]

apply for \( k = 5 \), so the result is supported by the method.
\begin{equation}
(a_{1.0} + a_{0.1} \beta + a_{2.0} \alpha^2 + a_{1.1} \alpha \beta + a_{3.0} \alpha^3 + a_{2.1} \alpha^2 \beta + a_{1.2} \alpha \beta^2 + a_{0.3} \beta^3 + a_{4.0} \alpha^4 + a_{3.1} \alpha^3 \beta
+ a_{1.3} \alpha^3 \beta + a_{2.2} \alpha^2 \beta^2) \frac{\partial V}{\partial \alpha} \\
+ (b_{1.0} \alpha + b_{0.1} \beta + b_{2.0} \alpha^2 + b_{1.1} \alpha \beta + b_{0.2} \beta^2 + b_{3.0} \alpha^3 + b_{2.1} \beta \alpha + b_{1.2} \beta \alpha^2 + b_{0.3} \beta^3 \\
+ b_{4.0} \alpha^4 + b_{3.1} \beta \alpha^3 + b_{1.3} \alpha^3 \beta + b_{2.2} \alpha^2 \beta^2 + b_{0.4} \beta^4) \frac{\partial V}{\partial \beta} \\
= (a_{1.0} + 2a_{2.0} \alpha + a_{1.1} \beta + 3a_{3.0} \alpha^2 + 2a_{2.1} \alpha \beta + a_{1.2} \beta^2 + 4a_{4.0} \alpha^3 + 3a_{3.1} \alpha^3 \beta + a_{1.3} \beta^3 + 2a_{2.2} \alpha^2 \beta \\
+ b_{0.1} + b_{1.1} \alpha + 2b_{0.2} \beta + b_{2.1} \alpha^2 + 2b_{1.2} \beta \alpha + 3b_{0.3} \beta^2 + b_{3.1} \alpha^3 + b_{3.1} \alpha^2 \beta + 2b_{2.2} \alpha^2 \beta \\
+ 4b_{0.4} \beta^3) V
\end{equation}

After calculating, the coefficients of terms are:
\begin{align*}
-2a_{2.0} + b_{1.1} v_{0.2} + b_{0.1} v_{1.2} = 0 \\
2b_{2.0} v_{1.2} - 2b_{1.2} v_{2.0} + (2a_{1.1} - 2b_{0.2}) v_{3.0} + 2b_{3.0} v_{0.2} + b_{1.0} v_{2.2} + 4a_{0.1} v_{4.0} = 0 \\
(a_{1.1} - 2b_{0.2}) v_{2.0} + 2b_{2.0} v_{0.2} + b_{1.0} v_{1.2} + 3a_{0.1} v_{3.0} = 0 \\
2b_{0.3} v_{0.2} - 2a_{1.1} v_{0.2} + 2a_{0.1} v_{2.2} + 4b_{1.0} v_{0.4} = 0 \\
-3a_{3.0} v_{0.2} + b_{0.1} v_{2.2} + b_{1.1} v_{1.2} + a_{1.2} v_{2.0} + a_{1.0} v_{0.2} - 2a_{0.2} v_{1.2} + b_{2.1} v_{0.2} - 3b_{3.0} v_{2.0} = 0 \\
2b_{1.0} v_{0.2} + 2a_{0.1} v_{2.0} = 0 \\
2a_{1.0} v_{3.0} - b_{0.1} v_{3.0} - b_{1.1} v_{2.0} = 0 \\
a_{1.0} v_{0.2} - b_{0.1} v_{0.2} = 0 \\
a_{0.1} v_{1.2} - a_{1.1} v_{0.2} = 0 \\
-b_{2.1} v_{2.0} + 3a_{1.0} v_{4.0} - b_{0.1} v_{4.0} + a_{2.0} v_{3.0} - a_{3.0} v_{2.0} - b_{1.1} v_{3.0} = 0 \\
b_{0.1} v_{2.0} - a_{1.0} v_{2.0} = 0 \\
-3b_{0.2} v_{2.0} + 3b_{1.1} v_{0.4} - a_{2.0} v_{0.4} - 2a_{1.0} v_{0.4} = 0 \\
-3b_{0.3} v_{3.0} + b_{3.1} v_{0.2} + b_{1.1} v_{2.2} + 2a_{1.2} v_{3.0} + 2b_{1.1} v_{1.2} - 4a_{4.0} v_{0.2} + b_{0.1} v_{3.2} + 2a_{1.0} v_{3.2} - 2a_{3.0} v_{1.2} \\
-3b_{3.1} v_{2.0} = 0 \\
-b_{0.3} v_{1.2} + 3b_{1.1} v_{0.4} + 3b_{0.1} v_{1.4} - 2a_{2.0} v_{0.4} - b_{1.3} v_{0.2} - 2a_{2.2} v_{0.2} = 0 \\
-b_{0.1} v_{5.0} - 2a_{4.0} v_{2.0} - b_{3.1} v_{2.0} + 2b_{1.1} v_{3.0} + 2a_{2.0} v_{4.0} - b_{1.1} v_{4.0} + 4a_{1.0} v_{5.0} = 0 \\
b_{0.3} v_{0.4} - a_{1.2} v_{0.4} = 0 \\
a_{0.3} v_{1.4} - a_{1.3} v_{0.4} = 0 \\
-2b_{0.4} v_{0.2} + a_{0.3} v_{1.2} - a_{1.3} v_{0.2} + a_{1.1} v_{0.4} + a_{0.1} v_{1.4} + 4b_{0.2} v_{0.4} = 0 \\
a_{3.0} v_{4.0} - b_{1.1} v_{5.0} - b_{1.1} v_{3.0} - a_{4.0} v_{3.0} + 3a_{2.0} v_{5.0} - b_{2.1} v_{4.0} = 0 \\
a_{4.0} v_{5.0} - b_{3.1} v_{5.0} = 0 \\
-b_{2.1} v_{5.0} + 2a_{3.0} v_{5.0} - b_{3.1} v_{4.0} = 0 \\
2b_{0.2} v_{3.2} - 2b_{2.2} v_{3.2} + 2a_{3.1} v_{5.0} = 0 \\
2b_{3.0} v_{3.2} - 2b_{2.2} v_{4.0} + 3a_{2.1} v_{5.0} + 4b_{1.0} v_{2.2} - 2b_{2.2} v_{5.0} + a_{3.1} v_{4.0} = 0 \\
4b_{4.0} v_{0.4} + a_{2.0} v_{3.2} - 4b_{0.4} v_{4.0} + 3a_{1.1} v_{4.0} - a_{3.1} v_{2.2} + 5a_{0.3} v_{5.0} + 4b_{3.0} v_{1.4} = 0 \\
-2b_{0.4} v_{3.2} + 2b_{2.2} v_{1.4} - 2a_{1.0} v_{3.1} + 2a_{1.3} v_{3.2} = 0 \\
4a_{0.1} v_{4.0} - 4b_{0.4} v_{3.0} + 4b_{1.1} v_{3.2} + 3b_{1.2} v_{3.0} - a_{3.1} v_{1.2} = 0 \\
2b_{2.0} v_{2.2} - 2b_{1.2} v_{3.0} + 2b_{1.0} v_{0.2} + a_{1.1} v_{1.2} + 3b_{1.0} v_{3.0} - 2b_{2.2} v_{2.0} + 2b_{1.0} v_{3.2} + 3a_{1.1} v_{4.0} + a_{0.1} v_{5.0} - 2b_{0.2} v_{4.0} \\
+ 2b_{3.0} v_{1.2} - a_{1.3} v_{0.2} = 0 \\
4a_{1.1} v_{5.0} + 2b_{2.0} v_{3.2} + 2a_{3.0} v_{4.0} + 2a_{3.1} v_{1.2} + 2b_{3.0} v_{2.2} - 2b_{1.2} v_{4.0} - 2b_{2.2} v_{3.0} - 2b_{0.2} v_{5.0} = 0 \\
2b_{2.2} v_{0.4} - a_{1.1} v_{1.4} - 2b_{0.4} v_{2.2} + 2b_{1.2} v_{1.4} - 2a_{3.1} v_{0.4} + a_{1.0} v_{2.2} + 3a_{0.3} v_{3.2} = 0 \\
3a_{0.1} v_{3.2} + a_{1.3} v_{2.0} + a_{1.1} v_{2.2} - a_{2.1} v_{1.2} - 4b_{0.4} v_{2.0} + 4b_{1.0} v_{1.4} + 4b_{2.0} v_{0.4} + 3a_{0.3} v_{3.0} - 3a_{3.1} v_{0.2} = 0
\end{align*}
Theorem 2

The quartic system (2.6) after substituting the above outcome: the system (2.6), the following corollary will be our

Proof: By substituting the above conditions in

After solving the system, an infinite number of solutions are obtained, but the optimal solutions are as follows:

From the above system, this system must be generated

After solving the system, an infinite number of solutions are obtained, but the optimal solutions are as follows:

Theorem 3

The quartic system (2.6) after substituting the above conditions with one center and one limit cycle, are based on the following system:

The nonlinear system

Theorem 7

The presence of an inverse integrals factor for Quintic polynomial systems.

(27) has an inverse integrating factor
\[ V(\alpha, \beta) = v_{6,0}\alpha^6 + v_{4,2}\alpha^4\beta^2 + v_{2,4}\beta^4\alpha^2 + v_{0,6}\beta^6. \]

Find the optimal solutions.

**Proof:** Let \( V(\alpha, \beta) \) be the inverse integrating factor so by (Definition 2) and equation (1.2). Will be resulted as follows

\[
P_0 \frac{\partial V}{\partial \alpha} + Q \frac{\partial V}{\partial \beta} = \left( \frac{\partial P}{\partial \alpha} + \frac{\partial Q}{\partial \beta} \right) V
\]

\[
(a_{2,1}\alpha^2\beta + a_{0,3}\beta^3 + a_{4,1}\alpha^4\beta + a_{2,3}\alpha^2\beta^3
\]

\[
+ a_{0,5}\beta^5) \frac{\partial V}{\partial \alpha}
\]

\[
+ (b_{3,0}\alpha^3 + b_{1,2}\beta^2 \alpha + b_{5,0}\alpha^5
\]

\[
+ b_{3,2}\alpha^3\beta^2 + b_{1,4}\beta^4\alpha) \frac{\partial V}{\partial \beta}
\]

\[
= (2a_{1,2}\alpha\beta + 4a_{4,2}\alpha^2\beta
\]

\[
+ 2a_{2,3}\alpha^3\beta^3 + 2b_{1,2}\beta\alpha + 2b_{3,2}\alpha^3\beta
\]

\[
+ 4b_{1,4}\beta^3\alpha^2)^V (2.8)
\]

In general, it is extremely difficult to solve quasi linear equation. Thus, the method is ignored. So by Method, \( V(\alpha, \beta) = \sum_{i=0}^{k} V(\alpha^i \beta^j) \), and apply for \( k = 6 \), the result will be found as follows

\[ V(\alpha, \beta) = v_{6,0}\alpha^6 + v_{4,2}\alpha^4\beta^2 + v_{2,4}\beta^4\alpha^2 + v_{0,6}\beta^6. \]

applying equation (2.8),

\[
\begin{pmatrix}
4a_{2,1} - 2b_{1,2} & 2b_{3,0} & 0 & 4b_{3,0} & 0 & 0 \\
6a_{0,3} & 2a_{2,1} & 0 & 2b_{1,2} & 6b_{3,0} & 0 \\
0 & 4a_{0,3} & 2a_{0,3} & 4b_{1,2} - 2a_{2,1} & 0 & 0 \\
2a_{4,1} - 2b_{3,2} & 2b_{5,0} & 0 & 4b_{5,0} & 0 & 0 \\
-4b_{1,4} + 4a_{2,3} & 0 & 2a_{2,3} - 2b_{1,4} & 6b_{5,0} & 0 & 0 \\
6a_{0,5} & 0 & 4a_{0,5} & 2b_{3,2} - 2a_{4,1} & 0 & 0 \\
0 & 0 & 2a_{0,5} & -2a_{4,1} + 4b_{3,2} & -2a_{2,3} + 2b_{1,4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

After solving the above system, the infinite numbers of solutions are obtained but the optimal solutions from the following conditions are:

\[ a_{2,1} = a_{2,3} = -2, a_{0,3} = a_{0,5} = a_{4,1} = -1, \]

\[ b_{3,0} = b_{5,0} = b_{1,4} = -a_{2,1}, b_{1,2} = -a_{4,1} \]

\[ b_{3,2} = b_{5,0} + 2, v_{6,0} = b_{5,0} v_{0,6}, \]

\[ v_{4,2} = b_{3,2} + 1, \]

\[ v_{2,4} = b_{3,2} v_{0,6}, \]

and \( v_{0,6} \) is an independent constant.
Application

Substituting the above conditions in \( V(\alpha, \beta) = \sum_{i+j=0}^{k} V_{i,j} \alpha^i \beta^j \), and also to the system (2.7), these results will be obtained:

by applying the (Theorem 7) to the Quintic system

\[
\dot{\alpha} = -2\alpha^2\beta - \beta^3 - \alpha^4\beta - 2\alpha^2\beta^3 - \beta^5,
\]
\[
\dot{\beta} = 2\alpha^3 + \beta^2\alpha + 2\alpha^5 + 4\alpha^3\beta^2 + 2\beta^4\alpha.
\]

Hence the polynomial inverse integrating factors of the above system is:
\[
V(\alpha, \beta) = v_{0,6}(2\alpha^6 + 5\alpha^4\beta^2 + 4\beta^4\alpha^2 + \beta^6).
\]

This application gives us the center and also has no limit cycle.

To prove the above result, (Theorem 2) must be used, and since the inverse integrating factor is homogeneous function, then the center is located at the point (0, 0), by (Definition 3) of the center.

Conclusions:

Conclusion must be based on the following points:

1) Finding a polynomial inverse integrals factor for some systems like system (1).
2) All theories of cubic and Quintic systems do not have boundary cycles by (Theorem 2).
3) If a non-zero homo polynomials are the inverse integrate factor of the system (1), then it has no limit cycle.
4) For the quartic system, one center and one limit cycle are found and also one center of Quintic system is concluded.

Authors' declaration:

- Conflicts of Interest: We have no conflicts of interest to disclose.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Duhok.

References:

وجود مفكوك التكامل لممكوس متعدد الحدود ودراسات حول الدورات المنتهية لأنظمة متعددة الحدود الكعبيّة والرباعية والخامسية

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الخلاصة:

تمثل الأهداف الرئيسية لدراسةنا وجود مفكوك التكامل لممكوس متعدد الحدود ودراسات حول الدورات المنتهية لممكوس متعدد الحدود الحقيقية. في نظام مستقل ثنائي الأبعاد للشكل

\[
\dot{\alpha} = \sum_{j=0}^{m} \sum_{i=0}^{j} a_{i,j-i} \alpha^i \beta^{j-i} = P_m(\alpha, \beta)
\]

\[
\dot{\beta} = \sum_{j=0}^{m} \sum_{i=0}^{j} b_{i,j-i} \alpha^i \beta^{j-i} = Q_m(\alpha, \beta)
\]

حيث

\[P_m(\alpha, \beta)\] و \[Q_m(\alpha, \beta)\] هما متعدد الحدود من الدرجة \[m\] ، \[m\] ، \[m\] ، \[m\] ، \[m\] بالكلمات المفتاحية: دورات منتهية، مفكوك التكامل الممكوس، أنظمة متعددة الحدود.

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