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Phase Fitted And Amplification Fitted Of Runge-Kutta-Fehlberg Method Of Order 4(5) For Solving Oscillatory Problems

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Abstract:

In this paper, the proposed phase fitted and amplification fitted of the Runge-Kutta-Fehlberg method were derived on the basis of existing method of 4(5) order to solve ordinary differential equations with oscillatory solutions. The recent method has null phase-lag and zero dissipation properties. The phase-lag or dispersion error is the angle between the real solution and the approximate solution. While the dissipation is the distance of the numerical solution from the basic periodic solution. Many of problems are tested over a long interval, and the numerical results have shown that the present method is more precise than the 4(5) Runge-Kutta-Fehlberg method.

Key words: Amplification fitted, Ordinary differential equations, Oscillatory problems, Phase fitted, Runge-Kutta-Fehlberg method.

Introduction:

Many efforts have been carried out for the resolution of second order ordinary differential equations (ODEs) with oscillatory solutions. Special second order ordinary differential equations can be written as follows:

$$y'' = f(x, y) \qquad (1)$$

This kind of equations appear in numerous scientific fields like physical chemistry, quantitative chemistry, theoretical physics, and molecular physics. Differential equations of oscillatory in nature cannot be solved efficiently using traditional methods. These problems need to be integrated over a period of the oscillation, so oscillatory problem require larger steps and can be extended over a long-time step.

Several researchers such (1), (2) and (3) have proposed an application of phase-lag to solve oscillatory problems. Many of projects are also concentrated with methods containing high dissipative order. In their paper, Nazari and Mohammadian (2) analyzed and introduced high-order low-dissipation low-dispersion diagonally implicit Runge–Kutta schemes for solving oscillatory problems. Van de Vyver (3) also suggested a symplectic RKN method with minimum phase-lag for solving problems with oscillatory solutions. Senu et al. (4) derived diagonally implicit RKN methods for solving oscillatory problems with high order of dispersion and dissipation errors. Recently Ahmad et al. (5) created the semi-implicit hybrid method with minimum phase-lag to solve problems with oscillatory solutions.

Another work related to phase fitting can be seen in Hussain et al. (6), Abdulganiy et al. (7), Fawzi et al. (8), Demba et al. (9) and Adel et al. (10). The objective of the phase-fitted method is to develop a method that has variable coefficients, which rely on the produce of hesitation “v” and step-size “h”. The creative method presides for the original method as “v” approaches to zero.

In this article basing on Runge–Kutta-Fehlberg method of order 4(5); we develop phase-fitted and amplification fitted method for solving problems with oscillatory solutions. The prospective method is utilized to solve the second order ODEs that are oscillatory in nature. To do that the equations of second order are first lowered to first-order ODEs system.

Derivation of the Method

The s-stage Runge-Kutta-Fehlberg method can be defined:

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$$y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$$
$$\bar{y}_{n+1} = \bar{y}_n + h\sum_{i=1}^{s} \bar{b}_i k_i \qquad (2)$$

where

$$k_i = f(x_n + c_i h, y_n + h\sum_{j=1}^{i} a_{ij} k_j) \qquad (3)$$

$i = 1,...,s$

This method is applied for the approximate computation of y_{n+1} when y_n is recognized.

Also, the method above can be shown using Butcher Table 1:

Table 1. Butcher Table

c_2	a_{21}			
c_3	a_{31}	a_{32}		
.	.	.	.	
c_s	a_{s1}	a_{s2}	.	$a_{s,s-1}$
	b_1	b_2	.	b_s
	$\bar{b}_1$	$\bar{b}_2$	.	$\bar{b}_s$

The coefficients $c_2,.......,c_s$ should satisfy the relation:

$$c_i = \sum_{j=1}^{i-1} a_{ij}, i = 2,...,s \qquad (4)$$

An appropriate style to obtain an assured algebraic arrangement is to fulfill equations number obtained from tree theory (11). Those equations can be exhibited during the prospective method structure.

The phase-lag assessment of Runge-Kutta methods is dependent on trial equation:

$$y' = I\omega y \qquad (5)$$

When Runge-Kutta-Fehlberg method defined in equation (3) is applied to equation (5), we get the following:

$$y_{n+1} = a_*^n y_n, \quad a_* = U(v^2) + ivV(v^2) \qquad (6)$$

where the variables $v = \omega h$ and *U*, *V* were polynomials in v^2 fully known from Runge-Kutta-Fehlberg coefficients $a_{ij}, b_i, \bar{b}_i, c_i$ displayed in the table 1.

Definition:

In *s*-stage Runge-Kutta-Fehlberg method, defined in equations (2) and (3), the equations:

$$P(v) = v - \arctan\left(v\frac{V(v^2)}{U(v^2)}\right) \qquad (7)$$
$$D(v) = 1 - \sqrt{U(v^2)^2 + V(v^2)^2}$$

are known as the dispersion error and the dissipative error respectively (5).

If $P(v) = O(v^{q+1})$ and $D(v) = O(v^{r+1})$ then the method becomes the dispersive of order *q* and the dissipative of order *r*.

The Runge-Kutta-Fehlberg method of 4(5) order is considered, given in following Table 2:

Table 2. Rkf45 Order Runge-Kutta-Fehlberg

1/4	1/4					
3/8	3/32	9/32				
12/13	1932/2197	-7200/2197	7296/2197			
1	439/216	-8	3680/513	-845/4104		
½	-8/27	2	-3544/2565	1859/4104	-11/40	
	25/216	0	1408/2565	2197/4104	-1/5	
	16/135	0	6656/12825	28561/56430	-9/50	2/55

To improve the proposed method, we put b_4 and b_5 as free parameters while the other parameters are the same as the current method. We intend to apply the phase-lag (*PL)* and the dissipation (*DS)* of the current method above to be $PL = 0$ and $DS = 0$ to solve for b_4 and b_5, where:

$$PL = \tan(v)\Big(1 - \left(\frac{176}{855} + \frac{12}{13}b_4 + b_5\right)v^2 + \left(\frac{1026}{2197}b_4 - \frac{1}{104} + \frac{85}{78}b_5\right)v^4 - \frac{1}{1040}v^6\Big) - v\Big(\frac{13639}{20520} + b_4 + b_5 - \left(\frac{22}{285} - \frac{864}{2197}b_4 - \frac{3}{2}b_5\right)v^2 + \left(-\frac{5}{52}b_5 + \frac{1}{195}\right)v^4\Big)$$

$$DS = -\frac{16879}{15412800}v^8 - \frac{4988993}{38013300}v^4 - \frac{24}{13}v^2 b_4 + \frac{118}{16055}v^8 b_5$$
$$-2v^2 b_5 + \frac{1315252}{626145}v^4 b_4 + \frac{3}{2704}v^{10} b_5 + \frac{513}{571220}v^{10} b_4$$
$$+\frac{186661}{8447400}v^6 - \frac{113813}{133380}v^6 b_5 + \frac{503}{855}v^2 - \frac{192}{28561}v^8 b_4$$
$$\frac{2070}{2197}v^8 b_4 b_5 + \frac{25}{2704}v^{10} b_5^2 - \frac{3880}{2197}v^6 b_4 b_5 + \frac{24}{13}v^4 b_4 b_5$$
$$+\frac{1}{1081600}v^{12} + \frac{144}{169}v^4 b_4^2 - \frac{9819}{41743}v^6 b_4 + v^4 b_5^2$$
$$-\frac{3414960}{4826809}v^6 b_4^2 + \frac{2735}{3042}v^8 b_5^2 + \frac{11}{156}v^6 b_5^2 + \frac{19}{243360}v^{10}$$
$$+\frac{1052676}{4826809}v^8 b_4^2 + \frac{62146}{11115}v^4 b_5$$

By using Taylor expansion series b_4 becomes:

$$b_4 = \frac{2197}{4104} + \frac{1}{4}v^2 + \frac{133399}{15210}v^4 + \frac{1082730921527}{3454738560}v^6 + \ldots.$$

and b_5 becomes:

$$b_5 = -\frac{1}{5} - \frac{1}{4}v^2 - \frac{1682647}{197730}v^4 - \frac{13646504396939}{44911601280}v^6 - \ldots.$$

where $v = \omega h$, ω denotes the prevailing hesitance and h indicates the step size.

The Test Problems and The Numerical Results

The proposed method will be applied to some problems with oscillatory solutions and the results are compared with the Runge-Kutta-Fehlberg method of order 4(5).

These problems are first decreased to first-order ODEs system. For the following test problems, the interval of integration is [0, 100].

The Tables 3,4 and 5 below show the comparison between the proposed 4(5) order Runge-Kutta-Fehlberg method and the current Runge-Kutta-Fehlberg of order 4(5) for the following problems starting with initial step size h_0.

Problem 1: (Homogeneous problem investigated by Chakravarti and Worland (12)).

$$y'' = -y, y(0) = 1, y'(0) = 2$$
$$\textit{the exact solution is } y = 2\sin(x) + \cos(x)$$

Problem 2: (Problem studied by Papadopoulos et al (13))

$$y'' = -100y + 99\sin(x), y(0) = 1, y'(0) = 11$$
$$\textit{the exact solution is } y = \cos(10x) + \sin(10x) + \sin(x)$$

Problem 3: (Problem studied by Chawla and Rao (14))

$$y'' = -100y, y(0) = 1, y'(0) = 2$$
$$\textit{the exact solution is } y = \frac{1}{5}\sin(10x) + \cos(10x)$$

The following notations are used:

TOL	the tolerance chosen
METHOD	the method used
FCN	function evaluations
STEP No.	successful steps
MAXE	maximum errors $\lvert y(x_i) - y_i \rvert$
NRKF45 PLDS	the proposed method
RKF45	the Runge-Kutta-Fehlberg of order 4(5)

Table 3. Comparison between the Nrkf45 And Rkf45 Methods for Problem 1 Starting with Initial Step Size $h_0 = 0.01$

TOL	METHOD	FCN	STEP No.	MAXE
10^{-6}	NRKF45 PLDS	9816	818	1.3395e-005
	RKF45	10176	848	1.2427e-004
10^{-8}	NRKF45 PLDS	25656	2138	1.3149e-007
	RKF45	48000	4000	5.5974e-006
10^{-9}	NRKF45 PLDS	46656	3888	2.8802e-009
	RKF45	60000	5000	5.2412e-007

Table 4. Comparison between the Nrkf45 and Rkf45 Methods for Problem 2 Starting with Initial Step Size $h_0 = 0.001$

TOL	METHOD	FCN	STEP No.	MAXE
10^{-5}	NRKF45 PLDS	93996	7833	1.6489e-004
	RKF45	96000	8000	2.6977e-003
10^{-6}	NRKF45 PLDS	96000	8000	1.9403e-006
	RKF45	102000	8500	6.4428e-005
10^{-7}	NRKF45 PLDS	108000	9000	1.1983e-008
	RKF45	114000	9500	4.5230e-006
10^{-8}	NRKF45 PLDS	114000	9500	1.0482e-008
	RKF45	120000	10000	1.8697e-007

Table 5. Comparison between the Nrkf45 And Rkf45 Methods for Problem 3 Starting with Initial Step Size $h_0 = 0.001$

TOL	METHOD	FCN	STEP No.	MAXE
10^{-5}	NRKF45 PLDS	93288	7774	1.9829e-004
	RKF45	96000	8000	1.6888e-003
10^{-6}	NRKF45 PLDS	96000	8000	2.3569e-006
	RKF45	102000	8500	6.6328e-005
10^{-7}	NRKF45 PLDS	108000	9000	9.3609e-009
	RKF45	114000	9500	3.1809e-006
10^{-8}	NRKF45 PLDS	114000	9500	6.1155e-009
	RKF45	120000	10000	1.7509e-007

Conclusion:

The number of function evaluations for the prospective method with phase-fitting and amplification-fitting is less than the number for the standard method. Also, the maximum errors of the potential method are lower than the present method depending on the tolerance chosen.

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Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Mosul.

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الطور المناسب والتوسيع المناسب لطريقة رنج-كوتا-فهلبيرج ذو الرتبة (5)4 لحل المسائل ذات الحل المتذبذب

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الخلاصة:

في هذا البحث تم إشتقاق الطور المناسب والتوسيع المناسب لطريقة رنج-كوتا-فهلبيرج القائمة على الطريقة الحالية ذات الرتبة (5)4 لحل المعادلات التفاضلية الإعتيادية ذات الحلول المتذبذبة. الطريقة الحالية لها خاصية تأخير الطور الصفري وخاصية التبدد الصفري. خطأ تأخر الطور أو التشتت هو الزاوية بين الحل الحقيقي والحل التقريبي. بينما التبديد هو الفترة أو المسافة للحل العددي من الحل الدوري الأساسي. يتم إختبار مجموعة من المسائل على مدى فترة زمنية طويلة وقد أظهرت النتائج العددية أن الطريقة الحديثة أكثر دقة من طريقة رنج-كوتا-فهلبيرج الحالية.

الكلمات المفتاحية: التوسيع المناسب، المعادلات التفاضلية الإعتيادية، المسائل ذات الحل المتذبذب، الطور المناسب، طريقة رنج-كوتا-فهلبيرج.