Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
2021, Vol. 18 No.1 (Suppl. March) **E-ISSN: 2411-7986**

DOI: http://dx.doi.org/10.21123/bsj.2021.18.1(Suppl.).0812

Fixed Point Theorems in General Metric Space with an Application

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Received 31/7/2019, Accepted 26/10/2020, Published 30/3/2021

Abstract

This paper aims to prove an existence theorem for Voltera-type equation in a generalized G- metric space, called the ϑ_v-metric space, where the fixed-point theorem in ϑ_v- metric space is discussed and its application. First, a new contraction of Hardy-Rogess type is presented and also then fixed point theorem is established for these contractions in the setup of ϑ_v-metric spaces. As application, an existence result for Voltera integral equation is obtained.

Keywords: Contraction mappings , Fixed point, Integral inclusion, ϑ_v - metric space.

Introduction

The Banach's contraction concept is the most essentail outcome in non-linear analysis(1). Many researchers have generalized and utilized this principle, such as, (2-7). Various applications of Banach Principle have been presented. One of these applications is solving Voltera integral equations via fixed point theorems. As known, Banach, is the first to do this in his Ph.D. thesis(1) . For many other results in this branch see,(8-15). It is worth noting to citation a new Al-Bundy's results (16)about constructing a fractal in $\vartheta -$ metric spaces. In the current paper, Following previous researchers the authors achieve new results in this work.

Definition 1 (5) : Let B be an non-empty set, a mapping ϑ_v: $B^3 \to \mathcal{R}_+$ is said tobe (ϑ_v–metric) for all $m_1, m_2, m_3, z \in B$ and $v \geq 1$ be a given real number satisfy the following

i. $\vartheta_v(m_1,m_2,m_3) = 0$ *if and only if* $m_1 = m_2 = m_3$
ii. $\vartheta_v(m_2,m_1,m_3) = \vartheta_v(m_1,m_2,m_3) = \vartheta_v(m_1,m_3,m_2)$
iii. $\vartheta_v(m_2,m_1,m_2) \leqslant \vartheta_v(m_1,m_2,z)$ with $m_1 \neq m_2$
iv. $\vartheta_v(m_2,m_1,m_2) \succ 0$ with $m_1 \neq m_2$
v. $\vartheta_v(m_1,m_2,m_3) \leq v\,(\vartheta_v(m_1,z,z) + \vartheta_v(z,m_2,m_3))$

Then (B,ϑ_v) is called a ϑ_v–metric.

Proposition 2 (5):

1. $\vartheta_v(m_1,m_2,m_3) \leq v\,(\vartheta_v(m_1,z,m_3) + \vartheta_v(z,m_2,m_3))$
2. $\vartheta_v(m_1,m_1,m_2) \leq 2v\,\vartheta_v(m_1,m_2,m_2)$.

Let $C(B)$ = the class of all non-empty closed sub-sets and Ӈ= Hausdorff ϑ_a–metric

Now Hausdorff ϑ_a–Metric is defined as follows (16)
Let Q , Ғ, X ∈ B , Ӈ : $C(B) \times C(B) \to \mathcal{R}_+$ such that

$$Ӈ(Q,Ғ,X) = max\{Sup_{x\in Q}\vartheta_v(x,Ғ,X), Sup_{x\in Ғ}\vartheta_v(x,Q,X), Sup_{x\in X}\vartheta_v(x,Ғ,Q)\}$$

where , $\vartheta_v(x,Ғ,Q) = d_{\vartheta_v}(x,Ғ) + d_{\vartheta_v}(Ғ,Q) + d_{\vartheta_v}(x,Q)$,
$d_{\vartheta_v}(x,Ғ) = inf\{d_{\vartheta_v}(x,f): f \in Ғ\}$.

Definition 3 (12): Let Ғ = the family of all fiunctions, Ꮃ: $\mathcal{R}^{++} \to \mathcal{R}$ be function suchthat

Ꮃ$_1$:- Ꮃis strictly nondecreasing, $d_1 < d_2 \to$ Ꮃ (d_1) < Ꮃ (d_2) ,∀ $d_2, d_1 \in (0,\infty)$;
Ꮃ$_2$:- $\lim_{i\to\infty}$ Ꮃ $(x_i) = -\infty$ ⟺ $\lim_{i\to\infty} x_i = 0$, foreach sequence$\langle x_i\rangle$ of positive real numbers ;
Ꮃ$_3$:- If $\lim_{i\to\infty} x_i = 0$, there exists $0 < a < 1$ such that $\lim_{i\to\infty}(x_i)^a$Ꮃ $(x_i) = 0$
Ꮃ$_4$:- ∀ sequence $\langle\beta_i\rangle \in \mathcal{R}^+$such tht ϵ + Ꮃ$(v\beta_i) \leq$ Ꮃ$(v\beta_{i-1})$. Some $\epsilon > 0$ and ∀i ∈N, so
ϵ + Ꮃ$(v^i\beta_n) \leq$ Ꮃ$(v^{i-1}\beta_{i-1})$.

Note (12):- For each $x > 0$

- Ꮃ$(d) = ln_d + d$

- $Ш(d) = ln_d$.

Main Results:

Initially, the following must be proved .

Lemma 4 : Let (B,ϑ_v) be a ϑ_v–metric ,and give any sequence in B (take $\langle z_i\rangle$), $\exists\ t > 0$ *with* $Ш \in Ғ$ and $i \in N$

$$t + Ш[\ v\ \vartheta_v(z_i, z_{i+1}, z_{i+2})] \le Ш[\ \vartheta_v(z_{i-1}, z_i, z_{i+1})] \quad \ldots(1)$$

Then the sequence is Cauchy.

Proof - Suppose that $\vartheta_{v_i} = \vartheta_v(z_i, z_{i+1}; z_{i+1})$,

$\Rightarrow\ t + Ш[\ v^i\ \vartheta_{v_i}] \le Ш[\ v^{i-1}\ \vartheta_{v_{i-1}}]$, from (1) and $Ш_4$.

Now, $t + Ш[\ v^{i-1}\ \vartheta_{v_{i-1}}] \le Ш[\ v^{i-2}\ \vartheta_{v_{i-2}}]$,

$t + Ш[\ v^{i-2}\ \vartheta_{v_i}] \le Ш[\ v^{i-3}\ \vartheta_{v_{i-3}}]$,

$\vdots$

$$Ш[\ v^i\ \vartheta_{v_i}] \le Ш[\ v^{i-1}\ \vartheta_{v_{i-1}}] - t \le Ш[\ v^{i-2}\ \vartheta_{v_{i-2}}] - 2t \le Ш[\ v^{i-1}\ \vartheta_{v_{i-1}}] \le \cdots \le Ш[\ \vartheta_{v_0}] - it \quad \ldots(2)$$

$\Rightarrow\ Ш[\ v^i\ \vartheta_{v_i}] \le Ш[\ \vartheta_{v_0}] - it$ when $i \to \infty \Rightarrow \lim_{i\to\infty} Ш(\ v^i\ \vartheta_{v_i}) = -\infty$.

From $Ш_2 \Rightarrow \lim_{i\to\infty}\ v^i\ \vartheta_{v_i} = 0$. And by condition $Ш_3$, *there exsits* $0 < u < 1$ such that $\lim_{i\to\infty}(\ v^i\ \vartheta_{v_i})^u\ Ш(\ v^i\ \vartheta_{v_i}) = 0$

By using (2)

$$(\ v^i\ \vartheta_{v_i})^u\ Ш(\ v^i\ \vartheta_{v_i}) \le (\ v^i\ \vartheta_{v_i})^u\ Ш(\ \vartheta_{v_0}) - (\ v^i\ \vartheta_{v_i})^u it$$

$$\le -(\ v^i\ \vartheta_{v_i})^u ir \le 0 \quad \ldots(3)$$

when $i \to \infty$, then

$$\lim_{i\to\infty} i(\ v^i\ \vartheta_{v_i})^u = 0 \quad \ldots(4)$$

Then there exists $p \in N$ such that $i(\ v^i\ \vartheta_{v_i})^u \le 1$, *for each* $i \ge p$.

$$\Rightarrow\ v^i\ \vartheta_{v_i} \le \frac{1}{(i)^{\frac{1}{p}}} \quad \ldots(5)$$

Let $i, f \in N$ since $p < i < f$ .from (5) and definition (1-v), the lead to

$$\vartheta_v(z_i, z_f, z_f) \le \sum_{a=1}^{f-1}\ v^a\ \vartheta_{v_a} \le \sum_{a=1}^{\infty}\ v^a\ \vartheta_{v_a} \le \sum_{a=1}^{\infty} \frac{1}{(a)^{\frac{1}{p}}} \prec \epsilon$$

That is $\langle z_i\rangle$is Cauchy.

Definition 5 : Let B is ϑ_v–metric, $S: B \to C(B)$ mapping with a function $\gamma: B \times B \to R_+$ is called $(Ш,\ \vartheta_v)$ -contraction if $\exists Ш \in Ғ, t > 0$ which that

$$t + Ш[\ vҤ(S_k, S_c, S_d)] \le Ш[\underline{\Delta}(k, c, d)] \quad \ldots(6)$$

with $\underline{\Delta}(k, c, d) = l_1\vartheta_v(k, c, d) + l_2\vartheta_v(k, S_k, S_d) + l_3\ \vartheta_v(c, S_c, S_d)$ Since $min\{\underline{\Delta}(k, c, d), Ҥ(S_k, S_c, S_d)\} > 0$. Satisfying the condition $l_1 + 2v\ l_2 + l_3 = 1, l_3$ *not equal one* and $l_1, l_2, l_3 \ge 0$.

Theorem 6 : let $S: B \to C(B)$ be $(Ш,\ \vartheta_v)$-contraction, B is complete ϑ_v– *metric and* $t > 0$ such that the following

let $c_0 \in S,\ \vartheta_v(c_0, c_1, c_2)$ since $c_1 \in B_{c_0}, c_2 \in S_{c_1}$

a. S_C is closed for any sequence $\langle c_i\rangle$, *then* $(c_i, c_{i+1}, c_{i+2}) \Rightarrow (c_i, c, c)$.

Then S has a fixed point .

Proof- by(a) , $c_1 \in B_{c_0},\ \vartheta_v(c_0, c_1, c_2)$ and $\exists c_3 \in B_{c_2}$

$$v\ \vartheta_v(c_1, c_2, c_3) \le\ vҤ(S_{c_0}, S_{c_1}, S_{c_2})$$

Since $Ш_1$, $Ш(\ v\ \vartheta_v(c_1, c_2, c_3)) \le Ш\left(vҤ(S_{c_0}, S_{c_1}, S_{c_2})\right) \quad \ldots.(7)$

By (6,7) , then

$$t + Ш(\ v\ \vartheta_v(c_1, c_2, c_3)) \le t + Ш\left(vҤ(S_{c_0}, S_{c_1}, S_{c_2})\right) \le Ш[l_1\ \vartheta_v(c_0, c_1, c_2) + l_2\ \vartheta_v(c_0, c_1, c_3) + l_3\ \vartheta_v(c_1, c_2, c_3)]$$

From Proposition (2)

$$\le Ш[l_1\vartheta_v(c_0, c_1, c_2) + l_2(v\ \vartheta_v(c_0, c_1, c_2) + v\ \vartheta_v(c_1, c_2, c_3)) + l_3\vartheta_v(c_1, c_2, c_3)]$$

where $l_1 + 2l_2 + l_3 = 1$ and$Ш_1$,

$$\Rightarrow\ v\ \vartheta_v(c_1, c_2, c_3) - vl_2\ \vartheta_v(c_1, c_2, c_3) - l_3\vartheta_v(c_1, c_2, c_3) \le l_1(c_0, c_1, c_2) + vl_2\ \vartheta_v(c_0, c_1, c_2)$$

$$(1 - vl_2 - l_3)\ \vartheta_v(c_1, c_2, c_3) \leqslant (l_1 + vl_2)\ \vartheta_v(c_0, c_1, c_2) .$$

$$\Rightarrow t + Ш(\ v\ \vartheta_v(c_1, c_2, c_3)) \le Ш[\ \vartheta_v(c_0, c_1, c_2)]$$

By continuous in this away , leads to

$$r + Ш(\ v\ \vartheta_v(c_i, c_{i+1}, c_{i+2})) \le Ш[\ \vartheta_v(c_{i-1}, c_i, c_{i+1})], \forall i \in N$$

From lemma (4) , then$\langle c_i\rangle$is cauchy sequence. Since B is complete $\exists c \in B \Rightarrow c_i \to c$.

Using the condition (b), that is $\vartheta_v(S_c, c, c) = 0 \Rightarrow c \in S_c$. If converse that $c \notin S_c$, then $\exists m \in N$ *such that* $\vartheta_v(c_i, c, S_c) > 0\ \forall m < i$.

Then

$$\vartheta_v(c, S_c, S_c) \le v\ \vartheta_v(c, c_{i+1}, c_{i+1}) + v\ \vartheta_v(c_{i+1}, S_c, S_c)$$

$$\le v\ \vartheta_v(c, c_{i+1}, c_{i+1}) + Ҥ(S_{c_i}, S_c, S_c)$$

$$\le v\ \vartheta_v(c, c_{i+1}, c_{i+1}) + l_1\ \vartheta_v(c_i, c, c) + l_2\ \vartheta_v(c_i, c_{i+1}, S_c) + l_3\ \vartheta_v(c, S_c, S_c)$$

$$\le l_2\ \vartheta_v(c, c, S_c) + l_3\ \vartheta_v(c, S_c, S_c)$$

$\le 2v\ l_2\ \vartheta_v(c, S_c, S_c) + l_3\ \vartheta_v(c, S_c, S_c)$, by Proposition (2)

$\Rightarrow\ \vartheta_v(c, S_c, S_c) \le (2v\ l_2 + l_3)\ \vartheta_v(c, S_c, S_c) < \vartheta_v(c, S_c, S_c)$. This is contraction.

Collaray 7 : Let $S: B \to C(B)$ be $(\mathrm{W}, \vartheta_v)$-contraction, B is complete ϑ_v– metric ,$v > 0$ and satisfying all conditions with

$$v\mathrm{H}(S_k, S_c, S_d) \Big/ \underline{\Delta}(k,c,d) \quad e^{v\mathrm{H}(S_k,S_c,S_d)-\underline{\Delta}(k,c,d)} \leq e^{-t}$$

*T*hen S has a fixedpoint.

Example 8: Let B={0,1,2,3} , $\vartheta_v(a,b,b) = [2|a-b|]^2$ is ϑ_v–metric for each a, b$\in B$ since $v = 2$. Let $f: B \to C(B)$, define

$$f_a = \begin{cases} \{0,1\} & if a = 0,1 \\ \{2,3\} & if\ a > 1 \end{cases}.$$

Solution:- $\forall\, a, b > 1$ and $a \neq b$, suppose $l_1 = 1, and\ l_2, l_3$ is equal zero .

That is, $min\{\vartheta_v(a,b,b), \mathrm{H}(f_a, f_b, f_b)\} > 0$. Since $\vartheta_v(a,b,b) = 4$, $\mathrm{H}(f_a, f_b, f_b) = 4$,$t = \frac{1}{2}$ and from corollary(7).

$$\Rightarrow \quad e^0 \leq e^{-\frac{1}{2}}$$

with $\mathrm{W}_a = \ln a + a$, $\forall\, a \in R^+$.

Then satisfies all conditions theorem 6), $\in f_a$. ◘

Application

In this section the gotten outcomes were used to attain the existence of solutions for a specific Fredholmtype integral consolidation. The application is motivated by(12)

Express the Fredholm-type as follows

$y(u) \in \int_a^u ⍰\,(u, x, y(x)) + \alpha(u)$. u ∈ [a, c]

Let $G_{CV}(R)$ = the family of non-empty convex and compact subset R, $⍰: [a,c]^2 \times R \to G_{CV}$, the operator $⍰_y := ⍰(u, x, y(x))$ is continuous since $\alpha: [a, c] \to R$is continuous for all $y \in C[a,c]$.

Now, B is complete $\vartheta_v - metric$ by considering

$\vartheta_v(x_1, x_2, x_3)$=$\sup_{u\in[a,c]}$ $[|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1|]^2$,$for\ v = 2$.

Theorem 9: Let $\delta = C([a,c], R)$ and let the set-valued operator $f: \delta \to C(\delta)$ defined by

$$f_{y(u)}\left\{ s \in \delta: s \in \int_a^u ⍰\,(u, x, y(x)) + K(u),\ u \in [a,c] \right\}.$$

Since $K(u)$ is continuos

And assume the following:

1- There exists $h: [a,c] \to R$ is a continuous function such that

$$\mathrm{H}\left(⍰(u,x,b_1(x)), ⍰(u,x,b_2(x)), ⍰(u,x,b_3(x))\right) \leq h(x)[|b_1(x) - b_2(x)| + |b_2(x) - b_3(x)| + |b_3(x) - b_1(x)|]$$

2- For each $b_1, b_2, b_3 \in B$,$\exists t > 0$.let that

$$\int_a^u h(x) \leq \sqrt{e^{-t}}$$

Then the operator has a fixed point.

Proof - the operator f should be satisfied all hypothesis of Theorem(6). Initially, the equation (6) must be inspected. Let $b_1, b_2, b_3 \in B such\ that\ s \in f_{b_1}$.

$\Rightarrow$ $⍰_{b_1}(u,x) \in ⍰_{b_1}(u,x), such\ that\ s_u = \int_a^u ⍰_{b_1}(u,x)dx + K(u) for\ u \in [a,c]$.

However, put $⍰_{b_2}(u,x) \in ⍰_{b_2}(u,x) = ⍰_{b_3}$ (u,x) ,by condition (i) makes sure that $\exists y\,(u,x) \in ⍰_{b_2}$(u,x) such that

$|⍰_{b_1}(u,x) - y(u,x)| + |y(u,x) - ⍰_{b_1}(u,x)| \leq h(x)[|b_1(x) - b_2(x)| + |b_2 x) - b_1(x)|]$ $for all\ x \in$ [a,c]

Let us take into consideration the multivalued operator T defined by

$$T_{(u,x)} = ⍰_{b_2}(u,x) \cap \left\{ \begin{matrix} q \in R: |⍰_{b_2}(u,x) - q| + |q - ⍰_{b_2}(u,x)| \\ \leq h(x)[|b_1(x) - b_2(x)| + |b_2(x) - b_1(x)|] \end{matrix} \right\}.$$

$$f_u = \int_a^u ⍰_{b_2}(u,x)dx + h(x) since\ ⍰_{b_2}(u,x) \in ⍰_{b_2}(u,x)$$

Then

$$[\,|s_u - f_u| + |f_u - s_u|\,]^2 \leq \left[\int_a^u \left(|⍰_{b_1}(u,x) - ⍰_{b_2}(u,x)| + |⍰_{b_2}(u,x) - ⍰_{b_1}(u,x)|\right) dx\right]^2$$

$$\leq \left[\int_a^u h(x)\left(|b_1(x) - b_2(x)| + |b_2(x) - b_1 x|\right) dx\right]^2$$

$$\leq \left[\sqrt{\delta\sup_{x\in[a,c]}\left(|b_1(x) - b_2(x)| + |b_2(x) - b_1(x)|\right)^2} \int_a^u h(x)\,dx\right]^2$$

$\leq e^{-t}\ \vartheta_v(b_1, b_2, b_2) \Rightarrow \vartheta_v(s, f, f) \leq e^{-t}\ \vartheta_v(b_1, b_2, b_2)$

By simply swapping the role of b_1& b_2, and applied natural logarithm, the lead to

$$r + \mathrm{W}[\,\mathrm{H}(f_{b_1}, f_{b_2}, f_{b_2})] \leq \mathrm{W}[\,\vartheta_v(b_1, b_2, b_2)]$$

Since $\mathrm{W}(d) = \ln\ d$ and f is [$(\mathrm{W}, \vartheta_v)$ –contraction with $l_1 = 1, l_2 = l_3 = 0$.After that all the condition for theorem are satisfied (6). Hence the operator f has afixed point.

Conclusion:

The effect of this study indicates that the integral Volterra equations satisfying the contractive condition have a fixed point. The solution of an integral equation by the fixed point method is approximated by showing some suitable conditions guarantee the convergence of the method.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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مبرهنات النقطة الصامدة في فضاء متري معمم مع تطبيق

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الخلاصة:

يهدف هذا البحث إلى إثبات مبرهنة وجود لمعادلة من نوع فولتيرا في تعميم فضاء G- متري يسمى فضاء ϑ_v - المتري، حيث تتم مناقشة مبرهنة النقطة الصامدة في فضاء ϑ_v - المترية وتطبيقها. أولاً ، تم تقديم انكماش جديد من نوع هاردي روجيس ثم تم كذلك إنشاء مبرهنة النقطة الصامدة لهذه الانكماشات في حالة فضاء ϑ_v - المترية. كتطبيق ، تم الحصول على نتيجة وجود معادلة فولتيرا التكاملية.

الكلمات المفتاحية: تطبيقات انكماشية ، النقطة الصامدة، فضاء ϑ_v - المترية ، التضمين التكاملي .