Fixed Point Theorems in General Metric Space with an Application

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Abstract

This paper aims to prove an existence theorem for Volterra-type equation in a generalized G- metric space, called the \( \vartheta_v \)-metric space, where the fixed-point theorem in \( \vartheta_v \)- metric space is discussed and its application. First, a new contraction of Hardy-Rogers type is presented and also then fixed point theorem is established for these contractions in the setup of \( \vartheta_v \)-metric spaces. As application, an existence result for Volterra integral equation is obtained.

Keywords: Contraction mappings , Fixed point, Integral inclusion, \( \vartheta_v \) - metric space.

Introduction

The Banach’s contraction concept is the most essential outcome in non-linear analysis(1). Many researchers have generalized and utilized this principle, such as, (2-7). Various applications of Banach Principle have been presented. One of these applications is solving Volterra integral equations via fixed point theorems. As known, Banach, is the first to do this in his Ph.D. thesis(1) . For many other results in this branch see,(8-15). It is worth noting to mention a new Al-Bundy’s results (16)about constructing a fractal in \( \vartheta \) - metric spaces. In the current paper, Following previous researchers the authors achieve new results in this work.

Definition 1 (5) : Let \( B \) be an non-empty set, a mapping \( \vartheta_v : B^3 \rightarrow \mathcal{R}_{+} \) is said to be a \( \vartheta_v \)-metric for all \( m_1, m_2, m_3, z \in B \) and \( v \geq 1 \) be a given real number satisfy the following

1. \( \vartheta_v(m_1, m_2, m_3) = 0 \) if and only if \( m_1 = m_2 = m_3 \)
2. \( \vartheta_v(m_1, m_2, m_3) = \vartheta_v(m_2, m_1, m_3) = \vartheta_v(m_1, m_3, m_2) \)
3. \( \vartheta_v(m_2, m_1, m_3) \leq \vartheta_v(m_1, m_2, z) \) with \( m_1 \neq m_2 \)
4. \( \vartheta_v(m_2, m_1, m_3) \geq 0 \) with \( m_1 \neq m_2 \)
5. \( \vartheta_v(m_1, m_2, m_3) \leq v(\vartheta_v(m_1, z, z) + \vartheta_v(z, m_2, m_3)) \)

Then \( (B, \vartheta_v) \) is called a \( \vartheta_v \)-metric.

Proposition 2 (5): 
1. \( \vartheta_v(m_1, m_2, m_3) \leq v(\vartheta_v(m_1, z, m_3) + \vartheta_v(z, m_2, m_3)) \)
2. \( \vartheta_v(m_1, m_3, m_2) \leq 2v(\vartheta_v(m_1, m_2, m_2)) \).

Let \( C(B) = \) the class all non-empty closed sub-sets and \( H\) Hausdorff \( \vartheta_v \)-metric

Now Hausdorff \( \vartheta_v \)-Metric is defined as follows

Let \( Q, F, X \in B \), \( H : C(B) \times C(B) \rightarrow \mathcal{R} \), such that

\[ H(Q, F, X) = \max\{\vartheta_v(x, F, X), \vartheta_v(x, Q, X), \vartheta_v(x, Q, F)\} \]

where , \( \vartheta_v(x, F, X) = d_{\vartheta_v}(x, F) + d_{\vartheta_v}(F, Q) + d_{\vartheta_v}(x, Q) \).

Definition 3 (12): Let \( F \) = the family of all functions, \( W : \mathcal{R}^+ \rightarrow \mathcal{R} \) be function such that

Wi: \( W \) is strictly nondecreasing, \( d_1 < d_2 \rightarrow W(d_1) < W(d_2) \) \( \forall d_1, d_2 \in (0, \infty) \);

Wii: \( \lim_{i \rightarrow \infty} W(x_i) = -\infty \iff \lim_{i \rightarrow \infty} x_i = 0 \) foreach sequence \( x_i \) of positive real numbers;

Wiii: If \( \lim_{i \rightarrow \infty} x_i = 0 \), there exists \( 0 < a < 1 \) such that \( \lim_{i \rightarrow \infty} (x_i)^a W(x_i) = 0 \);

Wiv: \( \forall \) sequence \( (\beta_i) \in \mathcal{R}^+ \) such that \( \epsilon + W(v \beta_i) \leq W(v \beta_{i-1}) \). Some \( \epsilon > 0 \) and \( \forall i \in \mathcal{N} \), so \( \epsilon + W(v \beta_i) \leq W(v \beta_{i-1}) \).

Note (12):- For each \( x > 0 \)

\[ W(d) = \ln d + d \]
\[ W(d) = \ln d. \]

**Main Results:**
Initially, the following must be proved.

**Lemma 4:** Let \((B, \theta)\) be an \(\theta\)-metric space and any sequence in \(B\) (take \(z_i\)), \(\exists t > 0\) with \(W \in F\) and \(i \in N\)

\[ t + W[v \theta_1(q_{i+1}, z_{i+2})] \leq W[\theta_1(q_{i+1}, z_{i+1})] \quad \ldots (1) \]

Then the sequence is Cauchy.

Proof - Suppose that \(\theta_{v_i} = \theta_{v}(q_{i+1}, z_{i+1})\).

\[ \Rightarrow t + W[v^{-1} \theta_{v_i-1}] \leq W[v^{-1} \theta_{v_{i-1}}] \quad \text{from (1)} \]

and \(W_4\).

Now, \(t + W[v^{-2} \theta_{v_{i-1}} - 2t] \leq W[v^{-1} \theta_{v_{i-1}}] \leq \ldots \leq W[v^{-i} \theta_{v_{i-1}}] \quad \text{from (2)} \]

\[ \Rightarrow W[v^{-i} \theta_{v_i}] = W[v^{-i} \theta_{v_i}] - it \quad \text{when} \]

\(i \to \infty \Rightarrow \lim_{i \to \infty} W[v^{-i} \theta_{v_i}] = -\infty. \)

From \(W_2 \Rightarrow \lim_{i \to \infty} v^{-i} \theta_{v_i} = 0.\)

And by condition \(W_3\), there exists \(u < 1 < \infty\) such that \(\lim_{i \to \infty} u^{-1} W(v^{-i} \theta_{v_i}) = 0.\)

By using (2)

\[ (v^{-i} \theta_{v_i})^u W(v^{-i} \theta_{v_i}) \leq \left( v^{-i} \theta_{v_i} \right)^u W(v^{-i} \theta_{v_i}) \]

\[ \leq -(v^{-i} \theta_{v_i})^u \eta r \leq 0 \quad \ldots (3) \]

when \(i \to \infty\), then

\[ \lim_{i \to \infty} i(v^{-i} \theta_{v_i})^u = 0 \quad \ldots (4) \]

Then there exists \(p \in N\) such that \(i(v^{-i} \theta_{v_i})^u \leq 1\), for each \(i \geq p\),

\[ \Rightarrow v^{-i} \theta_{v_i} \leq \frac{1}{i^p} \quad \ldots (5) \]

Let \(i, f \in N\) since \(p < i < f\). from (5) and definition (1-v), the lead to

\[ \theta_{v}(z_i, z_f, z_i) \leq \sum_{a=1}^{f-i} v^{-a} \theta_{v_a} \leq \sum_{a=1}^{\infty} v^{-a} \theta_{v_a} \leq \sum_{a=1}^{\infty} \frac{1}{(a)^p} \leq \varepsilon \]

That is \((z_i)\) is Cauchy.

**Definition 5:** Let \(B\) be a \(\theta\)-metric space. \(S: B \to C(B)\) mapping with a function \(y:B \times B \to R_{+}\) is called \((W, \theta)\)-contraction if \(\exists W \in F, t > 0\) which that

\[ t + W[H(S_k, S_c, S_d)] \leq W[\Delta(k, c, d)] \quad \ldots (6) \]

with \(\Delta(k, c, d) = l_1 \theta_{v}(k, c, d) + l_2 \theta_{v}(k, S_k, S_d) + l_3 \theta_{v}(c, S_c, S_d)\) Since

\[ \min\{\Delta(k, c, d), H(S_k, S_c, S_d)\} > 0. \]

Satisfying the condition \(l_1 + 2l_2 + l_3 = 1, l_3 \neq 0\) and \(l_1, l_2, l_3 \geq 0.\)

**Theorem 6:** let \(S: B \to C(B)\) be \((W, \theta)\)-contraction, \(B\) is complete \(\theta\)-metric and \(t > 0\) such that the following

\[ \exists c \in S, \theta_{v}(0, c, c) \leq \theta_{v}(S_0, S_c, S_d) \]

Since \(W_1: W(\theta_{v}(0, c, c)) \leq W(\theta_{v}(S_0, S_c, S_d)) \quad \ldots (7) \]

By (6,7), then

\[ t + W[\theta_{v}(0, c, c)] \leq t + W[\theta_{v}(S_0, S_c, S_d)] \leq W[l_1 \theta_{v}(0, c, c) + l_2 \theta_{v}(0, c, c) + l_3 \theta_{v}(0, c, c)] \]

From Proposition (2)

\[ \leq W[l_1 \theta_{v}(0, c, c) + l_2 \theta_{v}(0, c, c) + l_3 \theta_{v}(0, c, c)] \]

where \(l_1 + 2l_2 + l_3 = 1\) and \(W_1\).

\[ \Rightarrow \theta_{v}(0, c, c) \leq \theta_{v}(0, c, c) \]

\[ (1 - 2l_2 - l_3) \theta_{v}(0, c, c) \leq (l_1 + 2l_2) \theta_{v}(0, c, c) \]

\[ \Rightarrow W[\theta_{v}(0, c, c)] \leq W(\theta_{v}(0, c, c)) \quad \ldots (3) \]

By continuous in this way, leads to

\[ r + W[\theta_{v}(0, c, c)] \leq W[l_1 \theta_{v}(0, c, c) + l_2 \theta_{v}(0, c, c) + l_3 \theta_{v}(0, c, c)] \]

From Lemma (4), then \((c_i)\) is a Cauchy sequence.

Since \(B\) is complete \(\exists cl \in B \Rightarrow c \in c\).

Using the condition (b), that is \(\theta_{v}(S_c, c, c) = 0 \Rightarrow c \in S_c\).

If conversely \(c \in S_c\), then \(\exists m \in N\) such that \(\theta_{v}(0, c, c) > 0 \forall m < i\).

Then

\[ \theta_{v}(c, S_c, S_c) \leq \theta_{v}(c, S_c, S_c) \]

\[ + \theta_{v}(c, S_c, S_c) \]

\[ \leq \theta_{v}(c, S_c, S_c) + \theta_{v}(c, S_c, S_c) \]

\[ \leq 2 \theta_{v}(c, S_c, S_c) \]

by Proposition (2)

\[ \Rightarrow \theta_{v}(c, S_c, S_c) \leq 2 \theta_{v}(c, S_c, S_c) \]

This is contraction.
Collary 7: Let $S:B \to C(B)$ be $(\mathcal{W}, \vartheta_q)$- contraction, B is complete $\vartheta_q$-metric for each a, b $\in B$ since $\nu = 2$. Let $f:B \to C(B)$, define

$$f_a = \begin{cases} \{0,1\} & \text{if } a = 0,1 \\ \{2,3\} & \text{if } a > 1 \end{cases}$$

Solution:- $\forall a, b > 1$ and $\vartheta_q(a, b) > 0$. Suppose $\vartheta_q(a, b) = 4$, $H(f_a, f_b, f_b) = 4$, $t = \frac{1}{2}$ and from corollary 7.

$$e^0 \leq e^{-\frac{1}{2}}$$

with $\mathcal{W}_a = \ln a + a$, $\forall a \in \mathbb{R}^+$. Then satisfies all conditions theorem 6, $\in f_a$. □

Application

In this section the gotten outcomes were used to attain the existence of solutions for a specific Fredholm type integral consolidation. The application is motivated by $y(12)$.

Express the Fredholm-type as follows:

$$y(u) = \int_a^u \mathbb{K}(u, x, y(x)) + \alpha(u). u \in [a, c]$$

Let $G_C(R) = \text{the family of non-empty convex and compact subset } R, \mathbb{E} : [a, c]^2 \times R \to G_C\text{, the operator } \mathbb{E}(y) = \mathbb{K}(u, x, y(x)) \text{ is continuous since } \alpha : [a, c] \to R \text{ is continuous for all } y \in C[a, c].$

Now, B is complete $\vartheta_q$-metric by considering $\vartheta_q(x_1, x_2, x_3) = \sup_{x \in [a, c]} |x_1 - x_2| + |x_2 - x_3| + |x_3 - x_1| \leq 2.$

Theorem 9: Let $\delta = C([a, c], R)$ and let the set-valued operator $f: \delta \to C(\delta)$ defined by

$$f(y)(u) = \begin{cases} 0 & u \in [a, c] \\ \mathbb{K}(u, x, y(x)) + \alpha(u) \end{cases}$$

Since $\mathbb{K}(u)$ is continuous and continuously differentiable.

And assume the following:

1. There exists $h: [a, c] \to R$ is a continuous function such that

$$H \big( \mathbb{K}(u, x, b_1(x)), \mathbb{K}(u, x, b_2(x)), \mathbb{K}(u, x, b_3(x)) \big) \leq h(x)[|b_1(x) - b_2(x)| + |b_2(x) - b_3(x)| + |b_3(x) - b_1(x)|]$$

2. For each $b_1, b_2, b_3 \in B, \exists t_0 > 0$, let that

$$\int_a^u h(x) \leq \sqrt{e^{-2t}}$$

Then the operator has a fixed point.

Proof - the operator $f$ should be satisfied all hypothesis of Theorem 6. Initially, the equation (6) must be inspected. Let $b_1, b_2, b_3 \in B$ such that $s \in f_{b_1}$.

$$\Rightarrow \quad \square_{b_1}(u, x) \cap \square_{b_1}(u, x), \text{ such that } s_u = \int_a^u \square_{b_1}(u, x) + K(u) \text{ for } u \in [a, c].$$

However, put $\square_{b_1}(u, x) \in \square_{b_2}(u, x) = \square_{b_2}(u, x)$ by condition (i) makes sure that $\exists y(u, x) \in \square_{b_2}(u, x)$ such that

$$\square_{b_2}(u, x) - y(u, x) \leq h(x)[|b_1(x) - b_2(x)| + |b_2(x) - b_3(x)|] \text{ for all } x \in [a, c].$$

Let us take into consideration the multivalued operator $T$ defined by

$$T(u, x) = \square_{b_2}(u, x) \square_{b_2}(u, x).$$

Then

$$|s_u - f_u| + |f_u - s_u|^2$$

$$\leq \left[ \int_a^u \left( \square_{b_2}(u, x) - \square_{b_2}(u, x) \right) \right] dx$$

$$\leq \left[ \int_a^u h(x)[|b_1(x) - b_2(x)| + |b_2(x) - b_3(x)|] \right] dx$$

By simply swapping the role of $b_1$ & $b_2$, and applied natural logarithm, the lead to

$$r + \mathcal{W} \left[ H(f_{b_1}, f_{b_2}, f_{b_2}) \right] \leq \mathcal{W} \left[ \vartheta_q(b_1, b_2, b_2) \right]$$

Since $W(d) = \ln d$ and $f$ is $[(\mathcal{W}, \vartheta_q)^{-}\text{contraction]}$ with $l_1 = 1, l_2 = l_3 = 0$. After that all the condition for theorem are satisfied (6). Hence the operator $f$ has an fixed point.
Conclusion:

The effect of this study indicates that the integral Volterra equations satisfying the contractive condition have a fixed point. The solution of an integral equation by the fixed point method is approximated by showing some suitable conditions guarantee the convergence of the method.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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