Open Access
Baghdad Science Journal **Vol.16(3)2019**

DOI: http://dx.doi.org/10.21123/bsj.2019.16.3.0654

Common Fixed Point of a Finite-step Iteration Algorithm Under Total Asymptotically Quasi-nonexpansive Maps

Salwa Salman Abed * ***Zahra Mahmood Mohamed Hasan***

Received 20/8/2018, Accepted 12/3/2019, Published 1/9/2019

Abstract:

Throughout this paper, a generic iteration algorithm for a finite family of total asymptotically quasi-nonexpansive maps in uniformly convex Banach space is suggested. As well as weak / strong convergence theorems of this algorithm to a common fixed point are established. Finally, illustrative numerical example by using Matlab is presented.

Key words: Banach space, Common fixed points, Strong convergence, Total asymptotically quasi-nonexpansive map, Weak convergence.
MSC: 49J40; 47J20

Introduction:

The nonlinear equations $T(x) = y$ appearing in physical formulations can similarly be transformed into a fixed point equation of the form $x = Fx$. An approximate fixed point theorem is applied to get results on existence and uniqueness of such equations' solution. To decide whether an iteration method is useful for application, it is of paramount importance to answer the following question: Does it converge to fixed point of an operator?

Throughout this paper we examine essential concept based on the above question for a new finite-step iteration algorithm when applied to finite family of total asymptotically quasi-nonexpansive maps. Fixed point iteration algorithms may exhibit radically different behaviors for various classes of maps. While a particular fixed point iteration algorithm is convergent for an appropriate class of maps, it could not be convergent for others. Therefore, it is important to determine whether an iteration algorithm converges to fixed point of a map. In this field, there are numerous works regarding convergence of various iteration methods, as one can see in (1–13). Let M be a Banach space, B subset of M and T be a selfmap of B with set of all fixed points $F(T)$. In this work, we construct an iteration algorithm for a finite family of total asymptotically quasi-nonexpansive maps and give appropriate conditions for strong/weak convergence of this algorithm to common fixed points of the maps in uniformly convex Banach space.

Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq.
*Corresponding author: salwaalbundi@yahoo.com

Definition(1): A map $T: B \to B$ is called :

i-Nonexpansive map[1] if $\|Ta - Tb\| \le \|a - b\|$ for all $a, b \in B$.

ii-Quasi-nonexpansive map [2] if $F(T) \neq \emptyset$ and $\|Ta - a^*\| \le \|a - a^*\|$ for all $a \epsilon B$and for all $a^* \epsilon F(T)$.

iii-Asymptotically nonexpansive[1] if there is a sequence $(f_n) in$ $[0, +\infty)$ with $\lim_{n\to\infty} f_n = 0$ and $\|T^n a - T^n b\| \le (1 + f_n)\|a - b\|$ for all $a, b \in B$ and $n = 1, 2 \ldots$

iv-Asmptotically quasi-nonexpansive map [3] if $F(T) \neq \emptyset$ *and* there is a sequence (f_n)in $[0, +\infty)$ with $\lim_{n\to\infty} f_n = 0$ so that $\|T^n a - a^*\| \le (1 + f_n)\|a - a^*\|$, for all $a \in B, a^* \in F(T)$and $n = 1,2 \ldots$

v-Total asymptotically nonexpansive map [4] if there are null sequences of nonnegative real number $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, n \ge 1$ and nondecreasing continuous function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$, for all $a, b \in B$ so that
$\|T^n a - T^n b\| \le \|a - b\| + f_n\psi\|a - b\| + g_n$.

vi-Total asymptotically quasi-nonexpansive map if $F(T) \neq \emptyset$ *and* there are null sequences of nonnegative real number $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, n \ge 1$,
$\sum_{n=1}^{\infty} f_n < \infty$ and $\sum_{n=1}^{\infty} g_n < \infty$, and nondecreasing continuous function $\psi: [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$, for all $a \in B$ and $a^* \in F(T)$ so that
$\|T^n a - a^*\| \le \|a - a^*\| + f_n\psi\|a - a^*\| + g_n$.

vii-Uniformly K-Lipschitzain [5] if there is a constant $K > 0$ for all $a, b \in B$, so that

$$\|T^n a - T^n b\| \leq K\|a - b\|$$

Note(2):

$$\begin{array}{ccccc} \textit{Nonexpansive} & \Rightarrow & \textit{Asymptotically nonexpansive} & \Rightarrow & \textit{Total asymptotically nonexpansive} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \textit{Quasi-nonexpansive} & \Rightarrow & \textit{Asymptotically quasi-nonexpansive} & \Rightarrow & \textit{Total asymptotically quasi-nonexpansive} \end{array}$$

Definition(3)[6]: A Banach space M is called satisfying:

i-Opial's condition if any sequence (a_n) in M, is weakly convergent to a implies that $\lim_{n\to\infty} \inf \|a_n - a\| < \lim_{n\to\infty} \inf \|a_n - b\|$ for all $b \in M$ with $a \neq b$.

ii-Kadec-Klee property if each sequence (a_n) in M converging weakly to (a) jointly with $\|a_n\|$ converging strongly to $\|a\|$ imply that (a_n) converges strongly to a point $a \in M$.

Definition(4)[7]: A map $T: B \to M$ is said to be demiclosed with respect to $b \in M$ if for any sequence $(a_n)\, in\, B$, (a_n) converges weakly to a and $T(a_n)$ converges strongly to b. Therefore $a \in B$ and $T(a) = b$. If $(I - T)$ is demiclosed i.e if (a_n) converges weakly to a in B and $(I - T)$ converges strongly to 0 . Hence $(I - T)(a) = 0$.

Lemma(5)[8]: Let$(\mu_n)_{n=1}^{\infty}$, $(\delta_n)_{n=1}^{\infty}$ and $(e_n)_{n=1}^{\infty}$ be sequences of non negative numbers accomplishing the following inequality:

$$\mu_{n+1} \leq (1 + \delta_n)\mu_n + e_n \;, \forall\, n \geq 1$$

if $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} e_n < \infty$, then $(a_n)$is bounded and $\lim_{n\to\infty} \mu_n$ exists. Additionally if $\lim_{n\to\infty} inf\, a_n = 0$, then $\lim_{n\to\infty} \mu_n = 0$.

Lemma(6)[9]: Let M be a Banach space and $p > 1$ and $K > 0$ be two fixed numbers. Then M is uniformly convex if exists a continuous strictly nondecreasing and convex function $\zeta: [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ so that

$$\|\omega a + (1 - \omega)b\|^p \leq \omega\|a\|^p + (1 - \omega)\|b\| - x_p(\omega)\zeta(\|a - b\|)$$

for each $a, b \in B_K(0) = \{a \in M: \|a\| \leq K\}$ and $\omega \in [0,1]$, where

$$x_p(\omega) = \omega(1 - \omega)^p + \omega^p(1 - \omega)$$

Lemma(7)[10]: Let M be a real reflexive Banach space with its dual M^* accomplishing the Kadec-Klee property. Suppose that (a_n) bounded sequence in M so that $\lim_{n\to\infty}\|ta_n + (1 - t)p_1 - p_2\|$ exists $\forall\, t \in [0,1]$ and $p_1, p_2 \in W_w(a_n)$, where $W_w(a_n)$ attend to a set of all weak subsequential limits of (a_n), then $p_1 = p_2$.

Lemma(8)[11]: Let M be a uniformly convex Banach space, $\emptyset \neq B \subseteq M$. Therefore there is a strictly nondecreasing continuous function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ so that each lipschitzain map $T: B \to B$ for lipschitz constant K:

$$\|tTx + (1 - t)Tz - T(tx + (1 - t)z\| \leq Kf^{-1}\left(\|x - z\| - \frac{1}{K}\|Tx - Tz\|\right),$$

$\forall\, x, z \in B$ and $\forall t \in [0,1]$.

Main Results:

Let B be a nonempty closed convex subset of a Banach space M, $T_j: B \to B, \forall j = 1, 2, \ldots, k$ be total asymptotically quasi-nonexpansive maps. The iteration algorithm (a_n) is defined by:

$$a_1 \in B$$
$$a_{n+1} = (1 - \alpha_{kn})a_n + \alpha_{kn}T_k^n b_{kn}$$
$$b_{kn} = (1 - \alpha_{kn})a_n + \alpha_{kn}T_k^n b_{(k-1)n}$$
$$b_{(k-1)n} = \left(1 - \alpha_{(k-1)n}\right)a_n + \alpha_{(k-1)n}T_{k-1}^n b_{(k-2)n}$$
$$.$$
$$.$$
$$b_{2n} = (1 - \alpha_{2n})a_n + \alpha_{2n}T_2^n b_{1n}$$
$$b_{1n} = (1 - \alpha_{1n})a_n + \alpha_{1n}T_1^n b_{0n} \quad \ldots(1)$$

where $b_{0n} = a_n$ for all n and $(\alpha_n)_{n=1}^{\infty}$ is asquence in [0,1].

Lemma(9): Let M be a Banach sapace $\emptyset \neq B \subseteq M$, T_j, $j = 1, 2, \ldots, k$ be a family of total asymptotically quasi-nonexpansive selfmaps of B. Suppose $\bigcap_{j=1}^{k} F\,(T_j) \neq \emptyset$ and the sequence (a_n) be as in (1). Then

i-There are sequences $(u_n)$and (v_n)in $[0, \infty)$ such that $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} v_n < \infty$ and

$$\|a_{n+1} - a^*\| \leq (1 + u_n)^{j+1}\|a_n - a^*\| + v_n^{j+1},$$

$\forall\, a^* \in \bigcap_{j=1}^{k} F\,(T_j)$ and $\forall\, n$.

ii-There exist constants $Q_1, Q_2 > 0$ such that $\|a_{n+m} - a^*\| \leq Q_1\|a_n - a^*\| + Q_2$, $\forall\, a^* \in \bigcap_{j=1}^{k} F\,(T_j)$ and $n, m = 1, 2, \ldots$.. Suppose that there is $Z > 0$ such that $\psi(\lambda_j) \leq Z\lambda_j$, $j = 1, 2, \ldots, k$.

Proof:

i-Let $a^* \in \bigcap_{j=1}^{k} F\,(T_j)$, $u_n = \max_{1\leq j\leq k} f_{jn}$ and $v_n = \max_{1\leq j\leq k} g_{jn}$, since $\sum_{n=1}^{\infty} f_{jn} < \infty \Rightarrow \sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} g_{jn} < \infty \Rightarrow \sum_{n=1}^{\infty} v_n < \infty$.

Now,

$$\begin{aligned} \|b_{1n} - a^*\| &\leq (1 - \alpha_{1n})\|a_n - a^*\| + \alpha_{1n}\|T_1^n a_n - a^*\| \\ &\leq (1 - \alpha_n)\|a_n - a^*\| + \alpha_{1n}\{\|a_n - a^*\| + f_{1n}\psi(\|a_n - a^*\|) + g_{1n}\} \\ &\leq (1 - \alpha_{1n})\|a_n - a^*\| + \alpha_{1n}\{(1 + f_{1n}Z)\|a_n - a^*\| + g_{1n}\} \\ &\leq (1 + \alpha_{1n}f_{1n}Z)\|a_n - a^*\| + \alpha_{1n}g_{1n} \\ &\leq (1 + u_n)\|a_n - a^*\| + v_n \end{aligned}$$

Assume that $\|b_{jn} - a^*\| \leq (1 + u_n)^j\|a_n - a^*\| + v_n^j$. Then,

$$\begin{aligned} \|b_{(j+1)n} - a^*\| &\leq \left(1 - \alpha_{(j+1)n}\right)\|a_n - a^*\| + \alpha_{(j+1)n}\|T_{j+1}^n b_{jn} - a^*\| \\ &\leq \left(1 - \alpha_{(j+1)n}\right)\|a_n - a^*\| + \alpha_{(j+1)n}\left(1 + f_{(j+1)n}Z\right)\|b_{jn} - a^*\| + \alpha_{(j+1)n}g_{(j+1)n} \end{aligned}$$

$$\leq \left(1-\alpha_{(j+1)n}\right)\|a_n-a^*\| + \alpha_{(j+1)n}(1+u_n)(1+u_n)^j\|a_n - a^*\| + \alpha_{(j+1)n}(1+u_n)v_n^j + \alpha_{(j+1)n}v_n$$
$$\leq (1+u_n)^{j+1}\|a_n-a^*\| + v_n^{j+1}$$

Thus, by induction , we obtain

$$\left\|b_{jn}-a^*\right\| \leq (1+u_n)^j\|a_n-a^*\| + v_n^j$$

for all $j = 1,2,\dots,k$. ... (2)

Now, by (2) we have

$$\|a_{n+1}-a^*\|$$
$$\leq \left(1-\alpha_{jn}\right)\|a_n-a^*\| + \alpha_{jn}\left\|T_j^n b_{jn}-a^*\right\|$$
$$\leq \left(1-\alpha_{jn}\right)\|a_n-a^*\| + \alpha_{jn}\left\{\left(1+f_{jn}Z\right)\left\|b_{jn}-a^*\right\| + g_{jn}\right\}$$
$$\leq \left(1-\alpha_{jn}\right)\|a_n-a^*\| + \alpha_{jn}(1+u_n)(1+u_n)^j \|a_n-a^*\| + \alpha_{jn}(1+u_n)v_n^j + \alpha_{jn}v_n$$
$$\leq \left(1-\alpha_{jn}\right)\|a_n-a^*\| + \alpha_{jn}\left\{1+\sum_{m=1}^{j+1}\frac{j(j+1)\dots(j+2-m)}{m!}u_n^m\right\} \|a_n-a^*\| + v_n^{j+1}\|a_{n+1}-a^*\|$$
$$\leq \left\{1+\sum_{m=1}^{j+1}\frac{j(j+1)\dots(j+2-m)}{m!}u_n^m\right\} \|a_n-a^*\| + v_n^{j+1}$$
$$\leq (1+u_n)^{j+1}\|a_n-a^*\| + v_n^{j+1}$$

ii-From part (i), we obtain

$$\|a_{n+m}-a^*\| \leq (1+u_{n+m-1})^{j+1}\|a_{n+m-1}-a^*\| + v_{n+m-1}^{j+1}$$
$$\leq e^{(1+u_{n+m-1})^{j+1}\|a_{n+m-1}-a^*\|} + e^{v_{n+m-1}^{j+1}}$$
$$\leq e^{(j+1)(1+u_{n+m-1})\|a_{n+m-1}-a^*\|} + e^{(j+1)v_{n+m-1}}$$
$$\leq e^{(i+1)u_{n+m-1}\|a_{n+m-1}-a^*\|} + e^{(i+1)v_{n+m-1}}$$
$$\leq e^{(j+1)\sum_{k=1}^{n+m-1}u_k}\|a_n-a^*\| + e^{(j+1)\sum_{k=1}^{n+m-1}v_k}$$
$$\leq Q_1\|a_n-a^*\| + Q_2$$

Lemma(10): Let M be a uniformly convex Banach space, $\emptyset \neq B \subseteq M$ and $T_j, \forall j = 1,2,\dots,k$ be a family of total asymptotically quasi-nonexpansive selfmaps of B. Assume $\bigcap_{j=1}^{k} F(T_j) \neq \emptyset$ and (a_n) be as in (1). Then $\lim_{n\to\infty}\|a_n-a^*\|$ exists for all $a^* \in \bigcap_{j=1}^{k} F(T_j)$.

Proof: By using Lemma (9.i)

$$\|a_{n+1}-a^*\| \leq (1+u_n)^{j+1}\|a_n-a^*\| + v_n^{j+1}$$
$$\leq (1+u_n)\|a_n-a^*\| + v_n$$

and $\sum_{n=1}^{\infty}u_n < \infty, \sum_{n=1}^{\infty}v_n < \infty$.

So by Lemma (5), we get $\lim_{n\to\infty}\|a_n-a^*\|$ exists for all $a^* \in \bigcap_{j=1}^{k} F(T_j)$.

Lemma(11): Let M be a real Banach space, $\emptyset \neq B \subseteq M$ and $T_j, \forall j = 1,2,\dots,k$ be a family of Lipschitzain and total asymptotically quasi-nonexpansive selfmaps of B. Let (a_n) be as in (1), for all $a_1^*, a_2^* \in \bigcap_{j=1}^{k} F(T_j)$, then $\lim_{n\to\infty}\|ta_n + (1-t)a_1^* - a_2^*\|$ exists for each $t \in [0,1]$.

Proof: By Lemma (5), $\lim_{n\to\infty}\|a_n-a^*\|$ esists for all $a^* \in \bigcap_{j=1}^{k} F\left(T_j\right)$ and (a_n) is bounded. Letting $\gamma_n(t) = \|ta_n + (1-t)a_1^* - a_2^*\|$ for all $t \in [0,1]$. Then, $\lim_{n\to\infty}\gamma_n(0) = \|a_1^*-a_2^*\|$ and $\lim_{n\to\infty}\gamma_n(1) = \|a_n-a_2^*\|$ exist by Lemma (10).

Therefore, for $t \in [0,1]$ and for all $a \in B$, we define the map $R_n: B \to B$ by :

$$b_{1n} = (1-\alpha_{1n})a_n + \alpha_{1n}T_1^n b_{0n}$$
$$b_{2n} = (1-\alpha_{2n})a_n + \alpha_{2n}T_2^n b_{1n}.$$
$$b_{jn} = \left(1-\alpha_{jn}\right)a_n + \alpha_{jn}T_j^n b_{(j-1)n}$$
$$R_n a = \left(1-\alpha_{jn}\right)a + \alpha_{jn}T_j^n b_j$$

Now,

$$\|R_n a - R_n x\| \leq \left(1-\alpha_{jn}\right)\|a-x\| + \alpha_{jn}\left(1+f_{jn}Z\right)\left\|b_j-y_j\right\| + \alpha_{jn}g_{jn}$$
$$\leq \left(1-\alpha_{jn}\right)\|a-x\| + \alpha_{jn}(1+u_n)(1+u_n)^j\|a-x\| + \alpha_{jn}(1+u_n)v_n^j + \alpha_{jn}v_n$$
$$\leq (1+u_n)^{j+1}\|a-x\| + v_n^{j+1}$$
$$\leq (1+u_n)\|a-x\| + v_n$$

with $\sum_{n=1}^{\infty}u_n < \infty, \sum_{n=1}^{\infty}v_n < \infty$ and $s_n = 1+u_n$, it follows that $s_n \to 1$ *as* $n \to \infty$.

Setting $W_{n,m} = R_{n+m-1}R_{n+m-1}\dots R_n$, and $b_{n,m} = \left\|W_{n,m}(ta_n + (1-t)a_1^* - (tW_{n,m}a_n(1-t)a_1^*)\right\|$.

Thus,

$$\left\|W_{n,m}a - W_{n,m}x\right\|$$
$$= \|R_{n+m-1}R_{n+m-2}\dots R_n(a) - R_{n+m-1}R_{n+m-2}\dots R_n(x)\|$$
$$\leq s_{n+m-1}\|R_{n+m-2}\dots R_n(a) - R_{n+m-2}\dots R_n(x)\| + v_{n+m-1}$$
$$\leq \prod_{i=n}^{n+m-1}s_i\|a-x\| + \sum_{i=n}^{n+m-1}v_i$$
$$= A_n\|a-x\| + \sum_{i=n}^{n+m-1}v_i$$

for all $a, x \in B$, where $A_n = \prod_{i=n}^{n+m-1}s_i$, $W_{n,m}a_n = a_{n+m}$ and $a^* = a^*$ for all $a^* \in \bigcap_{j=1}^{k} F\left(T_j\right)$.

Hence, $\gamma_{n+m}(t) = \|ta_{n+m} + (1-t)a_1^* - a_2^*\|$

$$= \left\|\begin{matrix} tW_{n,m}a_n + ((1-t)a_1^* - a_2^* + W_{n,m}(ta_n + \\ (1-t)a_1^*) - a_2^* + a^* - a^* \\ -W_{n,m}(ta_n + (1-t)a_1^*) - a_2^* \end{matrix}\right\|$$

$\leq b_{n,m} + \|W_{n,m}(ta_n + (1-t)a_1^*) - a_2^*\|$

$\leq b_{n,m} + A_n\gamma_n(t) + \sum_{i=n}^{n+m-1} v_i$

by using Lemma (8), we have

$b_{n,m} \leq Kf^{-1}(\|a_n - a^*\| - \frac{1}{K}\|W_{n,m}a_n - W_{n,m}a^*\|$

$\leq Kf^{-1}(\|a_n - a^*\| - \frac{1}{K}(\|a_{n+m} - a^*\|$

$- \|W_{n,m}a_n - a^*\|)$

and so $(b_{n,m})$ converges uniformly to zero. Since $\lim_{n\to\infty} A_n = 1$ and $\lim_{n\to\infty} g_n = 0$, we get

$\lim_{n\to\infty} sup\gamma_{n+m}(t) \leq \lim_{n\to\infty} b_{n,m} + \lim_{n\to\infty} inf\gamma_n(t)$

$= \lim_{n\to\infty} inf\gamma_n(t)$

thus $\lim_{n\to\infty}\gamma_n(t)$ exists for all $t \in [0,1]$.

Theorem(12): Let M be a Banach space, $\emptyset \neq B \subseteq M$ and T_j , for each $j = 1,2,\dots,k$ be a family of total asymptotically quasi-nonexpansive selfmap of B. Suppose that $\bigcap_{j=1}^{k} F(T_j) \neq \emptyset$ and (a_n) be as in (1) convergence strongly to a common fixed point of T_j iff $\lim_{n\to\infty} inf\ d(a_n, F) = 0$, where $d(a, F) = inf_{a^*\in F}\|a - a^*\|$.

Proof: To show $\lim_{n\to\infty} inf\ d(a_n, F) = 0$ lead to (a_n) convergence strongly to a common fixed point of T_j , for each $j = 1,2,\dots,k$, by (2)

$$d(a_{n+1}, F) \leq (1 + u_n)^{j+1}d(a_n, F) + v_n^{j+1}$$
$$\leq (1 + u_n)d(a_n, F) + v_n$$

by Lemma (5), we obtain $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} inf a_n = 0$. Hence $\lim_{n\to\infty} a_n = 0$.

Now to prove the sufficiency, first show that (a_n) cauchy sequence. By using Lemma (9.ii), we get

$\|a_{n+m} - a^*\| \leq Q_1\|a_n - a^*\| + Q_2$ ... (3)

$\forall a^* \in \bigcap_{j=1}^{k} F(T_j), n = m = 1,2, \dots$since $\lim_{n\to\infty} a_n = 0, \forall\epsilon > 0, \exists N$ such that

$$d(a_n, F) \leq \frac{\epsilon}{3Q_1} - \frac{Q_2}{Q_1}, \quad \forall n \geq N$$

hence, there is $h \in F(T_j)$ such that

$\|a_N - h\| \leq \frac{\epsilon}{2Q_1} - \frac{Q_2}{Q_1}$ ... (4)

from (3) and (4), $\forall n \geq N$, we get

$\|a_{n+m} - a_n\| \leq \|a_{n+m} - h\| + \|a_n - h\|$

$\leq Q_1\|a_N - h\| + Q_2$

$+Q_1\|a_N - h\| + Q_2$

$\leq Q_1\frac{\epsilon}{2Q_1} - Q_2 + Q_2$

$+Q_1\frac{\epsilon}{2Q_1} - Q_2 + Q_2$

$= \epsilon$

Then (a_n) is a cauchy sequence and converges to $b \in M$. To show $b \in F(T_j)$, for any $\epsilon^* > 0$ such that

$\|a_n - b\|$

$\leq \frac{\epsilon^*}{2(2 + f_jZ)} - \frac{3g_j}{(2 + f_jZ)}, \forall n \geq N_1$ ... (5)

since $\lim_{n\to\infty} a_n = 0$ implies that $N_2 \geq N_1$ such that $d(a_n, F) \leq \frac{\epsilon^*}{3(4+3f_jZ)}, \forall n \geq N_2$

then $\exists h_1 \in F(T_j)$ such that

$\|a_N - h_1\| \leq \frac{\epsilon^*}{2(4 + 3f_jZ)}$ ... (6)

From (5) and (6) for any T_j, we obtain

$\|T_jb - b\| \leq \|T_jb - h_1\| + 2\|T_ja_{N_2} - h_1\|$

$+ \|a_{N_2} - h_1\| + \|a_{N_2} - b\|$

$\leq \|b - h_1\| + f_j\psi\|b - h_1\| + g_j$

$+ 2\|a_{N_2} - h_1\| + 2f_j\psi\|a_{N_2} - h_1\|$

$+ 2g_j + \|a_{N_2} - h_1\| + \|a_{N_2} - b\|$

$\leq (1 + f_jZ)\|b - h_1\|$

$+ 2(1 + f_jZ)\|a_{N_2} - h_1\| + 3g_j$

$+ \|a_{N_2} - h_1\| + \|a_{N_2} - b\|$

$\leq (1 + f_jZ)\|a_{N_2} - b\|$

$+ (1 + f_jZ)\|a_{N_2} - h\|$

$+ 2(1 + f_jZ)\|a_{N_2} - h_1\| + 3g_j$

$+ \|a_{N_2} - h_1\| + \|a_{N_2} - b\|$

$\leq (2 + f_jZ)\|a_{N_2} - b\|$

$+ (4 + 3f_jZ)\|a_{N_2} - h_1\| + 3g_j$

$\leq (2 + f_jZ)\frac{\epsilon^*}{2(2 + f_jZ)} - 3g_j + 3g_j$

$+ (4 + 3f_jZ)\frac{\epsilon^*}{2(4 + 3f_jZ)} = \epsilon^*$

Therefore, $\|T_jb - b\| = 0\ \forall j$. Thus $b \in F(T_j)$.

Theorem(13): Let M be a real Banach space, $\emptyset \neq B \subseteq M$, T_j , for each $j = 1,2,\dots,k$ be a uniformly continuous total asymptotically quasi-nonexpansive selfmaps of B and the sequence (a_n) be as in (1). Suppose that there is $Z > 0$ such that $\psi((\lambda_j) \leq Z\lambda_j, j = 1,2,\dots,k$. Then $\lim_{n\to\infty}\|a_n - T_j^n a_n\| = 0$.

Proof: By Lemma (10) $\lim_{n\to\infty}\|a_n - a^*\|$ exists . Assume that

$$\lim_{n\to\infty}\|a_n - a^*\| = c, \quad \forall c \geq 0.$$

If $c = 0$, so no prove is needed.

Now assume $c > 0$.

$$a_{n+1} = (1 - \alpha_{jn})a_n + \alpha_{jn}T_j^n b_{jn}$$

by Lemma (6), we have $K_1 > 0$

$\|a_{n+1} - a^*\|^2 \leq (1 - \alpha_{jn})\|a_n - a^*\|^2$

$+ \alpha_{jn}\{\|b_{jn} - a^*\|$

$+ f_{jn}\psi\|b_{jn} - a^*\| + g_{jn}\}^2$

$- x_2(\alpha_{jn})\zeta(\|T_j^n j - a_n\|)$

$\leq \|a_n - a^*\|^2 + (f_{jn}$

$+ g_{jn})K_1 - \epsilon^2\zeta(\|T_j^n b_{jn} - a_n\|)$

Then,

$\epsilon^2\zeta(\|T_j^n b_{jn}-a_n\|)$
$\le \|a_n-a^*\|^2-\|a_{n+1}-a^*\|^2+(f_{jn}+g_{jn})K_1$

that implies

$$\epsilon^2\sum_{n=1}^{\infty}\zeta(\|T_j^n b_{jn}-a_n\|) \le \|a_n-a^*\|^2+K_1\sum_{n=1}^{\infty}(f_{jn}+g_{jn}) < \infty$$

Hence, $\lim_{n\to\infty}\zeta(\|T_j^n b_{jn}-a_n\|)=0$, then

$$\lim_{n\to\infty}\|T_j^n b_{jn}-a_n\|=0.$$

Next,

$$\begin{aligned}\|b_{jn}-a_n\| &\le (1-\alpha_{jn})\|a_n-a_n\|+\alpha_{jn}\|T_j^n b_{(j-1)n}-a_n\|\\ &\le \alpha_{jn}(1+f_{jn}Z)\|b_{(j-1)n}-a_n\|+\alpha_{jn}g_{jn}\\ &\le \alpha_{jn}(1-\alpha_{(j-1)n})(1+f_{jn}Z)\|a_n-a_n\| +\alpha_{jn}\alpha_{(j-1)n}(1+f_{jn}Z)\|T_{j-1}^n b_{(j-2)n}-a_n\| + \alpha_{jn}g_{jn}\\ &\le \alpha_{jn}\alpha_{(j-1)n}(1+f_{jn}Z)(1+f_{(j-1)n}Z)\|b_{(j-2)n}-a_n\|+\alpha_{jn}\alpha_{(j-1)n}(1+f_{jn}Z) g_{(j-1)n}+\alpha_{jn}g_{jn}\\ &\quad .\\ &\le \alpha_{jn}\alpha_{j-1}\dots\alpha_1(1+f_{jn}Z)(1+f_{(j-1)n}Z)\dots(1+f_{1n}Z)\|a_n-a_n\|+\alpha_{j-1}(1+f_{jn}Z) g_{(j-1)n}+\cdots \alpha_{jn}\alpha_{j-1}\dots\alpha_1(1+f_{jn}Z)(1+f_{(j-1)n}Z)\dots(1+f_{1n}Z) g_{(j-1)n}\cdots g_{1n}+\alpha_{jn}g_{jn}\\ &< \infty\end{aligned}$$

since $\sum_{n=1}^{\infty}f_{jn}<\infty$ and $\sum_{n=1}^{\infty}g_{jn}<\infty$, hence $\lim_{n\to\infty}\|b_{jn}-a_n\|=0$.

Then,

$$\|T_j^n a_n-a_n\|\le\|T_j^n a_n-T_j^n b_{jn}\|+\|T_j^n b_{jn}-a_n\| \to 0 \; as\; n\to\infty.$$

Theorem(14): Let M be a real unformily convex Banach space, $\emptyset\ne B\subseteq M$ and T_j, for each $j=1,2,\dots,k$ be a family of lipschitzain and total asymptotically quasi-nonexpansive selfmaps of B. If the dual space $M^*\, of\, M$ has the Kadec-klee property and maps $I-T_j$, $\forall j$ is demiclosed to zero, then the sequence (a_n) be as in (1) converges weakly to a common fixed point of T_j.

Proof: As proved by Lemma (10) that $\lim_{n\to\infty}\|a_n-a^*\|$ exists. Since (a_n) is bounded in B and M is reflexive .

Then, there is a subsequence (a_{ni}) of (a_n) that converges weakly to a point $a^*\in B$. By Theorem (13)

$\lim_{n\to\infty}\|a_n-T_j^n a_n\|=0$, for all j=1, 2,…, k .Since by the hypothesis the maps $I-T_j$, $j=1,2,\dots,k$ is demiclosed to zero.

Therefore, $a^*\in F(T_j)$. Now, to prove (a_n) converge weakly to a point a^*.Assume that (a_{nk}) is another subsequence of (a_n) that converge weakly to a point $b^*\in F(T_j)$. By same argument as above, we obtain $b^*\in F(T_j)$.Then by Lemma (11) $\lim_{n\to\infty}\|ta_n+(1-t)a^*-b^*\|$ exists for all $t\in[0,1]$. By Lemma (7) $a^*=b^*$. Hence, (a_n) converges weakly to the point $a^*\in F(T_j)$.

Theorem(15): Let M be a uniformly convex Banach space, $\emptyset\ne B\subseteq M$ and T_j, for each $j=1,2,\dots,k$ be a family of total asymptotically quasi-nonexpansive selfmaps of B. If M accomplishes Opial's condition and the maps $I-T_j$, $\forall j=1,2,\dots,k$ is demiclosed to zero, therefore (a_n) be as in (1) converges weakly to a common fixed point of T_j, for each $j=1,2,..,k$.

Proof: Let $a^*\in F(T_j)$. As proved in Lemma (10) $\lim_{n\to\infty}\|a_n-a^*\|$ exists.

Now, we must prove that (a_n) converges weakly to a unique weak subsequential limit in $F(T_i)$.Since (a_n) is bounded sequence in M, there exists convergent subsequence (a_{nj}) of (a_n) converges weakly to $a^*\in B$. From Theorem (13), we have $\lim_{n\to\infty}\|a_n-T_j^n a_n\|=0$, for each j=1, 2,…,k. Since by the hypothesis the maps $I-T_j$, for each $j=1,2,\dots,k$ is demiclosed to zero, hence $T_j a^*=a^*$, that means $a^*\in F(T_i)$. Now to prove (a_n) converge weakly to a^*. Assume there is a subsequence (a_{ni}) of (a_n) converges weakly to $b^*\in F(T_j)$ and $a^*\ne b^*$. By the same argument as above we can show $b^*\in F(T_j)$.

To belay the uniqueness, assume $a^*\ne b^*$. Therefore by Opial's condition:

$$\begin{aligned}\lim_{n\to\infty}\|a_n-a^*\| &= \lim_{nj\to\infty}\|a_{nj}-a^*\|\\ &< \lim_{nj\to\infty}\|a_{nj}-b^*\|\\ &= \lim_{ni\to\infty}\|a_{ni}-b^*\|\\ &< \lim_{ni\to\infty}\|a_{ni}-a^*\|\\ &= \lim_{n\to\infty}\|a_n-a^*\|\end{aligned}$$

This is contradiction, so $a^*\ne b^*$. Hence (a_n) converges weakly to a^*.

The following corollaries as particular cases of total asymptotically quasi-nonexpansive maps are now obvious.

Corollary(16): Let M be a Banach space, $\emptyset\ne B\subseteq M$ and T_j, $j=1,2,\dots,k$ be a family of total asymptotically nonexpansive selfmaps of B. Spposse $F(T_j)\ne\emptyset$ and $\sum_{n=1}^{\infty}f_n<\infty$, $\sum_{n=1}^{\infty}g_n<\infty$. Suppose that (a_n) be as in (1) converges strongly to a common fixed point of T_j iff $\lim_{n\to\infty} inf\, d(a_n,F)=0$, where $d(a,F)=inf_{a^*\in F}\|a-a^*\|$.

Corollary(17): Let T_j, B, f_n and $g_n \forall j = 1,2,\dots,k$ be as in corollary (16). Therefore, (a_n) be as in (1) converges strongly to $a^* \in F(T_j)$, $j = 1,2,..,k$ iff (a_{nj}) of (a_n) that converges to a^*.

Corollary(18): Let $T_j, j = 1,2,\dots,k$ be a family of lipschitzain and total asymptotically nonexpansive selfmaps of B. If the dual space M^* of M possess the Kadec-klee property, maps $I - T_j, j = 1,2,\dots,k$ is demiclosed to zero, therefore the sequence (a_n) be as in (1) converges weakly to a common fixed point of T_j.

Corollary(19): Let $T_j, j = 1,2,\dots,k$ be a family of total asymptotically nonexpansive selfmaps of B. If M accomplishes Opial's condition and the maps $I - T_j$, $j = 1,2,\dots,k$ is demiclosed to zero, therefore (a_n) be as in (1) converges weakly to a common fixed point of T_j.

Numerical Example

Example: Let $T_j: R \to R, \forall j = 1,2,\dots,k$ be maps defined by $T_j a = \frac{2a}{3j}$ $\forall a \in R$. Choose $\alpha_{jn} = \frac{n}{3(n+1)}$ $\forall n$ with initial value $a_1 = 10$. Let (a_n) be as in (1) that converge to the fixed point $a^* = 0$. So it's clear from Table 1 and Figure 1.

Table 1. Numerical results corresponding to $a_1 = 10$ for 36 steps

n	Iteration(1)	n	Iteration(1)
1	10.0000	19	0.0229
2	8.3371	20	0.0157
3	6.4884	21	0.0107
4	4.8697	22	0.0073
5	3.5737	23	0.0050
6	2.5830	24	0.0034
7	1.8464	25	0.0023
8	1.3089	26	0.0016
9	0.9219	27	0.0011
10	0.6458	28	0.0007
11	0.4505	29	0.0005
12	0.3131	30	0.0003
13	0.2170	31	0.0002
14	0.1499	32	0.0002
15	0.1034	33	0.0001
16	0.0711	34	0.0001
17	0.0489	35	0.0000
18	0.0335	36	0.0000

We can see from above mentioned Table 1 and the following Figure 1 that the iteration algorithm convergence to the fixed point $a^* = 0$, when n=35.

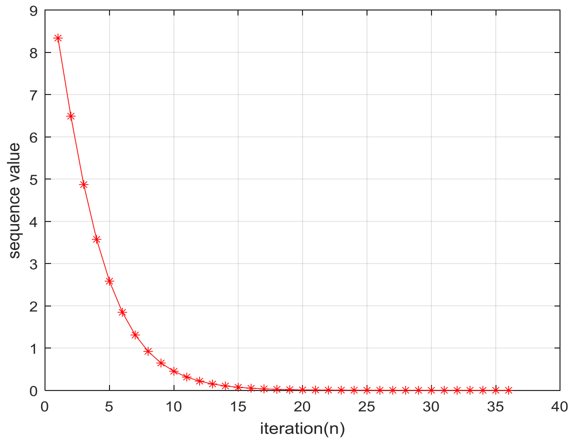

Figure 1. Convergence behavior corresponding to $a_1 = 10$ for 36 steps.

Conflicts of Interest: None.

References

1. Goebel K, Kirk WA. Fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 1972; 35:171-174.
2. Saluja GS, kim JK. Convergence of common fixed point of finite step iteration process for generalized asymptotically quasi-nonexpansive mappings, Nonlinear Funct. Anal. Appl. 2014; 19(1): 79-98, https://www.researchgate.net/publication/260968807
3. Saluja GS. Convergence to common fixed points for generalized asymptotically quasi-nonexpansive mappings, Bulletin of the International Mathematical Virtual Institute, 2014; 4: 69-79, doi: 1.7251/BIMVI140169S.
4. Alber YAI, Chidume CE, Zegeye H. Approximating fixed points of total asymptotically nonexpansive mappings, Fixed point theory Appl. 2006; article ID 10673.
5. Abkar A, Shekarbaigi M. A novel iteration algorithm applied to totally asymptotically nonexpansive mappings in CAT(0) spaces, Mathematices, 2017; 5(14), doi:10.3390/math5010014.
6. Mohammed LB, Kiliçman A. Strong Convergence for the Split Common Fixed-Point Problem for Total Quasi-Asymptotically Nonexpansive Mappings in Hilbert Space, Hindawi Publishing Corporation Abstract and Appl. Anal.2015; ID 412318, http://dx.doi.org/10.1155/2015/412318.
7. F.E.Browder FE. Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc. 1968; 74: 660-665.
8. Chidume CE, Ofoedu EU. Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings, J. Math. Anal. Appl. 2007; 333: 128-141, doi:10.1016/j.jmaa.2006.09.023.
9. Xu ZB, Roach GF. Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 1991; 157: 189-210.
10. Saluja GS. Weak convergence theorems for two asymptotically quasi-nonexpansive non-self

mappings in uniformly convex Banach spaces, J. Non. Sci. Appl. 2014; 7: 138–149.
11. Saluja GS. Weak convergence theorems for asymptotically nonexpansive mappings and total asymptotically nonexpansive non-self mappings, Sohag J. Math.2017; 4(2): 49-57. http://dx.doi.org/10.18576/sjm/040204
12. Abed SS, Abbas RF. S-iteration for general quasi multi valued contraction mappings, International J. Appl. Math. Statistical Sci. (IJAMSS), 2016; 5, Issue 4: 9-22, http://paper.researchbib.com/view/paper/78804
13. Abed SS, Abbas RF. Solving a System of Set-Valued Operator Equations, Solving a System of Set-Valued Operator Equations, 2016; 8(3), 13: 919-924 http://gpcpublishing.org/index.php/gjm/article/view/568

نقطة صامدة مشتركة لخوارزمية التكرار ذات خطوات منتهية لتطبيقات شبه لا متمددة مقاربة

سلوى سلمان عبد **زهرة محمود محمد حسن**

قسم الرياضيات، كلية التربية للعلوم الصرفة/ابن الهيثم، جامعة بغداد، بغداد، العراق.

الخلاصة:

خلال هذا البحث، تم اقتراح خوارزمية التكرار معممة لعائلة منتهية من التطبيقات شبه اللامتمددة المقاربة كليا في فضاء بناخ المحدب بشكل منتظم. وكذلك تم برهنة نظريات التقارب ضعيف وقوي لهذه الخوارزمية الى نقطة صامدة مشتركة. واخيرآ ، تم عرض مثال توضيحي عددي باستخدام ماتلاب.

الكلمات المفتاحية: فضاء بناخ، نقاط صامدة مشتركة، تقارب قوي، تطبيقات شبه اللامتمددة المقاربة، تقارب ضعيف.