On Blow-up Solutions of A Parabolic System Coupled in Both Equations and Boundary Conditions

Maan A. Rasheed

Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

ORCID ID: https://orcid.org/0000-0002-7955-1424
E-mail: maan.rasheed.edbs@uomustansiriyah.edu.iq

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Abstract:
This paper is concerned with the blow-up solutions of a system of two reaction-diffusion equations coupled in both equations and boundary conditions. In order to understand how the reaction terms and the boundary terms affect the blow-up properties, the lower and upper blow-up rate estimates are derived. Moreover, the blow-up set under some restricted assumptions is studied.

Key words: Blow-up rate estimate; Blow-up set; Comparison principle; Radial function; Reaction-Diffusion equation.

Introduction:
In this paper, the following parabolic problem is studied:

\[ \begin{align*}
& u_t = \Delta u + \lambda_1 e^{pu}, \quad v_t = \Delta v + \lambda_2 e^{qu}, \quad x \in B_R, \\
& \frac{\partial u}{\partial \eta} = e^{pu}, \quad \frac{\partial v}{\partial \eta} = e^{qu}, \quad x \in \partial B_R, \\
& u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in B_R,
\end{align*} \]

for \((x, t) \in \Omega \times (0, T)\), where \(F, G : R^2 \to R\), it can be said that a solution \((u, v)\) blows up in finite time, if there exist \(T < \infty\) such that either \(u\) or \(v\) blows up at \(t = T\), this means

\[ \sup_{x \in \Omega}|u(x, t)| \to \infty, \quad \text{or} \quad \sup_{x \in \Omega}|v(x, t)| \to \infty, \quad \text{as} \quad t \to T^- , \]

while

\[ \sup_{x \in \Omega}|u(x, t)| + |v(x, t)| \leq C < \infty, \quad t < T. \]

Moreover, one can say that \(u, v\) blow up simultaneously, if both \(u, v\) blow up at \(T\).

In fact, many systems of two coupled semilinear parabolic equations have been formulated from physical models arising in various fields of applied sciences, for example, in the chemical reaction process, chemical concentration and temperature. All of these examples can be mathematically represented by a coupled system of reaction diffusion equations in the form of (1), see (1).

Since several years ago, some authors have been interested in studying the blow-up properties for reaction-diffusion equations (systems) with nonlinear boundary conditions, see for instance (8,9,10,11,12,13).

In (8), it was considered a special case, where the reaction terms and boundary terms are of power type functions:
For the parabolic system as in (1), associated with the zero Dirichlet conditions:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + \lambda_1 e^{pu}, & \frac{\partial v}{\partial t} &= \Delta v + \lambda_2 e^{qu}, & x &\in B_R, \\
u(x, t) &= 0, & v(x, t) &= 0, & x &\in \partial B_R, \\
u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in B_R,
\end{aligned}
\]

for \( t \in (0, T) \).

It was proved in (15), that the upper (lower) blow-up rate estimates take the forms:

\[
\begin{aligned}
\log c - \frac{1}{q} \log(T - t) &\leq u(R, t) \\
&\leq \log c - \frac{1}{q} \log(T - t), \\
\log c - \frac{1}{p} \log(T - t) &\leq v(R, t) \\
&\leq \log c - \frac{1}{p} \log(T - t),
\end{aligned}
\]

\( t \in (0, T) \).

The purposes of this paper are: Firstly, deriving the upper and lower blow-up rate estimates for problem (1). Secondly, studying the blow-up set under some restricted assumptions. The rest of this paper is organized as follows: In section two, the local existence and blow-up with stating some properties of classical solutions of problem (1) are discussed. In section three, the upper and lower blow-up rate estimates are derived. In section four, the blow-up set is studied. In section five, some conclusions are stated.

**Preliminaries**

It is clear that the system (1) is uniformly parabolic, in addition, the reaction terms and the boundary terms are smooth functions. Moreover, the initial functions satisfy the compatibility conditions (2). Therefore, the local existence of unique classical solutions of problem (1) is known by standard parabolic theory (16). On the other hand, with any initial functions \((u_0, v_0)\), the solution of this system has to blow up in a finite time and the blow-up set contains the boundary \((\partial B_R)\). This result can be proved easily using the comparison principle, (17), and the known blow-up properties of problem (4), which has been studied in (14).

The next lemma shows some properties of the classical solutions of problem (1).

It will be denoted for simplicity \(u(r, t) = u(x, t)\), where \( r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \).

**Lemma 1** Let \((u, v)\) be a classical solution to problem (1) with (2). Then

1. \((u, v)\) is radial and \(u, v > 0\) in \(\bar{B}_R \times (0, T)\).
2. \(u_t, v_t \geq 0\) in \((0, R) \times (0, T)\).
3. \(u_t, v_t > 0\) in \(\bar{B}_R \times (0, T)\).

For the parabolic system as in (1), associated with the zero Dirichlet conditions:

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + \lambda_1 e^{pu}, & \frac{\partial u}{\partial t} &= \Delta v + \lambda_2 e^{qu}, & x &\in B_R, \\
u(x, t) &= 0, & v(x, t) &= 0, & x &\in \partial B_R, \\
u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x &\in B_R,
\end{aligned}
\]

for \( t \in (0, T) \).

It was proved in (15), that the upper (lower) blow-up rate estimates take the forms:

\[
\begin{aligned}
\log c - \frac{1}{q} \log(T - t) &\leq u(R, t) \\
&\leq \log c - \frac{1}{q} \log(T - t), \\
\log c - \frac{1}{p} \log(T - t) &\leq v(R, t) \\
&\leq \log c - \frac{1}{p} \log(T - t),
\end{aligned}
\]

\( t \in (0, T) \).
Next, the following lemma will be proved, which shows the relation between \( u \) and \( v \).

**Lemma 2** Let \((u, v)\) be a solution to problem (1), there exist \( M > 1 \) such that
\[
e^{pu} \leq Me^{qv}, \quad e^{qu} \leq Me^{pv},
\]
\((x, t) \in \overline{B}_T \times (0, T)\)

**Proof** Let
\[
J(x, t) = Me^{qu(x,t)} - e^{pv(x,t)}
\]
for \((x, t) \in B_R \times (0, T), \ r = |x|\).
A direct calculation shows
\[
J_t = qMe^{qu}u_t - pe^{pv}v_t,
J_{rr} = qMe^{qu}u_{rr} + q^2e^{qu}u_r^2 - pe^{pv}v_{rr} - p^2e^{pv}v_r^2.
\]
Thus
\[
\begin{align*}
J_t - J_{rr} &= \frac{n - 1}{r} \frac{1}{J_r} \\
&= qMe^{qu}u_t - pe^{pv}v_t \\
&\quad - qMe^{qu}u_r \\
&\quad - q^2Me^{qu}u_r^2 + pe^{pv}v_{rr} + p^2e^{pv}v_r^2 \\
&\quad - \frac{n - 1}{r}qMe^{qu}u_r + \frac{n - 1}{r^2}pe^{pv}v_r \\
&= qMe^{qu}(u_t - u_{rr} - \frac{n - 1}{r}u_r) \\
&\quad - pe^v(v_t - v_{rr} - \frac{n - 1}{r}v_r) - q^2Me^{qu}u_r^2 + \frac{r}{2}pe^{pv}v_r^2.
\end{align*}
\]
From (7), it follows that
\[
\begin{align*}
&u_r = \frac{1}{qMe^{qu}(pu_r e^{qu} + J_r)}, \\
u_r^2 = \frac{1}{q^2M^2e^{2qu}(p^2v_r^2 + 2pe^{pv}v_rJ_r + J_r^2)}.
\end{align*}
\]
Therefore
\[
\begin{align*}
J_t - \Delta J &= (q\lambda_1M - p\lambda_2)e^{qu+pv} \\
&+ e^{pu} - \frac{e^{pv}}{M^{eq} u^2} - \left(\frac{2e^{pv}}{M^{eq} u^2} v_r^2 + \frac{1}{M^{eq} J_r}\right)J_r.
\end{align*}
\]
Clearly,
\[
e^{pu} - \frac{e^{pv}}{M^{eq} u^2} = e^{pu} \frac{J_r}{M^{eq} u^2}
\]
Therefore, the last equation can be rewritten as follows:
\[
\begin{align*}
J_t - \Delta J &= (q\lambda_1M - p\lambda_2)e^{qu+pv} \\
&+ \left(\frac{2e^{pv}}{M^{eq} u^2} v_r + \frac{1}{M^{eq} J_r}\right)J_r.
\end{align*}
\]
It clear that, \( b, c \) are continuous functions and \( c \) is bounded in \( B_R \times (0, T^*) \), for \( T^* < T \).

Moreover,
\[
\frac{\partial J}{\partial v} |_{v=0} = (qMe^{qu}u_t - pe^{pv}v_t) \\
= qMe^{qu+pv} - pe^{qu+pv} \\
= (qM - p)e^{qu+pv} > 0,
\]
and
\[
f(x, 0) = Me^{qu_0} - e^{pv_0} \geq 0, \ x \in \overline{B}_R
\]
provided \( M \) is large enough.
From above and with using the Maximum principle (1), it follows that
\[
f(x, 0) \geq 0, \quad in \quad \overline{B}_R \times (0, T).
\]
Similarly, one can show that the function
\[H = Me^{pv} - e^{qu}\]
is nonnegative in \( \overline{B}_R \times (0, T)\).

**Blow-up Rate Estimates**

In this section, the upper and lower blow-up rate estimates for problem (1) with (2) are considered.

**Theorem 1** Let \((u, v)\) be a blow-up solution to problem (1); where \( \lambda_1 = \lambda_2 = \lambda; T \) is the blow-up time. Assume that the initial conditions \((u_0, v_0)\)
satisfy the conditions:
\[
u_0^2(x) - \frac{r}{2}\frac{e^{qu_0}}{u_0} \geq 0, \quad r \in (0, R)
\]
Then there is a positive constant \( c \) such that
\[
\log \left(1 - \frac{1}{2\eta} \right) \frac{\log(T - t)}{\log(R - t)} \leq \frac{1}{2\eta} \frac{\log(T - t)}{\log(R - t)} \leq \frac{1}{2\eta} \frac{\log(T - t)}{\log(R - t)}, \ t \in (0, T).
\]

**Proof** Define the functions \( J_1, J_2 \) as follows:
\[
J_1(x, t) = u_t(r(t), t) - \frac{r}{R}e^{pu(r(t), t)}, \\
J_2(x, t) = v_t(r(t), t) - \frac{r}{R}e^{qu(r(t), t)}.
\]
A direct calculation shows
\[
\begin{align*}
J_{tt} &= u_{tt}(r(t), t) - \frac{r}{R}e^{pu(t, t)} + \frac{n - 1}{r}u_r - \frac{1}{M^{eq} u^2} + \frac{1}{M^{eq} J_r}
J_{rr} &= u_{rr}(r(t), t) - \frac{r}{R}e^{pu(r(t), t)} - \frac{1}{M^{eq} u^2} + \frac{1}{M^{eq} J_r}
J_{t} &= (u_{tt} - \frac{n - 1}{r}u_r + \frac{1}{M^{eq} u^2} - \frac{1}{M^{eq} J_r})J_r.
\end{align*}
\]
From above, it follows that
\[
\begin{align*}
J_{tt} - J_{rr} &= \frac{n - 1}{r}J_r \\
&= \frac{n - 1}{r} \left(\frac{2e^{pv}}{M^{eq} u^2} v_r + \frac{1}{M^{eq} J_r}\right)J_r.
\end{align*}
\]
Thus
\[
\begin{align*}
J_{tt} &= \Delta J + \frac{n - 1}{r}J_r - \frac{r}{R}e^{pu} + \lambda pe^{pv}v_r - \frac{r}{R}e^{qu} \\
&\quad + \frac{r}{R}p^2e^{pv}v_r^2 + \frac{2}{R}pe^{pv}v_r.
\end{align*}
\]
It clear that, \( b, c \) are continuous functions and \( c \) is bounded in \( B_R \times (0, T^*) \), for \( T^* < T \).

Moreover,
\[
\frac{\partial J}{\partial v} |_{v=0} = (qMe^{qu}u_t - pe^{pv}v_t) \\
= qMe^{qu+pv} - pe^{qu+pv} \\
= (qM - p)e^{qu+pv} > 0,
\]
\[ J_1(0, t) = u_r(0, t) \geq 0, \]
\[ J_2(0, t) = v_r(0, t) \geq 0, \]
\[ J_1(R, t) = J_2(R, t) = 0, \quad t \in (0, T). \]

Since the supermoms of the functions \( \lambda e^{qu}, \lambda e^{pv} \) and \( \frac{1-n}{r^2} \) (on \( B_R \times (0, t) \)) for \( t < T \) are finite, therefore, from above and maximum principle, it follows
\[ J_1, J_2 \geq 0, \quad (x, t) \in B_R \times (0, T). \]

Moreover,
\[ \frac{\partial}{\partial \eta} |_{\partial B_R} \leq 0. \]

This means
\[ (u_r - \frac{r}{R} p e^{pv} v_r - \frac{1}{R} e^{pv}) |_{\partial B_R} \leq 0. \]

Thus
\[ u_t \leq \left( \frac{n-1}{R} u_r + \lambda e^{pv} + \frac{r}{R} p e^{pv} v_r + \frac{1}{R} e^{pv} \right) |_{\partial B_R} \leq 0. \]

This implies that
\[ u_t(R, t) \leq \frac{n-1}{R} e^{pv(R, t)} + \lambda e^{v(R, t)} + p e^{pv(R, t) + 4u(R, t), t \in (0, T).} \]

From the last inequality and Lemma 2, it follows
\[ u_t(R, t) \leq \frac{n-1}{R} Me^{qu(R, t)} + \lambda Me^{qu(R, t)} + M M e^{qu(R, t)} + M e^{qu(R, t)}, \quad t \in (0, T). \]

Thus, there exist a constant \( C \) such that
\[ u_t(R, t) \leq Ce^{e^{qu(R, t)}}, \quad t \in (0, T). \]

Integrate this inequality from \( t \) to \( T \) and since \( u \) blows up at \( R \), it follows
\[ \frac{\epsilon}{(T-t)^{\frac{1}{q}}} \leq e^{u(R, t)}, \quad t \in (0, T) \]

or
\[ \log e - \frac{1}{2q} \log(T-t) \leq u(R, t), \quad t \in (0, T). \]

It can be shown in a similar way that
\[ \log e - \frac{1}{2p} \log(T-t) \leq v(R, t), \quad t \in (0, T). \]

Next, the upper bounds are considered

**Theorem 2** Let \( u \) be a blow-up solution to problem (1)-(2), \( T \) is the blow-up time. Then there is a positive constant \( C \) such that
\[ u(R, t) \leq \log C - \frac{1}{q} \log(T-t), \]
\[ v(R, t) \leq \log C - \frac{1}{p} \log(T-t), \quad t \in (0, T). \]

**Proof**

\[ M(t) = \max u(x, t), \quad N(t) = \max v(x, t). \]

\[ M(t), N(t) \] are increasing in \((0, T)\) due to
\[ u_t, v_t > 0, \quad (x, t) \in B_R \times (0, T). \]

For \( 0 < z < t < T, x \in B_R \), as in (18), the integral equation for problem (1) with respect to \( u \) can be written as follows
\[ u(x, t) = \int_{B_R} \Gamma(x - y, t - z) u(y, z) \]
\[ + \lambda \int_{z}^{t} \int_{B_R} \Gamma(x - y, t - \tau) e^{pv(y, \tau)} d\gamma d\tau \]
\[ + \int_{z}^{t} \int_{S_R} \Gamma(x - y, t - \tau) e^{pv(y, \tau)} d\gamma d\tau \]
\[ - \int_{z}^{t} \int_{S_R} u(y, \tau) \frac{\partial}{\partial \eta} (x - y, t - \tau) d\gamma d\tau, \]
where \( \Gamma \) is the fundamental solution of the heat equation, which takes the form
\[ \Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \]

\[ \quad \text{(9)} \]

Letting \( x \to \partial B_R \) and using the jump relation, (19), for the fourth term on the right hand side of the last equation, it follows that
\[ \frac{1}{2} u(x, t) = \int_{B_R} \Gamma(x - y, t - z) u(y, z) d\gamma \]
\[ + \lambda \int_{z}^{t} \int_{B_R} \Gamma(x - y, t - \tau) e^{pv(y, \tau)} d\gamma d\tau \]
\[ + \int_{z}^{t} \int_{S_R} \Gamma(x - y, t - \tau) e^{pv(y, \tau)} d\gamma d\tau \]
\[ - \int_{z}^{t} \int_{S_R} u(y, \tau) \frac{\partial}{\partial \eta} (x - y, t - \tau) d\gamma d\tau, \]
for \( x \in \partial B_R, 0 < z < t < T. \)

Since, \( u, v \) are positive and radial, it follows
\[ \int_{B_R} \Gamma(x - y, t - z) u(y, z) d\gamma > 0, \]
\[ \int_{S_R} e^{pv(y, \tau)} \Gamma(x - y, t - \tau) d\gamma d\tau \]
\[ = \int_{S_R} e^{pv(y, \tau)} \Gamma(x - y, t - \tau) d\gamma d\tau. \]

Thus
\[ \frac{1}{2} M(t) \geq \int_{z}^{t} \int_{S_R} \Gamma(x - y, t - \tau) e^{pv(y, \tau)} d\gamma d\tau \]
\[ - \int_{z}^{t} \int_{S_R} \Gamma(x - y, t - \tau) d\gamma d\tau, \quad x \in S_R, 0 < z < t < T. \]

It is known that (see (19)) for \( 0 < t_2 < t_3, \) there is \( C^* > 0 \) such that
\[ \left| \frac{\partial}{\partial \eta} (x - y, t_2 - t_1) \right| \leq \frac{1}{(t_2 - t_1)^{\mu}} \cdot \frac{1}{|x - y|^{(n+1-2\mu-\sigma)}}, \]
for \( x, y \in S_R, \sigma \in (0, 1). \)

Choose \( 1 - \frac{\sigma}{2} < \mu < 1, \) from (19), there exist \( C_1 > 0 \) such that
\[ \int_{S_R} \frac{d\gamma}{|x - y|^{(n+1-2\mu-\sigma)}} < C_1. \]

Also, if \( t_1 \) close to \( t_2, \) then there exist a constant \( c \) such that
\[ \int_{S_R} \Gamma(x - y, t_2 - t_1) d\gamma \geq \frac{c}{\sqrt{t_2 - t_1}}. \]

Thus
\[ \frac{1}{2} M(t) \geq c \int_{z}^{t} \frac{e^{pv(y, \tau)}}{\sqrt{t - \tau}} d\tau - C \int_{z}^{t} M(t) \frac{d\tau}{|t - \tau|^\mu}. \]

Since for \( 0 < z < \tau < t < T, \)
it is clear that $M(t) \leq M(t)$, thus
\[
\frac{1}{z} M(t) \geq c \int_t^T e^{\frac{pN(t)}{x}} \frac{d\tau}{\sqrt{t-\tau}} - C_1 M(t) |T - z|^{1-\mu}
\]
(10)

Taking $z$ so that $C_1 |T - z|^{1-\mu} = 1/2$, it follows
\[
M(t) \geq c \int_t^T e^{\frac{pN(t)}{x}} \frac{d\tau}{\sqrt{t-\tau}} \equiv A(t)
\]
(11)

Clearly,
\[
A'(t) = c e^{\frac{pN(t)}{x}}
\]

From Lemma 2, there exist a constant $k > 1$ such that the last equation becomes
\[
A'(t) = c e^{\frac{pN(t)}{x}} \geq c e^{\frac{pN(t)}{k}} \frac{k}{\sqrt{T-t}}
\]

which leads to
\[
\int_0^T \frac{dA}{e^{\frac{pN(t)}{x}}} \geq \int_t^T c \frac{d\tau}{\sqrt{T-t}}
\]

Clearly,
\[
A(T) = \lim_{t \to T} \int_t^T e^{\frac{pN(t)}{x}} \frac{d\tau}{\sqrt{T-t}} = \infty.
\]

This leads to
\[
\frac{1}{qe^{\frac{pN(t)}{x}}} \geq \frac{2c}{k} \sqrt{T-t}.
\]

Therefore, there exist a constant $C_0 > 0$ such that
\[
e^{\frac{pN(t)}{x}} \leq C_0, \quad z < t < T.
\]
(12)

On the other hand, for $t_0 = 2t - T$ (Assuming that $t$ is close to $T$),
\[
A(t) \geq c \int_{t_0}^T e^{\frac{pN(t)}{x}} \frac{d\tau}{\sqrt{T-t}} \geq c e^{\frac{pN(t_0)}{x}} \int_{2t-T}^t \frac{1}{\sqrt{T-t}} \frac{d\tau}{\sqrt{T-t}}
\]

Combining the last inequality with (12), yields
\[
\frac{C_0}{\sqrt{T-t}} \geq e^{\frac{pN(t_0)}{x}} 2c(\sqrt{2} - 1) \sqrt{T-t},
\]

which leads to
\[
e^{\frac{pN(t_0)}{x}} \leq \frac{C_0}{c(\sqrt{2} - 1) \sqrt{T-t_0}}.
\]

Thus there exist a constant $C$ such that
\[
e^{\frac{pN(t)}{x}} \leq \frac{C}{(T-t)}, \quad 0 < t < T
\]
or
\[
v(R, t) \leq \log C - \frac{1}{p} \log(T - t), \quad t \in (0, T).
\]

In the same way, it can be shown that
\[
u(R, t) \leq \log C - \frac{1}{q} \log(T - t), \quad t \in (0, T).
\]

**Blow-up Set**

In this section, the blow-up set for problem (1) is studied, under some restricted assumptions on $\lambda_1, \lambda_2$.

**Theorem 3** Let $(u, v)$ be a blow-up solution to problem (1), such that the following conditions are satisfied:

i. $1 \leq p \leq 2, 1 \leq q \leq 2$.

ii. $\lambda \leq \frac{1}{4(4R^2(n + 1))^2 \min(Q, K, P)}$ (13)

where $Q = \frac{1}{C}, K = \frac{4(n+1)}{(R^2+4(n+1))^2} e^{-\|u_0\|_{\infty}}$.

$P = \frac{4(n+1)}{(R^2+4(n+1))^2} e^{-\|v_0\|_{\infty}}$,

$T$ is the blow-up time; and $C$ is given in Theorem 3.2. $\lambda = \max(\lambda_1, \lambda_2)$.

Then for any $0 \leq a < R$, there exists $A > 0$ such that
\[
u(x, t) \leq \log \left(\frac{1}{A(R^2-r^2)^2}\right)
\]
\[
v(x, t) \leq \log \left(\frac{1}{A(R^2-r^2)^2}\right),
\]
for $(x, t) \in B_R \times (0, T), \ r = |x|$

**Proof** Define the two functions $U, V$ as follows
\[
U(x, t) = V(x, t) = \log \left(\frac{1}{A(x) + B(T-t)}\right),
\]
(14)

for $(x, t) \in \overline{B_R} \times (0, T)$,

where $\alpha(x) = (R^2 - r^2)^2, \ r = |x|, \ B > 0, \ A \geq \lambda$.

A direct calculation shows:
\[
U_t = \frac{B}{(x) + B(T-t)}, \quad U_r = \frac{4\alpha(x) R^2 - r^2}{(x) + B(T-t)},
\]
\[
U_{rr} = \frac{B - 4\alpha(x)(R^2 - r^2)}{(x) + B(T-t)^2} + 16\alpha(x) \frac{(x) + B(T-t)}{\lambda}
\]

Thus
\[
U_t - U_{rr} - \frac{n-1}{r} U_r - \lambda e^{pq} V = \frac{(x) + B(T-t)^2}{\lambda}
\]
\[
\geq \frac{[B - 4\alpha(x)(R^2 - r^2) - 4\alpha(x)(R^2 - r^2)] u(x)}{[\alpha(x) + B(T-t)]}
\]
\[
\geq \frac{[B - 4\alpha(x)(R^2 - r^2) - 4\alpha(x)(R^2 - r^2)] u(x)}{[\alpha(x) + B(T-t)]}
\]

Provided $(T - t) \leq 1/2, A(R^2 - r^2) \leq 1/2, \ A \geq \lambda$.

\[
B \geq \frac{\lambda}{\lambda - \lambda e^U}
\]

$4\alpha(x)(R^2(n + 1)) \geq 4\alpha(x)(R^2(n + 1))$.

where $0 < r \leq a < R$.

So,
\[
U_t - U_{rr} - \frac{n-1}{r} U_r - \lambda e^{pq} \geq 0
\]
(16)

in $(0, R) \times (0, T)$

In the same way, it can be shown that
\[
V_t - V_{rr} - \frac{n-1}{r} V_r - \lambda e^{qU} \geq 0
\]
(17)

in $(0, R) \times (0, T)$

It follows that
\[
U_t - A - \lambda \leq \lambda e^{pq} \geq 0, \quad in B_R \times (0, T),
\]
\[
V_t - \Delta - \lambda_2 e^{qU} \geq 0, \quad in B_R \times (0, T)
\]
(18)

provided $B \geq A(4R^2(n + 1))$.

Moreover,
\[ U(x,0) = \log \frac{1}{[\alpha(x)+BT]} \geq \log \frac{1}{[AR^4+BT]} \geq \]

\[ u(x,0), \ x \in B_R \]  \hspace{1cm} (19)

Provided 

\[ \frac{1}{[AR^4+BT]} \geq e^{u(x,0)}, \ x \in B_R, \]

or

\[ \frac{1}{[AR^4+BT]} \geq e^{||u_0||_\infty}, \]

From condition (13), it is obtained that

\[ \frac{1}{AR^4+BT} \geq \frac{BR^2}{4(n+1)BT} + \frac{1}{4(n+1)} = \frac{B(R^2 + 4(n+1)T)}{R^2 + 4(n+1)T} \]

So that, (19) is satisfied if

\[ B \leq \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-||u_0||_\infty} \]

In the same way, it can be shown that:

\[ V(x,0) = \log \frac{1}{[\alpha(x)+BT]} \geq \log \frac{1}{[AR^4+BT]} \geq \]

\[ v(x,0), \ x \in B_R \]  \hspace{1cm} (20)

Provided that

\[ B \leq \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-||v_0||_\infty} \]

Moreover,

\[ U(R,t) = \log \frac{1}{B(T-t)} \geq \log \frac{c}{(T-t)^q}, \]

\[ V(R,t) = \log \frac{1}{B(T-t)} \geq \log \frac{c}{(T-t)^q}, \]  \hspace{1cm} (21)

t \in (0,T).

provided \( B \leq 1/c \)

From (21) and Theorem 2, it follows that

\[ U(R,t) \geq u(R,t), \quad V(R,t) \geq v(R,t), \]  \hspace{1cm} (22)

t \in (0,T).

From (18, 19, 20,22), and by the comparison principle (20), it follows that

\[ U(x,t) \geq u(x,t), \quad V(x,t) \geq v(x,t), \]

for \( (x,t) \in B_R \times (0,T). \)

Moreover, from (14), it is obtained that

\[ u(x,t) \leq \log \left( \frac{1}{A(R^2-v^2)} \right), \]

\[ v(x,t) \leq \log \left( \frac{1}{A(R^2-v^2)} \right), \]  \hspace{1cm} (23)

for \( (x,t) \in B_R \times (0,T). \)

Conclusion:

This paper is devoted to deriving the upper and lower blow-up rate estimates, and blow-up set for a system of two reaction-diffusion equations coupled in both equations and boundary conditions. The results show that; under the assumptions of theorems 1 and 2, the upper blow-up rate estimates of problem (1) are coincident with the upper blow-up rate estimates of problem (5), while the lower blow-up rate estimates of problems (1) are coincident with the lower blow-up rate estimates of problem (4), see (15). Moreover, from (23), it can be concluded that any point \( x \in B_R \) cannot be a blow-up point for problem (1) with (13). Therefore, the blow-up can only occur on the boundary. This means, if \( \lambda_1 \) and \( \lambda_2 \) are small enough, then the blow-up set is the same as that of problem (4), see (14).

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Author’s declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

References:


حول الحلول المنفجرة لنظام من نوع القطع المكافئ مقترن في كل من المعادلات والشروط الحدودية

معن عبد الكاظم رشيد
قسم الرياضيات، كلية التربية الأساسية، الجامعة المستنصرية، بغداد، العراق.

الخلاصة:
يتم هذا البحث بالحلول المنفجرة لنظام يتكون من معادلتين انتشار ورد الفعل مقترنين في كلا من المعادلات والشروط الحدودية. لغرض فهم كيفية تأثير مقاطع ورد الفعل والشروط الحدودية على خواص الالتفاف، تم القيام بتشميش الفيدي السعودي والاعلى للانفجار. علاوة على ذلك، تم دراسة مجموعة النقاط المنفجرة تحت شروط محددة.

الكلمات المفتاحية: تقييم نسبة الانفجار، مجموعة الانفجار، مبدأ المقارنة، الدالة نصف القطري، معادلة الانفجار ورد الفعل.