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The Fractional Local Metric Dimension of *Comb* Product Graphs

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Abstract:

For the connected graph G with vertex set $V(G)$ and edge set $E(G)$, the *local resolving neighborhood* $R_l\{u,v\}$ of two adjacent vertices u,v is defined by $R_l\{u,v\}=\{x\in V(G): d(x,u)\neq d(x,v)\}$. A *local resolving function* f_l of G is a real valued function $f_l: V(G)\to[0,1]$ such that $f_l(R_l\{u,v\})\geq 1$ for every two adjacent vertices $u,v\in V(G)$. The *fractional local metric dimension* of graph G denoted $dim_{fl}(G)$, is defined by $dim_{fl}(G)=\min\{|f_l|: f_l \text{ is a local resolving function of } G\}$. One of the operation in graph is the comb product graphs. The comb product graphs of G and H is denoted by $G\rhd H$. The purpose of this research is to determine the fractional local metric dimension of $G\rhd H$, for graph G is a connected graph and graph H is a complete graph (K_n). The result of $G\rhd K_n$ is $dim_{fl}(G\rhd K_n)=|V(G)|.dim_{fl}(K_{n-1})$.

Key words: Comb product graphs, Local fractional metric dimension, Resolving function.

Introduction:

The first authers to discuss the minimum resolving set and the metric dimension problems is (1, 2). They assumed that the graph used is a connected graph, simple graph and a finite graph. In (3), graph G is defined as a finite and non-empty set of $V(G)$ whose elements are called vertices and sets $E(G)$ (maybe empty) whose elements are called edges which are non-ordered pairs of two different elements of $V(G)$.

Let u and v be two vertices in G, $d_G(u,v)$ is the distance between two vertices u to v of G, defines as the shortest path between u to v. For an ordered subset $W=\{w_1,w_2,\dots,w_k\}\subseteq V(G)$ and $v\in V(G)$, the representation of v with respect to W is an ordered *k-tuple* $r(v|W)=\big(d(v,w_1),d(v,w_2),\dots,d(v,w_n)\big)$, where $d(v,w)$ is the distance between two vertices v to w. The set W is called a *resolving set* for G if each vertex in G has a different representation of W. A resolving set that has a minimal cardinality is called a *basis* of G. The number of vertex on the basis of graph G is called *dimension* of G and denoted by $dim(G)$.

In (4) introduced the local metric dimension of graph, they defined the local resolving set and the local metric dimension of a graph. In (5, 6) studied the commutative characterization of graph operations with respect to the local metric dimension and metric dimension, respectively.

The development of the metric dimension is the fractional metric dimension. The fractional metric dimensions were first examined by (7) they defined the concept of the fractional metric dimension of involving *resolving set*, *resolving function* and *fractional metric dimension*. Then their research was continued by (8). Furthermore, in (9) also found characterization $dim_f(G)=\frac{|V(G)|}{2}$ where G is a connected graphs. Meanwhile, the fractional metric dimension of trees and uncyclic graphs can be seen in (10).

Furthermore, research about the fractional metric dimensions of a product graph has been investigated by (11, 8), and in (12) who studied the fractional metric dimensions on permutation.

The latest development of fractional metric dimension of graphs was conducted by (13). In (14, 15) found the fractional metric dimension of comb product graph. Figure 1 shows examples of comb product graphs

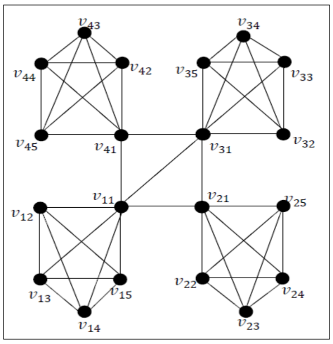

Figure 1. graph $G \rhd K_5$

Below is the notation (index) of the graph $G \rhd H$.

a. The set point of the parent is a member of $V(G)$. The point set of the parent is $U = \{v_{i1}: v_i \in V(G)\}$ with $i = 1,2,\ldots,n$.
b. The leaf set on the parent element v_{i1} is $U_i = \{v_{ij}: u_j \in V(H),\ j = 2,3,4,\ldots,m\}$.

In (14) discussed fractional metric dimension of comb product graph. In this paper discussing the fractional local metric dimension of *comb* product graphs of G and H, for G is an arbitrary graph and graph H is a complete graph.

Results:

In this research, we investigate the fractional local metric dimension of *comb* product graphs where H are some special graphs. We first recall some fractional local metric dimension of a special graphs.

Theorem (1) : For the P_n, $dim_{fl}(P_n) = 1$.

Proof. Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid i = 1,2,\ldots,n-1\}$. Given local resolving function $f_l: V(P_n) \to [0,1]$, for any two adjacent vertices $v_i, v_j \in V(P_n)$ with $v_i v_j \in E(P_n)$ then $R_l\{v_i, v_j\} = V(P_n)$, so

$$f_l(v_1) + f_l(v_2) + \cdots + f_l(v_n) \geq 1$$

$$\sum_{v \in V(P_n)} f_l(v_i) \geq 1$$

hence $dim_{fl}(P_n) = min\{\sum_{v \in V(P_n)} f_l(v_i)\} = 1$. Then $dim_{fl}(P_n) = 1$. ■

Theorem (2) : For the cycle graph C_n, then

$$dim_{fl}(C_n) = \begin{cases} 1 & if\ n\ is\ even \\ \dfrac{n}{n-1} & if\ n\ is\ odd \end{cases}$$

Proof. There are two cases

Case 1. n is odd.

Let $g: V(C_n) \to [0,1]$ be the constant function defined by $g(v_i) = \frac{1}{n-1}$. Let v_i, v_{i+1} be two adjacent vertices of C_n. Then $R_l\{v_i, v_{i+1}\} = V(C_n) - \{v_k\}$ is the unique vertex such that $d(v_k, v_i) = d(v_k, v_{i+1}) = \frac{n-1}{2}$. Hence $g(R_l\{v_i, v_j\}) = 1$. Thus g is a local resolving function of C_n and $dim_{fl}(C_n) \leq |g| = \frac{n}{n-1}$. Now let f be any local resolving function of C_n with $|f| = dim_{fl}(C_n)$. Then $f(R_l\{v_i, v_{i+1}\}) \geq 1$ for each edge $v_i v_{i+1}$. Adding these n inequalities is obtained $(n-1)|f| \geq n$. Hence $dim_{fl}(C_n) = |f| \geq \frac{n}{n-1}$. Thus $dim_{fl}(C_n) = \frac{n}{n-1}$.

Case 2. n is even

In this case $R_l\{v_i, v_j\} = V(C_n)$ for any edge $v_i v_{i+1}$. Hence the constant function $g: V(C_n) \to [0,1]$ defined by $g(v_i) = \frac{1}{n}$ is a local resolving function of G and $|g| = 1$. It follows from the definition of *local resolving function* that $1 \leq dim_{fl}(G) \leq dim_f(G)$ for any connected graph G, that $dim_{fl}(C_n) = 1$. ■

Theorem (3) : Let G be a connected graph , then

1. $dim_{fl}(G) = \frac{n}{2}$ *if only if G a complete graph* (K_n)
2. $dim_{fl}(S_n) = 1$
3. *for the wheel graph* (W_n), *then*

$$dim_{fl}\ (W_n) = \begin{cases} 2 & if\ n = 3 \\ \frac{n}{n-1} & if\ n \geq 4 \end{cases}$$

The fractional local metric dimension of comb product graph for some special graphs, is presented as below.

Theorem (4) : For $n, m \geq 3$, then $dim_{fl}(K_n \rhd K_m) = |V(K_n)| dim_{fl}(K_{m-1})$.

Proof. Let $f_l: V(K_n \rhd K_m) \to [0,1]$ be a local resolving function. Any two adjacent vertices $u, v \in V(K_n \rhd K_m)$. There are three possibilities u and v.

i. If u, v are in the same leaf, then there is $i \in \{1,2,\ldots,n\}$ and $j, k \in \{2,3,\ldots,m\}$ with $j \neq k$ such that $u = v_{ij}$ and $v = v_{ik}$, is obtained $R_l\{u, v\} = \{v_{ij}, v_{ik}\}$. So $f_l(v_{ij}) + f_l(v_{ik}) \geq 1$. The number of vertex on the same leaf is $m-1$ and the number of vertex on the parent is n, then

$$(m-2)\left(\sum_{z \in V(K_n \rhd K_m)} f_l(z) - \sum_{v \in U} f_l(v)\right) \geq n.\binom{m-1}{2}$$

$$(m-2) \sum_{z \in V(K_n \rhd K_m)} f_l(z) \geq \frac{n(m-1)!}{2!\,(m-3)!}$$

$$\sum_{z \in V(K_n \rhd K_m)} f_l(z) \geq n.\frac{(m-1)}{2}$$

ii. If u, v are in the parent, then there are $i, j \in \{1,2,\dots,n\}$ such that $u = v_{i1}$ and $v = v_{j1}$. Local resolving neighborhood $R_l\{u,v\} = \{v_{i1}, v_{i2}, \dots, v_{im}, v_{j1}, v_{j2}, \dots, v_{jm}\}$ so that

$f_l(R_l\{u, v\}) = \sum_{u\in U_i} f_l(u) + \sum_{u\in U_j} f_l(u) + f_l(v_{i1}) + f_l(v_{j1}) \geq 1$. Then

$$(n-1)\sum_{v\in U} f_l(v) + (n-1)\sum_{v\in U_i} f_l(u) \geq \binom{n}{2}$$

$$(n-1)\sum_{z\in V(K_n \rhd K_m)} f_l(z) \geq \frac{n!}{(n-2)!.2!}$$

$$(n-1)\sum_{z\in V(K_n \rhd K_m)} f_l(z) \geq \frac{n(n-1)}{2}$$

$$\sum_{z\in V(K_n \rhd K_m)} f_l(z) \geq \frac{n}{2}$$

iii. If u is in parent and v is in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $p \in \{2,3,\dots,m\}$ such that $u = v_{i1}$ and $v = v_{ip}$. Local resolving neighborhood $R_l\{u,v\} = V(K_n \rhd K_m) - (V(U_i)\backslash\{v_{ip}\})$ so that $f_l(R_l\{u, v\}) = \sum_{z\in V(K_n \rhd K_m)} f(z) - (\sum_{u\in U_i} f_l(u) - f_l(v_{ip})) \geq 1$. Then

$$n.(m-1)\sum_{z\in V(K_n \odot K_m)} f_l(z) - (m-1)\sum_{u\in U_i} f_l(u) \geq n.(m-1)$$

Because $\sum_{u\in U_i} f_l(u) \geq 0$, then

$$n.(m-1)\sum_{z\in V(K_n \rhd K_m)} f_l(z) - 0 \geq n.(m-1)$$

$$\sum_{z\in V(K_n \rhd K_m)} f_l(z) \geq 1$$

Based on the results of the description above, the maximum values taken from equations 1), 2) and 3) are:

$$\sum_{z\in V(K_n \rhd K_m)} f_l(z) \geq n.\frac{(m-1)}{2}$$

As a result:

$dim_{fl}(K_n \rhd K_m)$

$$= min\left\{\sum_{z\in V(K_n \rhd K_m)} f_l(z) : f_l \text{ local resolving function}\right\}$$

$$= n.\frac{(m-1)}{2}$$

Because ordo of K_n is n and $dim_{fl}(K_m) = \frac{m}{2}$ then $dim_{fl}(K_n \rhd K_m) = |V(K_n)| dim_{fl}(K_{m-1})$. ∎

Theorem (5): For $n, m \geq 3$, then $dim_{fl}(P_n \rhd K_m) = |V(P_n)| dim_{fl}(K_{m-1})$.

Proof. Let $f_l: V(P_n \rhd K_m) \to [0,1]$ be a local resolving. Any two adjacent vertices $u, v \in V(P_n \rhd K_m)$, there are three possibilities u and v.

i. If u, v are in the same leaf, then there is $i \in \{1,2,\dots,n\}$ and $j, k \in \{2,3,\dots,m\}$ with $j \neq k$ such that $u = v_{ij}$ and $v = v_{ik}$. $R_l\{u,v\} = \{v_{ij}, v_{ik}\}$. So that $f_l(v_{ij}) + f_l(v_{ik}) \geq 1$. The number of vertex on the same leaf is $m-1$ and the number of vertex on the parent is n, then

$$(m-2)\left(\sum_{z\in V(P_n \rhd K_m)} f_l(z) - \sum_{v\in U} f_l(v)\right) \geq n.\binom{m-1}{2} \quad (2)$$

$$(m-2)\sum_{z\in V(P_n \rhd K_m)} f_l(z) \geq \frac{n(m-1)!}{2!\,(m-3)!}$$

$$\sum_{z\in V(P_n \rhd K_m)} f_l(z) \geq n.\frac{(m-1)}{2}.$$

ii. If u, v are in parent, then there are $i, j \in \{1,2,\dots,n\}$ such that $u = v_{i1}$ and $v = v_{j1}$. Local resolving neighborhood $R_l\{u,v\} = V(P_n \rhd K_m)$ so that $f_l(R_l\{u, v\}) = \sum_{z\in V(P_n \rhd K_m)} f_l(z) \geq 1$. Then

$$(n-1)\sum_{z\in V(P_n \rhd K_m)} f_l(z) \geq (n-1)$$

$$\sum_{z\in V(P_n \rhd K_m)} f_l(z) \geq 1 \quad (3)$$

iii. If u is in parent and v is in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $p \in \{2,3,\dots,m\}$ such that $u = v_{i1}$ and $v = v_{ip}$. Local resolving neighborhood $R_l\{u,v\} = V(P_n \rhd K_m) - (V(U_i)\backslash\{v_{ip}\})$ so that $f_l(R_l\{u, v\}) = \sum_{z\in V(P_n \rhd K_m)} f(z) - (\sum_{u\in U_i} f_l(u) - f_l(v_{ip})) \geq 1$. Then

$$n.(m-1)\sum_{z\in V(P_n\rhd K_m)} f_l(z)-(m-1)\sum_{u\in U_i} f_l(u)\geq n.(m-1)$$

Because $\sum_{u\in U_i} f_l(u)\geq 0$, then

$$n.(m-1)\sum_{z\in V(P_n\rhd K_m)} f_l(z)-0\geq n.(m-1)$$

$$\sum_{z\in V(P_n\rhd K_m)} f_l(z)\geq 1$$

Based on the result of the above description, the maximum values taken from equations 1), 2) and 3) are

$$\sum_{z\in V(P_n\rhd K_m)} f_l(z)\geq n.\frac{(m-1)}{2}$$

As a result :

$$dim_{fl}(P_n\rhd K_m) = min\left\{\sum_{z\in V(P_n\rhd K_m)} f_l(z): f_l \ local\ resolving\ function\right\} = n.\frac{(m-1)}{2}$$

Because ordo of P_n is n and $dim_{fl}(K_m)=\frac{m}{2}$ then $dim_{fl}(P_n\rhd K_m)=|V(P_n)|dim_{fl}(K_{m-1})$. ∎

Theorem (6): For $n,m\geq 3$, then $dim_{fl}(C_n\rhd K_m)=|V(C_n)|dim_{fl}(K_{m-1})$.

Proof. Let $f_l: V(C_n\rhd K_m)\to[0,1]$ be a local resolving function. Any two adjacent vertices $u,v\in V(C_n\rhd K_m)$. There are three possibilities u and v.

i. If u,v are in the same leaf, then there is $i\in\{1,2,\dots,n\}$ and $j,k\in\{2,3,\dots,m\}$ with $j\neq k$ such that $u=v_{ij}$ and $v=v_{ik}$. $R_l\{u,v\}=\{v_{ij},v_{ik}\}$. So that $f_l(v_{ij})+f_l(v_{ik})\geq 1$. The number of vertex on the same leaf is $m-1$ and the number of vertex on the parent is n, then

$$(m-2)\left(\sum_{z\in V(C_n\rhd K_m)} f_l(z)-\sum_{v\in U} f_l(v)\right)\geq n.\binom{m-1}{2}$$

$$(m-2)\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq \frac{n(m-1)!}{2!\,(m-3)!}$$

$$\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq n.\frac{(m-1)}{2} \quad (1)$$

ii. If u,v are in parent, then there are $i,j\in\in\{1,2,\dots,n\}$ such that $u=v_{i1}$ and $v=v_{j1}$. Local resolving neighborhood

$$R_l\{u,v\}=\begin{cases} V(C_n\rhd K_m) & if\ n\ is\ even\\ V(C_n\rhd K_m)-V_k & if\ n\ is\ odd\end{cases}$$

with

$$V_k=\begin{cases}\{v_{k\left(i+\left(\frac{n+1}{2}\right)\right)}: k=1,2,\dots,m\} & for\ i<\left(\frac{n+1}{2}\right)\\ \{v_{k\left(i-\left(\frac{n+1}{2}\right)+1\right)}: k=1,2,\dots,m\} & for\ i\geq\left(\frac{n+1}{2}\right)\end{cases}$$

so that

$$f_l(R_l\{u,v\})=\begin{cases}\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq 1 & if\ n\ is\ even\\ \sum_{z\in V(C_n\rhd K_m)} f_l(z)-\sum_{u\in V_k} f_l(u)\geq 1 & if\ n\ is\ odd\end{cases} \quad (3)$$

Then

$$n\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq n$$

$$\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq 1$$

Or

$$n\sum_{z\in V(C_n\rhd K_m)} f_l(z)-\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq n$$

$$(n-1)\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq n$$

$$\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq\frac{n}{(n-1)}.$$

iii. If u is in parent and v is in leaf of u, then there are $i\in\{1,2,\dots,n\}$ and $p\in\{2,3,\dots,m\}$ such that $u=v_{i1}$ and $v=v_{ip}$. Local resolving neighborhood $R_l\{u,v\}=V(C_n\rhd K_m)-(V(U_i)\backslash\{v_{ip}\})$ so that $f_l(R_l\{u,v\})=\sum_{z\in V(C_n\rhd K_m)} f(z)-\left(\sum_{u\in U_i} f_l(u)-f_l(v_{ip})\right)\geq 1$. Then

$$n.(m-1)\sum_{z\in V(C_n\odot K_m)} f_l(z)-(m-1)\sum_{u\in U_i} f_l(u)\geq n.(m-1)$$

Because $\sum_{u\in U_i} f_l(u)\geq 0$, then

$$n.(m-1)\sum_{z\in V(C_n\rhd K_m)} f_l(z)-0\geq n.(m-1)$$
$$\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq 1.$$

Based on the results of the description above, the maximum values taken from equations 1), 2) and 3), are

$$\sum_{z\in V(C_n\rhd K_m)} f_l(z)\geq n.\frac{(m-1)}{2}$$

As a result

$$dim_{fl}(C_n\rhd K_m)$$
$$=min\left\{\sum_{z\in V(C_n\rhd K_m)} f_l(z): f_l \; local\; resolving\; function\right\}$$
$$=n.\frac{(m-1)}{2}$$

Because ordo of C_n is n and $dim_{fl}(K_m)=\frac{m}{2}$ then $dim_{fl}(C_n\rhd K_m)=|V(C_n)|dim_{fl}(K_{m-1})$. ∎

Lemma (7): For every $u,v\in V(G\rhd K_m)$ where $uv\in E(G\rhd K_{1,m})$ with $m\geq 3$ then there are x and y are in the same leaf that $R_l\{x,y\}\subseteq R_l\{u,v\}$.

Proof. Taken any $u,v\in V(G\rhd K_m)$ where $uv\in E(G\rhd K_m)$ then there are three possibilities u and v.

i. If u,v are in the same leaf or $u,v\in V(K_{m-1})$, then $R_l(u,v)=\{u,v\}$
ii. If u,v are in the parent or $uv\in E(G)$, then there are $i,j\in\{1,2,\dots,n\}$ such that $u=v_{i1}$ and $v=v_{j1}$. Local resolving neighborhood $R_l\{u,v\}$ is obtained by $U_i\cup U_j\subseteq R_l\{u,v\}$. Because $m\geq 3$ then $|U_i|=m-1\geq 2$ so that there are two vertices on a similiar leaf which are members of $R_l\{u,v\}$.
iii. If u is in parent and v is in leaf of u, then there are $i\in\{1,2,\dots,n\}$ and $p\in\{2,3,\dots,m\}$ such that $u=v_{i1}$ and $v=v_{ip}$. Local resolving neighborhood $R_l\{u,v\}$ is obtained $U_j\subseteq R_l\{u,v\}$ where $j\in\{1,2,\dots,n\}$ and $j\neq i$, or $R_l\{u,v\}=V(G\rhd K_m)-V(K_m)\backslash\{u,v\}$.

Based on the description above, it is proven that for every $u,v\in V(G\rhd K_m)$ and $uv\in E(G\rhd K_m)$, there are two vertices x,y are in the same leaf which are the local resolving neighborhood a pairs of vertices $\{u,v\}$ so that $x,y\in R_l\{u,v\}$.

Theorem (8): Let G be a connected graph of order n, then $dim_{fl}(G\rhd K_m)=|V(G)|dim_{fl}(K_{m-1})$ for $n,m\geq 3$.

Proof: Let $f_l: V(G\rhd K_m)\to[0,1]$ be a local resolving function of a graph G. Any two adjacent vertices x and y in K_m are in the same leaf satisfies $R_l\{x,y\}=\{x,y\}$ so that

$$f_l(x)+f_l(y)\geq 1$$

For any two adjacent vertices $u,v\in V(G\rhd K_m)$, by **Lemma (7)**. There are two different vertex x,y are in the same leaf so that $x,y\in R_l\{u,v\}$. Because $R_l\{x,y\}=\{x,y\}$ then $f_l(R_l\{u,v\})=f_l(x)+f_l(y)\geq 1$. As a result, $f_l(v_{i1})=0$ for $i\in\{1,2,3,\dots,n\}$. Because for every x,y in similar leaf $R_l\{x,y\}=\{x,y\}$, than (3)

$$\sum_{z\in V(G\rhd K_m)} f(z)=\sum_{v\in U} f(v)+\sum_{v\in U_1} f(u)+\sum_{v\in U_2} f(u)+\dots+\sum_{v\in U_n} f(u)$$

$$min\left(\sum_{z\in V(G\rhd K_m)} f(z)\right)=min\left(\sum_{v\in U} f(v)+\sum_{v\in U_1} f(u)+\sum_{v\in U_2} f(u)+\dots+\sum_{v\in U_n} f(u)\right)$$

$$dim_{fl}(G\rhd K_m)=min\left(\sum_{z\in V(G\rhd K_m)} f(z)\right)$$
$$=min\sum_{v\in U} f(v)+min\sum_{v\in U_1} f(u)+min\sum_{v\in U_2} f(u)+\dots+min\sum_{v\in U_n} f(u)$$
$$=min\sum_{v\in U_1} f(u)+min\sum_{v\in U_2} f(u)+\dots+min\sum_{v\in U_n} f(u)$$
$$=\sum_{i=1}^{n} min\sum_{v\in U_i} f(u)$$
$$=\sum_{i=1}^{n} dim_{fl}(K_{m-1})$$
$$=n.dim_{fl}(K_{m-1})$$
$$=|V(G)|.dim_{fl}(K_{m-1}).$$

So obtained, $dim_{fl}(G\rhd K_m)=|V(G)|.dim_{fl}(K_{m-1})$. ∎

Conclusion:

In this paper the results of the fractional local metric dimension of *comb* product graph $(G\rhd K_m)$, namely $dim_{fl}(G\rhd K_m)=|V(G)|.dim_{fl}(K_{m-1})$ where G is a connected graph. This research can be continued for G graph and H graph is arbitrary graph, and for further research Cartesian product can be used.

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Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- The author has signed an animal welfare statement.
- Ethical Clearance: The project was approved by the local ethical committee in University of Airlangga.

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البعد المحلي المتري الجزئي للرسوم البيانية لمنتج Comb

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الخلاصة:

يعرف الرسم البياني المتصل G مع قمة الرأس V (G) ومجموعة الحافة E (G)، (حي الحل المحلي) $R_l\{u,v\}$ لذرتين متجاورتين u، v بواسطة $R_l\{u,v\} = \{x \in V(G): d(x,u) \neq d(x,v)\}$ دالة الحل المحلية fi لـ G هي دالة ذات قيمة حقيقية $f_l: V(G) \rightarrow [0,1]$ بحيث يكون $f_l(R_l\{u,v\}) \geq 1$ لكل رأسين متجاورين $u,v \in V(G)$ البُعد المتري المحلي الجزئي لـ الرسم البياني G يشير إلى $dim_{fl}(G)$، وهو معرّف بواسطة $dim_{fl}(G) = \min\{|f_l|: f_l \text{ is a local resolving function of } G\}$. وهي دالة حل محلية لـ G}. إحدى العمليات في الرسم البياني هي الرسوم البيانية لمنتج Comb. الرسوم البيانية لمنتج Comb لـ G و H يشار إليه بواسطة $G \rhd H$ الهدف من هذا البحث هو تحديد البعد المتري المحلي الجزئي لـ $G \rhd H$ ، وذلك لان الرسم البياني G هو رسم بياني متصل والرسم البياني H هو رسم بياني كامل (K_n) نحصل من $G \rhd K_n$ على $dim_{fl}(G \rhd K_n) = |V(G)|.dim_{fl}(K_{n-1})$

الكلمات المفتاحية: البُعد المتري المحلي الجزئي ؛ حل الدالة ؛ الرسوم البيانية لمنتج Comb.