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On Blow-up Solutions of A Parabolic System Coupled in Both Equations and Boundary Conditions

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Abstract:

This paper is concerned with the blow-up solutions of a system of two reaction-diffusion equations coupled in both equations and boundary conditions. In order to understand how the reaction terms and the boundary terms affect the blow-up properties, the lower and upper blow-up rate estimates are derived. Moreover, the blow-up set under some restricted assumptions is studied.

Key words: Blow-up rate estimate; Blow-up set; Comparison principle; Radial function; Reaction-Diffusion equation.

Introduction:

In this paper, the following parabolic problem is studied:

$$\left.\begin{array}{ll} u_t = \Delta u + \lambda_1 e^{pv}, & v_t = \Delta v + \lambda_2 e^{qu}, x \in B_R, \\ \frac{\partial u}{\partial \eta} = e^{pv}, & \frac{\partial v}{\partial \eta} = e^{qu}, \ x \in \partial B_R, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), x \in B_R, \end{array}\right\} (1)$$

for $t \in (0,T)$

where $\lambda_1, \lambda_2 > 0; p, q > 0; B_R$ is a ball in R^n ; u_0, v_0 are smooth, nonnegative, radial non-decreasing, functions, satisfying the conditions:

$$\left.\begin{array}{lll} \frac{\partial u_0}{\partial \eta} = e^{pv_0}, & \frac{\partial u_0}{\partial \eta} = e^{qu_0}, & x \in \partial B_R, \\ \Delta u_0 + e^{pv_0} \geq 0, & \Delta v_0 + e^{qu_0} \geq 0, & x \in \overline{B}_R, \\ u_{0r}(|x|) \geq 0, & v_{0r}(|x|) \geq 0, & x \in \overline{B}_R. \end{array}\right\} (2)$$

The blow-up phenomenon in time-dependent problems has been studied over the past years by many authors, see for instance (1, 2 ,3, 4). For numerical studies of blow-up solutions, see (5,6,7).

In general, for a time- dependent equation (1), one can say that the classical solution u blows up in $L^\infty -$ norm or blows up (for short), if there exist $T < \infty$, called the blow-up time, such that u is well defined for all $0 < t < T$, while it becomes unbounded in $L^\infty -$ norm, when t approach to T, that is:

$$\sup_{x\in\Omega}|u(x,t)| \to \infty \quad as \quad t \to T^-.$$

For a system of two coupled semilinear heat equations, namely

$$u_t = \Delta u + F(u,v), \quad v_t = \Delta v + G(u,v),$$

for $(x,t) \in \Omega \times (0,T)$,

where $F, G: R^2 \to R$, it can be said that a solution (u,v) blows up in finite time, if there exist $T < \infty$ such that either u or v blows up at $t = T$, this means

$$\sup_{x\in\Omega}|u(x,t)| \to \infty,$$
$$or \quad \sup_{x\in\Omega}|v(x,t)| \to \infty, \quad as \quad t \to T^-,$$

while

$$\sup_{x\in\Omega}\{|u(x,t)| + |v(x,t)|\} \leq C < \infty, \quad t < T.$$

Moreover, one can say that u, v blow up simultaneously, if both u, v blow up at T.

In fact, many systems of two coupled semilinear parabolic equations have been formulated from physical models arising in various fields of applied sciences, for example, in the chemical reaction process, chemical concentration and temperature. All of these examples can be mathematically represented by a coupled system of reaction diffusion equations in the form of (1), see (1).

Since several years ago, some authors have been interested in studying the blow-up properties for reaction-diffusion equations (systems) with nonlinear boundary conditions, see for instance (8,9,10,11,12,13).

In (8), it was considered a special case, where the reaction terms and boundary terms are of power type functions:

$$\left.\begin{array}{ll} u_t = u_{xx} + v^{p_1}, & v_t = v_{xx} + u^{p_2}, \\ u_x(1,t) = v^{q_1}, & v_x(1,t) = u^{q_2}, \\ u_x(0,t) = 0, & v_x(0,t) = 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), \end{array}\right\} \quad (3)$$

$for\ \ x \in (0,1),\ \ t \in (0,T)$

where $p_1, p_2, q_1, q_2 > 0$, and u_0, v_0 are radial nondecreasing, positive smooth functions satisfying the conditions:

$$u_{0x}(0) = v_{0x}(0) = 0,$$
$$u_{0x}(1) = v_0^{q_1}(1),\quad v_{0x}(1) = u_0^{q_2}(1).$$

It was shown that if $\max\{p_1p_2, p_1q_2, p_2q_1, q_1q_2\} \leq 1$, then every solution of problem (3) exists globally; otherwise every solution blows up in finite time. Moreover, the blow-up occurs only at $x = 1$ and the blow-up rate estimates take the following forms

$$C_1(T-t)^{-\alpha} \leq u(1,t) \leq C_2(T-t)^{-\alpha},\quad t \in (0,T),$$
$$C_3(T-t)^{-\beta} \leq v(1,t) \leq C_4(T-t)^{-\beta},\quad t \in (0,T),$$

where

$$\alpha = \alpha(p_1,p_2,q_1,q_2), \beta = \beta(p_1,p_2,q_1,q_2).$$

In (9), the critical exponents and the conditions, under which blow-up can occur in a finite time, were studied for a system of two heat equations with inner absorption reaction terms and coupled boundary conditions of exponential type:

$$\left.\begin{array}{lll} u_t = \Delta u - \lambda_1 e^{pu}, & v_t = \Delta v - \lambda_2 e^{qv}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \eta} = e^{p_1 v + q_1 u}, & \dfrac{\partial v}{\partial \eta} = e^{p_2 v + q_2 u}, & x \in \partial\Omega \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \Omega, \\ for\ t \in (0,T) & & \end{array}\right\}$$

where Ω is a bounded domain with smooth boundary; $\lambda_1, \lambda_2 > 0$; $p_1, q_2 > 0$; $p_2, q_1 \geq 0$.

The problem (1) has been studied in (14,15), where $\lambda_1 = \lambda_2 = 0$, namely:

$$\left.\begin{array}{ll} u_t = \Delta u, & v_t = \Delta v,\ \ x \in B_R, \\ \dfrac{\partial u}{\partial \eta} = e^{pv}, & \dfrac{\partial v}{\partial \eta} = e^{qu},\ \ x \in \partial B_R, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), x \in B_R, \end{array}\right\} (4)$$

$for\ t \in (0,T)$

It has been proved that the blow-up in this problem can only occur on the boundary. Moreover, the upper (lower) blow-up rate estimates take the forms:

$$logc - \frac{1}{2q}log(T-t) \leq u(R,t) \leq logc - \frac{1}{2q}log(T-t),$$
$$logc - \frac{1}{2p}log(T-t) \leq v(R,t) \leq logc - \frac{1}{2p}log(T-t),$$

$t \in (0,T)$.

For the parabolic system as in (1), associated with the zero Dirichlet conditions:

$$\left.\begin{array}{lll} u_t = \Delta u + \lambda_1 e^{pv}, & v_t = \Delta v + \lambda_2 e^{qu}, & x \in B_R, \\ u(x,t) = 0, & v(x,t) = 0, & x \in \partial B_R, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in B_R, \\ for\ t \in (0,T) & & \end{array}\right\} (5)$$

It was proved in (15), that the upper (lower) blow-up rate estimates take the forms:

$$logc - \frac{1}{q}log(T-t) \leq u(R,t) \leq logc - \frac{1}{q}log(T-t),$$
$$logc - \frac{1}{p}log(T-t) \leq v(R,t) \leq logc - \frac{1}{p}log(T-t),$$

$t \in (0,T)$.

The purposes of this paper are: Firstly, deriving the upper and lower blow-up rate estimates for problem (1). Secondly, studying the blow-up set under some restricted assumptions. The rest of this paper is organized as follows: In section two, the local existence and blow-up with stating some properties of classical solutions of problem (1) are discussed. In section three, the upper and lower blow-up rate estimates are derived. In section four, the blow-up set is studied. In section five, some conclusions are stated.

Preliminaries

It is clear that the system (1) is uniformly parabolic, in addition, the reaction terms and the boundary terms are smooth functions. Moreover, the initial functions satisfy the compatibility conditions (2). Therefore, the local existence of unique classical solutions of problem (1) is known by standard parabolic theory (16). On the other hand, with any initial functions (u_0, v_0), the solution of this system has to blow up in a finite time and the blow-up set contains the boundary (∂B_R). This result can be proved easily using the comparison principle, (17), and the known blow-up properties of problem (4), which has been studied in (14).

The next lemma shows some properties of the classical solutions of problem (1).

It will be denoted for simplicity $u(r,t) = u(x,t)$, where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$

Lemma 1 Let (u, v) be a classical solution to problem (1) with (2). Then

1. (u,v) is radial and $u, v > 0$ in $\overline{B}_R \times (0,T)$.
2. $u_r, v_r \geq 0$ in $(0,R) \times (0,T)$.
3. $u_t, v_t > 0$, in $\overline{B}_R \times (0,T)$.

Next, the following lemma will be proved, which shows the relation between u and v.

Lemma 2 Let (u,v) be a solution to problem (1), there exist $M>1$ such that

$$e^{pv} \le Me^{qu}, \quad e^{qu} \le Me^{pv}, \quad (6)$$

$(x,t)\in \overline{B}_R\times(0,T)$

Proof Let

$$J(x,t)=Me^{qu(r,t)}-e^{pv(r,t)},$$

for $(x,t)\in B_R\times(0,T), \quad r=|x|$.

A direct calculation shows

$$J_t = qMe^{qu}u_t - pe^{pv}v_t,$$
$$J_r = qMe^{qu}u_r - pe^{pv}v_r, \qquad (7)$$
$$J_{rr} = qMe^{qu}u_{rr} + q^2Me^{qu}u_r^2 - pe^{pv}v_{rr} - p^2e^{pv}v_r^2.$$

Thus

$$J_t - J_{rr} - \frac{n-1}{r}J_r = qMe^{qu}u_t - pe^{pv}v_t - qMe^{qu}u_{rr} - q^2Me^{qu}u_r^2 + pe^{pv}v_{rr} + p^2e^{pv}v_r^2 - \frac{n-1}{r}qMe^{qu}u_r + \frac{n-1}{r}pe^{pv}v_r$$
$$= qMe^{qu}\left(u_t - u_{rr} - \frac{n-1}{r}u_r\right) - pe^{v}(v_t - v_{rr} - \frac{n-1}{r}v_r) - q^2Me^{qu}u_r^2 + p^2e^{pv}v_r^2$$
$$= qMe^{qu}(\lambda_1 e^{pv}) - pe^{v}(\lambda_2 e^{qu}) - q^2Me^{qu}u_r^2 + p^2e^{pv}v_r^2.$$

From (7), it follows that

$$u_r = \frac{1}{qMe^{qu}}(pv_r e^{pv} + J_r),$$
$$u_r^2 = \frac{1}{q^2M^2e^{2qu}}(p^2v_r^2e^{2pv} + 2e^{pv}v_rJ_r + J_r^2).$$

Therefore,

$$J_t - \Delta J = (q\lambda_1 M - p\lambda_2)e^{qu+pv} + (e^{pv} - \frac{e^{2pv}}{Me^{qu}})v_r^2 - (\frac{2e^{pv}}{Me^{qu}}v_r + \frac{1}{Me^{qu}}J_r)J_r.$$

Clearly,

$$e^{pv} - \frac{e^{2pv}}{Me^{qu}} = e^{pv}\frac{J}{Me^{qu}}$$

Therefore, the last equation can be rewritten as follows:

$$J_t - \Delta J - bJ_r - cJ = (q\lambda_1 M - p\lambda_2)e^{qu+pv} \ge 0, \quad (x,t)\in B_R\times(0,T)$$

provided $M > p\lambda_2/q\lambda_1$, where

$$b(x,t) = -\left[\frac{2e^{pv}}{Me^{qu}}v_r + \frac{1}{Me^{qu}}J_r\right],$$
$$c(x,t) = \frac{e^{pv}}{Me^{qu}}v_r^2$$

It clear that, b,c are continuous functions and c is bounded in $B_R\times(0,T^*)$, for $T^*<T$.

Moreover,

$$\frac{\partial J}{\partial \eta}|_{x\in\partial B_R} = (qMe^{qu}u_r - pe^{pv}v_r)$$
$$= qMe^{qu+pv} - pe^{qu+pv}$$
$$= (qM-p)e^{qu+pv} > 0,$$

and

$$J(x,0) = Me^{qu_0} - e^{pv_0} \ge 0, \quad x\in\overline{B}_R$$

provided M is large enough.

From above and with using the Maximum principle ,(1), it follows that

$$J\ge 0, \quad in \quad \overline{B}_R\times(0,T).$$

Similarly, one can show that the function $H = Me^{pv} - e^{qu}$ is nonnegative in $\overline{B}_R\times(0,T)$.

Blow-up Rate Estimates

In this section, the upper and lower blow-up rate estimates for problem (1) with (2) are considered.

Theorem 1 Let (u,v) be a blow-up solution to problem (1); where $\lambda_1=\lambda_2=\lambda$; T is the blow-up time. Assume that the initial conditions (u_0,v_0) satisfy the conditions:

$$u_{0r}(r) - \frac{r}{R}e^{pv_0(r)} \ge 0,$$
$$v_{0r}(r) - \frac{r}{R}e^{qu_0(r)} \ge 0, \quad r\in(0,R) \quad (8)$$

Then there is a positive constant c such that

$$\log c - \frac{1}{2q}\log(T-t) \le u(R,t),$$
$$\log c - \frac{1}{2p}\log(T-t) \le v(R,t), \quad t\in(0,T).$$

Proof Define the functions J_1, J_2 as follows:

$$J_1(x,t) = u_r(r,t) - \frac{r}{R}e^{pv(r,t)},$$
$$J_2(x,t) = v_r(r,t) - \frac{r}{R}e^{qu(r,t)}.$$

A direct calculation shows

$$J_{1t} = u_{rt} - \frac{r}{R}pe^{pv}(v_{rr} + \frac{n-1}{r}v_r + \lambda e^{qu}),$$
$$J_{1r} = u_{rr} - \frac{r}{R}pe^{pv}v_r - \frac{1}{R}e^{pv},$$
$$J_{1rr} = (u_{rt} - \frac{n-1}{r}u_{rr} + \frac{n-1}{r^2}u_r - \lambda pe^{pv}v_r) - \frac{r}{R}(pe^{pv}v_{rr} + p^2e^{pv}v_r^2) - \frac{2}{R}pe^{pv}v_r$$

From above, it follows that

$$J_{1t} - J_{1rr} - \frac{n-1}{r}J_{1r} = -\frac{n-1}{r^2}(u_r - \frac{r}{R}e^{pv}) + \lambda pe^{pv}(v_r - \frac{r}{R}e^{qu}) + \frac{r}{R}p^2e^{pv}v_r^2 + \frac{2}{R}pe^{pv}v_r$$

Thus

$$J_{1t} - \Delta J_1 + \frac{n-1}{r^2}J_1 - \lambda pe^{pv}J_2 = \frac{r}{R}p^2e^{pv}v_r^2 + \frac{2}{R}pe^{pv}v_r \ge 0,$$

for $(x,t)\in B_R\times(0,T)\cap\{r>0\}$.

In the same way, it can be shown that

$$J_{2t} - \Delta J_2 + \frac{n-1}{r^2}J_2 - \lambda qe^{qu}J_1 \ge 0,$$

for $(x,t)\in B_R\times(0,T)\cap\{r>0\}$.

Clearly, from (8), it follows that

$$J_1(x,0), J_2(x,0) \ge 0 \quad x\in B_R$$

And

$$J_1(0,t) = u_r(0,t) \geq 0,$$
$$J_2(0,t) = v_r(0,t) \geq 0$$
$$J_1(R,t) = J_2(R,t) = 0, \quad t \in (0,T).$$

Since, the supermoms of the functions $\lambda e^{qu}, \lambda e^{pv}$ and $\frac{1-n}{r^2}$ (on $B_R \times (0,t)$ for $t < T$) are finite, therefore, from above and maximum principle, it follows

$$J_1, J_2 \geq 0, \quad (x,t) \in B_R \times (0,T).$$

Moreover,

$$\frac{\partial J_1}{\partial \eta}|_{\partial B_R} \leq 0.$$

This means

$$(u_{rr} - \frac{r}{R} p e^{pv} v_r - \frac{1}{R} e^{pv})|_{\partial B_R} \leq 0.$$

Thus

$$u_t \leq (\frac{n-1}{r} u_r + \lambda e^{pv} + \frac{r}{R} p e^{pv} v_r + \frac{1}{R} e^{pv})|_{\partial B_R}$$

This implies that

$$u_t(R,t) \leq \frac{n-1}{R} e^{pv(R,t)} + \lambda e^{v(R,t)} + p e^{pv(R,t)+qu(R,t)} + \frac{1}{R} e^{pv(R,t)}, \quad t \in (0,T).$$

From the last inequality and Lemma 2, it follows

$$u_t(R,t) \leq \frac{n-1}{R} M e^{qu(R,t)} + \lambda M e^{qu(R,t)} + M p e^{2qu(R,t)} + \frac{M}{R} e^{qu(R,t)}, \quad t \in (0,T).$$

Thus, there exist a constant C such that

$$u_t(R,t) \leq C e^{2qu(R,t)}, \quad t \in (0,T).$$

Integrate this inequality from t to T and since u blows up at R, it follows

$$\frac{c}{(T-t)^{\frac{1}{2q}}} \leq e^{u(R,t)}, \quad t \in (0,T)$$

or $\quad \log c - \frac{1}{2q}\log(T-t) \leq u(R,t), \quad t \in (0,T).$

It can be shown in a similar way that

$$\log c - \frac{1}{2p}\log(T-t) \leq v(R,t), \quad t \in (0,T).$$

Next, the upper bounds are considered

Theorem 2 Let u be a blow-up solution solution to problem (1)-(2), T is the blow-up time. Then there is a positive constant C such that

$$u(R,t) \leq \log C - \frac{1}{q}\log(T-t),$$
$$v(R,t) \leq \log C - \frac{1}{p}\log(T-t), \quad t \in (0,T).$$

Proof Define

$$M(t) = \max_{\overline{B}_R} u(x,t), \quad N(t) = \max_{\overline{B}_R} v(x,t).$$

$M(t), N(t)$ are increasing in $(0,T)$ due to

$$u_t, v_t > 0, \quad (x,t) \in \overline{B}_R \times (0,T).$$

For $0 < z < t < T, x \in B_R$, as in (18), the integral equation for problem (1) with respect to u can be written as follows

$$u(x,t) = \int_{B_R} \Gamma(x-y,t-z) u(y,z)$$
$$+\lambda_1 \int_z^t \int_{B_R} \Gamma(x-y,t-\tau) e^{pv(y,\tau)} dy d\tau$$
$$+\int_z^t \int_{S_R} \Gamma(x-y,t-\tau) e^{pv(y,\tau)} ds_y d\tau$$
$$-\int_z^t \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau) ds_y d\tau,$$

where Γ is the fundamental solution of the heat equation, which takes the form:

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{(n/2)}} \exp(-\frac{|x|^2}{4t}) \qquad (9)$$

Letting $x \to \partial B_R$ and using the jump relation, (19), for the fourth term on the right hand side of the last equation, it follows that:

$$\frac{1}{2} u(x,t) = \int_{B_R} \Gamma(x-y,t-z) u(y,z) dy$$
$$+\lambda_1 \int_z^t \int_{B_R} \Gamma(x-y,t-\tau) e^{pv(y,\tau)} dy d\tau$$
$$+\int_z^t \int_{S_R} \Gamma(x-y,t-\tau) e^{pv(y,\tau)} ds_y d\tau$$
$$-\int_z^t \int_{S_R} u(y,\tau) \frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau) ds_y d\tau,$$

for $x \in \partial B_R, 0 < z < t < T$.

Since u, v are positive and radial, it follows

$$\int_{B_R} \Gamma(x-y,t-z) u(y,z) dy > 0,$$
$$\int_z^t \int_{S_R} e^{pv(y,\tau)} \Gamma(x-y,t-\tau) ds_y d_\tau$$
$$= \int_z^t e^{pv(R,\tau)} (\int_{S_R} \Gamma(x-y,t-\tau) ds_y) d\tau.$$

Thus

$$\frac{1}{2} M(t) \geq \int_z^t e^{pN(\tau)} (\int_{S_R} \Gamma(x-y,t-\tau) ds_y) d\tau$$
$$-\int_z^t M(\tau) (\int_{S_R} |\frac{\partial \Gamma}{\partial \eta_y}(x-y,t-\tau)| ds_y) d\tau, \quad x \in S_R, 0 < z < t < T.$$

It is known that (see (19)) for $0 < t_2 < t_2$, these is $C^* > 0$ such that

$$|\frac{\partial \Gamma}{\partial \eta_y}(x-y,t_2-t_1)| \leq \frac{C^*}{(t_2-t_1)^\mu} \cdot \frac{1}{|x-y|^{(n+1-2\mu-\sigma)}},$$

for $x, y \in S_R, \sigma \in (0,1)$.

Choose $1 - \frac{\sigma}{2} < \mu < 1$, from (19), there exist $C_1 > 0$ such that

$$\int_{S_R} \frac{ds_y}{|x-y|^{(n+1-2\mu-\sigma)}} < C_1.$$

Also, if t_1 close to t_2, then there exist a constant c such that

$$\int_{S_R} \Gamma(x-y,t_2-t_1) ds_y \geq \frac{c}{\sqrt{t_2-t_1}}.$$

Thus

$$\frac{1}{2} M(t) \geq c \int_z^t \frac{e^{pN(\tau)}}{\sqrt{t-\tau}} d\tau - C \int_z^t \frac{M(\tau)}{|t-\tau|^\mu} d\tau.$$

Since for $0 < z < \tau < t < T$,

it is clear that $M(\tau) \le M(t)$, thus

$$\frac{1}{2}M(t) \ge c\int_z^t \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau - C_1^* M(t)|T-z|^{1-\mu} \quad (10)$$

Taking z so that $C_1^*|T-z|^{1-\mu} = 1/2$, it follows

$$M(t) \ge c\int_z^t \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau \equiv A(t) \quad (11)$$

Clearly,

$$A'(t) = c\frac{e^{pN(t)}}{\sqrt{T-t}}.$$

From Lemma 2, there exist a constant $k > 1$ such that the last equation becomes

$$A'(t) = \frac{c}{k}\frac{e^{qM(t)}}{\sqrt{T-t}} \ge \frac{c}{k}\frac{e^{qA(t)}}{\sqrt{T-t}}$$

which leads to

$$\int_t^T \frac{dA}{e^{qA}} \ge \int_t^T \frac{c}{k}\frac{d\tau}{\sqrt{T-\tau}}.$$

Clearly,

$$A(T) = \lim_{t\to T} c\int_z^t \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau = c\int_z^t \lim_{\tau\to T}\frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau = \infty.$$

This leads to

$$\frac{1}{qe^{qA(t)}} \ge \frac{2c}{k}\sqrt{T-t}.$$

Therefore, there exist a constant $C_0 > 0$ such that

$$e^{qA(t)} \le \frac{C_0}{\sqrt{T-t}}, \quad z < t < T. \quad (12)$$

On the other hand, for $t_0 = 2t - T$ (Assuming that t is close to T),

$$A(t) \ge c\int_{t_0}^t \frac{e^{pN(\tau)}}{\sqrt{T-\tau}}d\tau \ge ce^{pN(t_0)}\int_{2t-T}^t \frac{1}{\sqrt{T-\tau}}d\tau = e^{pN(t_0)}2c(\sqrt{2}-1)\sqrt{T-t}.$$

Combining the last inequality with (12), yields

$$\frac{C_0}{\sqrt{T-t}} \ge e^{pN(t_0)}2c(\sqrt{2}-1)\sqrt{T-t},$$

which leads to

$$e^{pN(t_0)} \le \frac{C_0}{c(\sqrt{2}-1)(T-t_0)}.$$

Thus there exist a constant C such that

$$e^{pN(t)} \le \frac{C}{(T-t)}, \quad 0 < t < T$$

or

$$v(R,t) \le \log C - \frac{1}{p}\log(T-t), \quad t \in (0,T).$$

In the same way, it can be shown that

$$u(R,t) \le \log C - \frac{1}{q}\log(T-t), \quad t \in (0,T).$$

Blow-up Set

In this section, the blow-up set for problem (1) is studied, under some restricted assumptions on λ_1, λ_2.

Theorem 3 Let (u,v) be a blow-up solution to problem (1), such that the following conditions are satisfied:

i. $1 \le p \le 2, 1 \le q \le 2.$
ii. $\lambda \le \frac{1}{(4R^2(n+1))}\min\{Q, K, P\}$ (13)

$where\ Q = \frac{1}{C},\ K = \frac{4(n+1)}{(R^2+4(n+1)T)}e^{-||u_0||_\infty},$

$$P = \frac{4(n+1)}{(R^2+4(n+1)T)}e^{-||v_0||_\infty},$$

T is the blow-up time; and C is given in Theorem 3.2, $\lambda = \max\{\lambda_1, \lambda_2\}$.

Then for any $0 \le a < R$, there exists $A > 0$ such that: $u(x,t) \le \log\left(\frac{1}{A(R^2-r^2)^2}\right)$,

$$v(x,t) \le \log\left(\frac{1}{A(R^2-r^2)^2}\right),$$

for $(x,t) \in B_R \times (0,T),\ r = |x|$

Proof Define the two functions U, V as follows

$$U(x,t) = V(x,t) = log\frac{1}{(A\alpha(x)+B(T-t))}, \quad (14)$$

for $(x,t) \in \overline{B}_R \times (0,T)$,
where $\alpha(x) = (R^2-r^2)^2,\ r = |x|,\ B > 0,$ $A \ge \lambda.$

A direct calculation shows:

$$U_t = \frac{B}{(\alpha(x)+B(T-t))}, \quad U_r = \frac{4rA(R^2-r^2)}{(\alpha(x)+B(T-t))},$$

$$U_{rr} = \frac{[\alpha(x)+B(T-t)][4A(R^2-3r^2)] + 16A^2r^2(R^2-r^2)}{(\alpha(x)+B(T-t))^2}$$

Thus

$$U_t - U_{rr} - \frac{n-1}{r}U_r - \lambda e^{pV} = \frac{[B-4A(n-1)(R^2-r^2)](\alpha(x)+B(T-t))}{(\alpha(x)+B(T-t))^2} - \frac{[4A(R^2-3r^2)][\alpha(x)+B(T-t)] + 16r^2\alpha(x)}{[\alpha(x)+B(T-t)]^2} - \frac{\lambda}{[\alpha(x)+B(T-t)]^p}$$

$$\ge \frac{[B-4A(n-1)(R^2-r^2)-4A(R^2-3r^2)-16r^2]v(x)}{[\alpha(x)+B(T-t)]^2} - \frac{\lambda}{[\alpha(x)+B(T-t)]^2}$$

$$\ge \frac{[B-4AR^2n-4AR^2]\alpha(x)-\lambda}{[\alpha(x)+B(T-t)]^2} \ge 0$$

Provided $(T-t) \le 1/2$, $A(R^2-r^2) \le 1/2$, and

$$B \ge \frac{\lambda}{A(R^2-r^2)} + 4AR^2(n+1) \ge 4AR^2(n+1) \ge 4\lambda R^2(n+1), \quad (15)$$

where $0 < r \le a < R$.

So, $U_t - U_{rr} - \frac{n-1}{r}U_r - \lambda e^{pV} \ge 0,$ (16)

in $(0,\mathrm{R}) \times (0,T)$

In the same way, it can be shown that:

$$V_t - V_{rr} - \frac{n-1}{r}V_r - \lambda e^{qU} \ge 0, \quad (17)$$

in $(0,\mathrm{R}) \times (0,T)$

It follows that

$$\left.\begin{aligned} U_t - \Delta U - \lambda_1 e^{pV} \ge 0, \quad in\ B_R \times (0,T), \\ V_t - \Delta V - \lambda_2 e^{qU} \ge 0, \quad in\ B_R \times (0,T) \end{aligned}\right\} \quad (18)$$

provided $B \ge A(4R^2(n+1))$.

Moreover,

$U(x,0) = log\frac{1}{[\alpha(x)+BT]} \geq log\frac{1}{[AR^4+BT]} \geq u(x,0),\ \ x \in B_R$ (19)

Provided $\frac{1}{[AR^4+BT]} \geq e^{u(x,0)},\ \ x \in B_R,$

or $\frac{1}{[AR^4+BT]} \geq e^{||u_0||_\infty},$

From condition (13), it is obtained that

$$\frac{1}{[AR^4+BT]} \geq \frac{1}{\frac{BR^2}{4(n+1)}+BT} = \frac{4(n+1)}{B(R^2+4(n+1)T)}$$

So that, (19) is satisfied if

$$B \leq \frac{4(n+1)}{R^2+4(n+1)T}e^{-||u_0||_\infty}$$

In the same way, it can be shown that:

$V(x,0) = log\frac{1}{[\alpha(x)+BT]} \geq log\frac{1}{[AR^4+BT]} \geq v(x,0),\ \ x \in B_R$ (20)

Provided that

$$B \leq \frac{4(n+1)}{R^2+4(n+1)T}e^{-||v_0||_\infty}$$

Moreover,

$$\left.\begin{array}{l} U(R,t) = log\frac{1}{B(T-t)} \geq log\frac{c}{(T-t)^{\frac{1}{q}}}\ , \\ V(R,t) = log\frac{1}{B(T-t)} \geq log\frac{c}{(T-t)^{\frac{1}{p}}}\ , \\ t \in (0,T). \end{array}\right\} \quad (21)$$

provided $B \leq 1/c$

From (21) and Theorem 2, it follows that

$U(R,t) \geq u(R,t),\qquad V(R,t) \geq v(R,t),$ (22)

$t \in (0,T)$

From (18, 19, 20,22), and by the comparison principle (20), it follows that

$$U(x,t) \geq u(x,t),\ \ V(x,t) \geq v(x,t),$$
$$for\ (x,t) \in B_R \times (0,T).$$

Moreover, from (14), it is obtained that

$$\left.\begin{array}{l} u(x,t) \leq \log\left(\frac{1}{A(R^2-r^2)^2}\right), \\ v(x,t) \leq \log\left(\frac{1}{A(R^2-r^2)^2}\right), \end{array}\right\} \quad (23)$$

for $(x,t) \in B_R \times (0,T)$.

Conclusion:

This paper is devoted to deriving the upper and lower blow-up rate estimates, and blow-up set for a system of two reaction-diffusion equations coupled in both equations and boundary conditions. The results show that; under the assumptions of theorems 1and 2, the upper blow-up rate estimates of problem (1) are coincident with the upper blow-up rate estimates of problem (5), while the lower blow-up rate estimates of problems (1) are coincident with the lower blow-up rate estimates of problem (4), see (15). Moreover, from (23), it can be concluded that any point $x \in B_R$ cannot be a blow-up point for problem (1) with (13). Therefore, the blow-up can only occur on the boundary. This means, if λ_1 and λ_2 are small enough, then the blow-up set is the same as that of problem (4), see (14).

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- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

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حول الحلول المنفجرة لنظام من نوع القطع المكافئ مقترن في كل من المعادلات والشروط الحدودية

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الخلاصة:

يهتم هذا البحث بالحلول المنفجرة لنظام يتكون من معادلتي انتشارو رد الفعل مقترنتين في كلا من المعادلات والشروط الحدودية. لغرض فهم كيفية تاثير مقاطع رد العفل والشروط الحدودية على خواص الانفجار، تم القيام باشتقاق القيد السفلي والعلوي للانفجار. علاوة على ذلك، تمت دراسة مجموعة النقاط المنفجرة تحت شروط محددة.

الكلمات المفاتحية: تقدير نسبة الانفجار ، مجموعة الانفجار ، مبدأ المقارنة ، الدالة نصف القطرية، معادلة الانتشار ورد الفعل.