DOI: http://dx.doi.org/10.21123/bsj.2019.16.3(Suppl.).0786

New Approach for Solving Three Dimensional Space Partial Differential Equation

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Received 14/10/2018, Accepted 3/4/2019, Published 23/9/2019

Abstract:

This paper presents a new transform method to solve partial differential equations, for finding suitable accurate solutions in a wider domain. It can be used to solve the problems without resorting to the frequency domain. The new transform is combined with the homotopy perturbation method in order to solve three dimensional second order partial differential equations with initial condition, and the convergence of the solution to the exact form is proved. The implementation of the suggested method demonstrates the usefulness in finding exact solutions. The practical implications show the effectiveness of approach and it is easily implemented in finding exact solutions.

Finally, all algorithms in this paper are implemented in MATLAB version 7.12.

Key words: Convergence, Coupled two methods, Homotopy perturbation method, Partial differential equations, Transformation.

Introduction:

Many phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). In physics for example, the heat flow and the wave propagation phenomena are well described by PDEs (1, 2). So, it is a useful tool for describing natural phenomena of science and engineering models. A PDE is called linear if the power of the dependent variable and each partial derivative contained in the equation is one and the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear. Most of engineering problems are nonlinear, and it is difficult to solve them analytically. The importance of obtaining the exact or approximate solution of nonlinear PDEs in physics and mathematics is still a significant problem that needs new methods to get exact or approximate solutions. Various powerful mathematical methods have been proposed for obtaining exact and approximate analytic solutions. Some of the classic analytic methods are perturbation techniques (3) and Hirota's bilinear method (4).

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In recent years, many research workers have paid attention to study the solutions of non-linear PDEs by using various methods. Among these are the Adomian decomposition method (ADM) (5), the tanh method, the homotopy perturbation method (HPM), the homotopy analysis method (HAM) (6), the differential transform method, Laplace decomposition method (7, 8), and the variational iteration method (VIM) (9,10).

In this research, we will use the new method based on couple new transform with HPM which we will call the new transform homotopy perturbation method (NTHPM) to solve three dimensions second order partial differential equation of the form:

$$u_{xx} + u_{yy} + u_{zz} = \alpha u_t \; ; \; x, y, z \in R \;\&\; t > 0 \qquad (1)$$

with initial condition (IC): $u(x, y, z, 0) = f(x, y, z)$; α is constant.

This method provides an effective and efficient way of solving a wide range of non-linear PDEs. The advantage of this method is its capability of combining two powerful methods for obtaining exact solutions for non-linear partial differential equations. In this research we consider a method in solve non-linear three dimensional space PDEs.

New Transform

New transform was introduced by Luma and Alaa (11) to solve differential equations and engineering problems. part from other advantages of new transform (NT) over other integral transforms such as accuracy and simplicity illustrated in (12), it consists of a very interesting fact about this transform.

The new transform of a function *f*(*t*), defined by

$$\bar{f}(u) = \mathbb{T}\{f(t)\} = \int_0^\infty e^{-t} f\left(\frac{t}{u}\right) dt, \qquad (2)$$

Here some basic properties of the NT are introduced:

If a, b are constants, f(t) and g(t) are functions having NT, then

1. Linearity Property: $\mathbb{T}\{af(t) + bg(t)\} = a\mathbb{T}\{f(t)\} + b\mathbb{T}\{g(t)\}$

2. Convolution Property: $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ (3)
3. $\mathbb{T}\{t^n\} = \frac{n!}{v^n}$, $v \neq 0$, $n = 0,1,2,3, ...$
4. Differentiation Property: $\mathbb{T}\{f'\} = v(\mathbb{T}\{f\} - f(0))$

For more details see (11)

Solving Linear PDE by Suggested Method

To illustrate the ideas of suggested method new transform homotopy perturbation method (NTHPM) firstly rewrite the PDE (1) as following:

$$L[u(x,y,z,t)] + R[u(x,y,z,t)] = g(x,y,z,t) \qquad (4a)$$

with initial condition

$$u(x,y,z,0) = f(x,y,z) \qquad (4b)$$

where $all\ x, y, z\ in\ R$, L: is the linear differential operator ($L = \alpha\frac{\partial}{\partial t}$), R: is the remainder of the linear operator, g(x,y,z,t) is the inhomogeneous part.

We construct a Homotopy as:

$H(u(x,y,z,t),p) = (1-p)[L(u(x,y,z,t)) - L(u(x,y,z,0))] + p\,[A[u(x,y,z,t)] - g(x,y,z,t)] = 0$ (5)

Where p∈[0, 1] is an embedding parameter and A defined as A= L+ R.

It is clear that, if p=1, then the homotopy equation (5) is converted to the differential equation (4a).

Substituting equation (4) into equation (5) and rewrite it as:

$$L(u) - L(f) - pL(u) + pL(f) + pL(u) + pR(u) - p\,g = 0$$

Then

$$L(u) - L(f) + p[L(f) + R(u) - g] = 0 \qquad (6)$$

Since *f*(x, y, z) is independent of the variable *t* and the linear operator *L* dependent on t so, *L*(*f*(x, y, z)) = 0, i.e., equation (6) becomes:

$$L(u) + pR(u) - p\,g = 0 \qquad (7)$$

According to the classical perturbation technique, the solution of the equation (7) can be written as a power series of embedding parameter p, as follow:

$$u(x,y,z,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \qquad (8)$$

The convergence of series (8) at *p* =1 is discussed and proved in (13-15), which satisfies the differential equation (4).

The final step is determining the parts u_n (n= 0,1, 2,…), to get the solution $u(x,y,z,t)$.

Here, we couple the NT with HPM as follow:

Taking the NT (with respect to the variable *t*) for the equation (7) to get:

$$\mathbb{T}\{L(u)\} + p\,\mathbb{T}\{R(u)\} - p\,\mathbb{T}\{g\} = 0 \qquad (9)$$

Now by using the differentiation property of NT (property 4) and equation (4), (9) becomes:

$$v\alpha\mathbb{T}\{u\} - v\alpha f(x) + p\,\mathbb{T}\{R(u)\} - p\,\mathbb{T}\{g\} = 0 \qquad (10)$$

Hence:

$$\mathbb{T}\{u\} = f(x,y,z) - p\frac{\mathbb{T}\{R(u)\}}{v\alpha} + p\frac{\mathbb{T}\{g\}}{v\alpha} \qquad (11)$$

Taking the inverse of the NT on both sides of equation (11), to get:

$$u(x,y,z,t) = f(x,y,z) - p\,\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R(u(x,y,z,t))\}}{v\alpha}\right\} + p\,\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\} \qquad (12)$$

Then substituting equation (8) into equation (12) to obtain:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y,z) - p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\,\{R(\sum_{n=0}^{\infty} p^n u_n)\}}{v\alpha}\right\} + p\,\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\} \qquad (13)$$

By comparing the coefficient of powers of *p* in both sides of the equation (13) we have:

$$\begin{aligned}
u_0 &= f(x,y,z)\\
u_1 &= -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_0]\}}{v\alpha}\right\} + \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t)\}}{v\alpha}\right\}\\
u_2 &= -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_1]\}}{v\alpha}\right\}\\
u_3 &= -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_2]\}}{v\alpha}\right\}\\
&\vdots\\
u_{n+1} &= -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v\alpha}\right\}
\end{aligned} \qquad (14)$$

Illustrative Application

Here, the suggested method NTHPM will be used to solve the 3DS-PDE with initial condition as the following:

Problem 1

Consider the following 3DS - PDE

$u_{xx} + u_{yy} + u_{zz} = \alpha u_t$; $all\ x, y, z\ in\ R\ \&\ t > 0$

subject to IC: $u(x,y,z,0) = f(x, y, z) = 5 \sin(ax)\sin(by)\sin(cz)$,

where a, b and c are constants and α is any coefficient. According to the equation (14) the powers series of p as following:

$P^0 : u_0(x, y, z, t) = 5 \sin(ax)\sin(by)\sin(cz)$

$P^1 : u_1(x, y, z, t) = -(5 \sin(ax) \sin(by) \sin(cz)) (\frac{t}{\alpha})(a^2 + b^2 + c^2)$

$P^2 : u_2(x, y, z, t) = (5 \sin(ax) \sin(by) \sin(cz)) (\frac{t^2}{2\,\alpha^2}) (a^2 + b^2 + c^2)^2$

$P^3 : u_3(x, y, z, t) = -(5 \sin(ax) \sin(by) \sin(cz)) (\frac{t^3}{3!\,\alpha^3}) (a^2 + b^2 + c^2)^3$

$P^4 : u_4(x, y, z, t) = (5 \sin(ax) \sin(by) \sin(cz)) (\frac{t^4}{4!\,\alpha^4}) (a^2 + b^2 + c^2)^4$

$P^5 : u_5(x, y, z, t) = -(5 \sin(ax) \sin(by) \sin(cz)) (\frac{t^5}{5!\,\alpha^5}) (a^2 + b^2 + c^2)^5$

⋮

$P^n : u_n(x, y, z, t) = (-1)^n (5 \sin(ax) \sin(by) \sin(cz)) (\frac{t^n}{n!\,\alpha^n}) (a^2 + b^2 + c^2)^n$

Thus, we get the following series form:

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) = \sum_{n=0}^{\infty} (-1)^n 5 \sin(ax) \sin(by) \sin(cz) \left(\frac{t^n}{n!\ \alpha^n}\right) (a^2 + b^2 + c^2)^n$$

The closed form of the above series is

$$u(x,y,z,t) = 5 \sin(ax) \sin(by) \sin(cz)\ e^{-\frac{t}{\alpha}(a^2+b^2+c^2)}$$

This gives an exact solution of the problem.

Problem 2

Consider the following 3DS-PDE

$u_{xx} + u_{yy} + u_{zz} + e^{x+y} = \alpha u_t$; $all\ x, y, z\ in\ R\ \&\ t > 0$

with IC: $u(x,y,z,0) = f(x, y, z) = d$, where d is constants.

From equation (14) we get the powers series of p as follow:

$P^0 : u_0(x, y, z, t) = d$

$P^1 : u_1(x, y, z, t) = \frac{t}{\alpha} (e^{x+y})$

$P^2 : u_2(x, y, z, t) = \frac{t^2}{2\alpha^2} (e^{x+y})$

$P^3 : u_3(x, y, z, t) = \frac{t^3}{6\alpha^3} (e^{x+y})$

$P^4 : u_4(x, y, z ,t) = \frac{t^4}{24\alpha^4} (e^{x+y})$

⋮

$P^n : u_n(x, y, z ,t) = \frac{t^n}{n!\,\alpha^n} (e^{x+y})$

Thus, the follows series form is obtained

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) = d + \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!\ \alpha^{n+1}} (e^{x+y})$$

Therefore, the closed form of the above series is

$$u(x,y,z,t) = e^{x+y}\left(e^{\frac{t}{\alpha}} - 1\right) + d$$

The Convergence of the Solution

Now, we need to show the convergence of series form to the exact form as the following.

Lemma 1 If f be continues function then

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Proof

Suppose that

$$\int f(x)dx = F(x) + c$$

Assume that $x = t - \tau$ then $dx = -d\tau$ then

$$\frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = -\frac{\partial}{\partial t}\int_t^0 f(x)dx = \frac{\partial}{\partial t}\int_0^t f(x)dx$$

$$= \frac{\partial}{\partial t}[F(x)|_0^t] = \frac{\partial}{\partial t}[F(t) - F(0)]$$

$$= \frac{\partial}{\partial t}F(t) - \frac{\partial}{\partial t}F(0) = f(t)$$

$$so, \quad \frac{\partial}{\partial t}\int_0^t f(t-\tau)d\tau = f(t)$$

Lemma 2 Let $\mathbb{T}$ is new transform. Then

$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\{f(X,t)\}\right\}\right) = f(X,t)$, where $X = (x,y,z)$

Proof:

Using property 2 and 3 of NT, and lemma (1), we have

$$\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\left\{f(X,t)\right\}\right\}\right)$$
$$= \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{u}\ \mathbb{T}\{1\}\mathbb{T}\left\{f(X,t)\right\}\right\}\right)$$
$$= \frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\{\mathbb{T}\{1 * f(X,t)\}\}\right)$$
$$= \frac{\partial}{\partial t}(1 * f(X,t))$$
$$= \frac{\partial}{\partial t}\left(\int_0^t f(X,t-\tau)d\tau\right) = f(X,t)$$

Theorem 1 (Convergence Theorem)

If the series form given in equation (8) with p = 1, i.e.,

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (15)$$

is convergent. Then the limit point converges to the exact solution of equation (1), where u_n (n = 0, 1, …) are calculated by NTHPM, i.e.,

$$\left.\begin{aligned} u_0(x,y,z,t) + u_1(x,y,z,t) &= \mathbb{T}^{-1}\left\{f + \frac{1}{v\alpha}\ \mathbb{T}\{-R[u_0]\}\right\} \\ u_n(x,y,z,t) &= -\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}\mathbb{T}\{R[u_{n-1}]\}\right\}, n > 1 \end{aligned}\right\} \qquad (16)$$

Proof

Suppose that equation (15) converges to the limit point say as

$$w(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \qquad (17)$$

Now, from right hand side of equation (1) we have:

$$\alpha\frac{\partial w}{\partial t} = \alpha\ \frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n(x,y,z,t)$$
$$= \alpha\ \frac{\partial}{\partial t}\left[u_0 + u_1 + \sum_{n=2}^{\infty} u_n(x,y,z,t)\right]$$

$= \alpha\ \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{f + \frac{1}{v\alpha}\ \mathbb{T}\{-R[u_0]\}\right\} - \sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v\alpha}[\mathbb{T}\{R[u_{n-1}]\}]\right\}\right] = \alpha\ \frac{\partial f}{\partial t} - R[u_0] - \frac{\partial}{\partial t}\ \left(\sum_{n=2}^{\infty}\mathbb{T}^{-1}\left\{\frac{1}{v}[\mathbb{T}\{R[u_n]\}]\right\}\right) = 0 - R[u_0] - \sum_{n=0}^{\infty}\frac{\partial}{\partial t}\left(\mathbb{T}^{-1}\left\{\frac{1}{v}[\mathbb{T}\{R[u_n]\}]\right\}\right) \quad (18)$

By lemma (2), equation (18) becomes

$$\alpha\frac{\partial w}{\partial t} = -\sum_{n=0}^{\infty} R[u_n] = -R\left[\sum_{n=0}^{\infty} u_n\right] = -Rw$$
$$= w_{xx} + w_{yy} + w_{zz}$$

Then $w(x, y, z, t)$ satisfies equation (1). So, it is exact solution.

Solving Nonlinear PDE by Suggested Method

This section consists of the procedure of the combine NT algorithm with the HPM to solve equation (19)**,** i.e.,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \qquad (19)$$

To explain the main idea of NTHPM firstly the nonlinear PDE is written as follow:

$$L[u(x,y,z,t)] + R[u(x,y,z,t)] + N[u(x,y,z,t)] = g(x,y,z,t) \qquad (20)$$

with initial condition (IC)

$$u(x,y,z,0) = f(x,y,z) \qquad (21)$$

Where all $x, y, z\ in\ R$, L is a linear differential operator ($L = \frac{\partial}{\partial t}$), R is a remained of the linear operator, N is a nonlinear differential operator and $g(x,y,z,t)$ is the nonhomogeneous part.

We construct a Homotopy as: u(x, p): $R^n \times$ [0, 1] $\to R$, using the homotopy perturbation technique which satisfies

$$H(u(x,y,z,t),p) = (1-p)\left[L(u(x,y,z,t)) - L\big(u(x,y,z,0)\big)\right] + p[A(u(x,y,z,t)) - g(x,y,z,t)] = 0 \quad (22)$$

Where p∈[0, 1] is an embedding parameter and the operator A defined as:

$A = L + R + N.$

Obviously, if p = 0, equation (22) becomes $L(u(x,y,z,t)) = L\big(u(x,y,z,0)\big)$.

It is clear that, if p =1, then the homotopy equation (22) converts to the main differential equation (20). In topology, this deformation is called homotopic. Substitute equation (21) in equation (22) and it is rewritten as:

$L(u(x,y,z,t)) - L\big(f(x,y,z)\big) - pL(u(x,y,z,t)) + pL\big(f(x,y,z)\big) + pL(u(x,y,z,t)) + pR(u(x,y,z,t)) + pN(u(x,y,z,t)) - p\ g(x,y,z,t) = 0$

Then

$$L(u(x,y,z,t)) - L\big(f(x,y,z)\big) + p\left[L\big(f(x,y,z)\big) + R(u(x,y,z,t)) + N(u(x,y,z,t)) - g(x,y,z,t)\right] = 0 \quad (23)$$

Since $f(x, y, z)$ is independent of the variable t and the linear operator L dependent on t so, $L\big(f(x,y,z)\big) = 0$, i.e., equation (23) becomes:

$$L(u(x,y,z,t)) + p[R(u(x,y,z,t)) + N((x,y,z,t)) - g(x,y,z,t)] = 0 \qquad (24)$$

According to the classical perturbation technique, the solution of equation (24) can be written as a power series of embedding parameter p, in the form

$$u(x,y,z,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,z,t) \qquad (25)$$

For most cases, the series form in (25) is convergent and the convergent rate depends on the nonlinear operator $N(u(x,y,z,t))$.
Taking the NT (with respect to the variable t) for the equation (24) to get:

$$\mathbb{T}\{L(u)\} + p\,\mathbb{T}\{R(u) + N(u) - g\} = 0 \quad (26)$$

Now, by using the differentiation property of NT and IC in equation (21), equation (26) becomes:

$$v\mathbb{T}\{u\} - vf(x,y,z) + p\,\mathbb{T}\{R(u) + N(u) - g\} = 0 \quad (27)$$

Hence:

$$\mathbb{T}\{u\} = f(x,y,z) + p\,\frac{\mathbb{T}\{g - R(u) - N(u)\}}{v} \quad 28)$$

By taking the inverse of new transform on both sides of equation (28) to get:

$$u(x,y,z,t) = f(x,y,z) + p\,\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t) - R(u(x,y,z,t)) - N(u(x,y,z,t))\}}{v}\right\} \quad (29)$$

Then, substitute equation (25) in equation (29) to obtain:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y,z) + p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t) - R(\sum_{n=0}^{\infty} p^n u_n) - N(\sum_{n=0}^{\infty} p^n u_n)\}}{v}\right\} \quad (30)$$

The nonlinear part can be decomposed, as will be explained later, by substituting equation (25) in it as:

$$N(u) = N\left(\sum_{n=0}^{\infty} p^n u_n(x,y,z,t)\right) = \sum_{n=0}^{\infty} p^n A_n \quad (31)$$

Then equation (30) becomes:

$$\sum_{n=0}^{\infty} p^n u_n = f(x,y,z) + p\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t) - R(\sum_{n=0}^{\infty} p^n u_n) - \sum_{n=0}^{\infty} p^n A_n\}}{v}\right\} \quad (32)$$

By comparing the coefficient with the same power of p in both sides of the equation (32) we have:

$$u_0 = f(x,y,z)$$
$$u_1 = \mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{g(x,y,z,t) - R(u_0) - A_0\}}{v}\right\}$$
$$u_2 = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R(u_1) + A_1\}}{v}\right\} \quad (33)$$
$$\vdots$$
$$u_{n+1} = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R(u_n) + A_n\}}{v}\right\}$$

and so on. According to the series solution in equation (25), then at p=1 we can get

$$u(x,y,z,t) = u_0(x,y,z,t) + u_1(x,y,z,t) + \cdots = \sum_{n=0}^{\infty} u_n(x,y,z,t) \quad (34)$$

Illustrative Example

In this section, the suggested method will be used to solve equation (19), with appropriate initial condition IC: $f(x,y,z) = u(x,y,z,0) = \frac{2e^{\mu}}{e^{\mu}+1}$ where $\mu = \frac{-1}{3}(x+y+z)$, we have $L[u(x,y,z,t)] = \frac{\partial u(x,y,z,t)}{\partial t}$, $R[u(x,y,z,t)] = -\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$, $N[u(x,y,z,t)] = u\frac{\partial u}{\partial z}$ and $g(x,y,z,t) = 0$
First, compute A_n to the nonlinear part $N(u)$ to get:

$$N(u) = N(\textstyle\sum_{n=0}^{\infty} p^n u_n) = (\sum_{n=0}^{\infty} p^n u_n)\left(\frac{\partial}{\partial z}[\sum_{n=0}^{\infty} p^n u_n]\right) = (\sum_{n=0}^{\infty} p^n u_n)\left(\sum_{n=0}^{\infty} p^n \frac{\partial u_n}{\partial z}\right) = u_0\frac{\partial u_0}{\partial z} + p\left(u_0\frac{\partial u_1}{\partial z} + u_1\frac{\partial u_0}{\partial z}\right) + p^2\left(u_0\frac{\partial u_2}{\partial z} + u_1\frac{\partial u_1}{\partial z} + u_2\frac{\partial u_0}{\partial z}\right) + p^3\left(u_0\frac{\partial u_3}{\partial z} + u_1\frac{\partial u_2}{\partial z} + u_2\frac{\partial u_1}{\partial z} + u_3\frac{\partial u_0}{\partial z}\right) + \cdots$$

So,

$$A_0 = u_0\frac{\partial u_0}{\partial z}$$
$$A_1 = u_1\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_1}{\partial z}$$
$$A_2 = u_2\frac{\partial u_0}{\partial z} + u_1\frac{\partial u_1}{\partial z} + u_0\frac{\partial u_2}{\partial z}$$

and so on.
From IC, we can get

$$A_0 = -\frac{4\,e^{2\mu}}{3\,(e^{\mu}+1)^3}$$
$$A_1 = -t\,\frac{4\,e^{2\mu}\,(2 - e^{\mu})}{9\,(e^{\mu}+1)^4}$$
$$A_2 = -t^2\,\frac{e^{\mu}\,(8\,e^{\mu} - 14\,e^{2\mu} + 2\,e^{3\mu})}{27\,(e^{\mu}+1)^5}$$
$$A_3 = -t^3\,\frac{e^{\mu}\,(16\,e^{\mu} - 66\,e^{2\mu} + 36\,e^{3\mu} - 2\,e^{4\mu})}{243\,(e^{\mu}+1)^6}$$

and so on.
Moreover, the sequence of parts u_n is:

$$u_0 = \frac{2\,e^{\mu}}{(e^{\mu}+1)}$$
$$u_1 = t\,\frac{2\,e^{\mu}}{3(e^{\mu}+1)^2}$$
$$u_2 = t^2\,\frac{e^{\mu}\,(1 - e^{\mu})}{9\,(e^{\mu}+1)^3}$$
$$u_3 = t^3\,\frac{e^{\mu}\,(1 - 4\,e^{\mu} + e^{2\mu})}{81\,(e^{\mu}+1)^4}$$
$$u_4 = t^4\,\frac{e^{\mu}\,(1 - 11e^{\mu} + 11e^{2\mu} - e^{3\mu})}{972\,(e^{\mu}+1)^5}$$

and so on.
Substituting the above values in series form (34), hence the solution of the problem is close to the form:

$$u(x,y,z,t) = \frac{2\,e^{\mu + \frac{1}{3}t}}{e^{\mu + \frac{1}{3}t} + 1}$$

That presents the exact solution.

The Convergence of the Solution for Nonlinear Case

Now, we must prove the convergence of solution of equation (19) to the exact solution when we used the NTHPM, the solution is given in equation (34), where u_n, (n= 0, 1, …), are calculated by new transform, i.e.,

$$\left.\begin{aligned} &u_0 = f(x,y,z) \\ &u_n = -\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_{n-1}] + A_{n-1}\}}{v}\right\} \;, n \ge 1 \end{aligned}\right\} \quad (35)$$

and A_n, (n= 0, 1, …), are defined as

$$A_n = u_n \frac{\partial u_0}{\partial z} + u_{n-1}\frac{\partial u_1}{\partial z} + \cdots + u_0 \frac{\partial u_n}{\partial z} = \sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial z} \quad (36)$$

Theorem (2) (Convergence Theorem)

If the series (34) which was calculated by NTHPM is convergent then the limit point converges to the exact solution of moisture content equation (19).

Suppose that equation (34) converge, then we called the limit point as

$$w(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \quad (37)$$

Now, from left hand side of equation (19) we have:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} u_n(x,y,z,t)$$

$$= \frac{\partial}{\partial t}\left[u_0(x,y,z,t) + \sum_{n=1}^{\infty} u_n(x,y,z,t)\right]$$

$$= \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\{f\} - \sum_{n=1}^{\infty}\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_{n-1}] + A_{n-1}\}}{v}\right\}\right]$$

$$= \frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\{f\} - \sum_{n=0}^{\infty}\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\} - \sum_{n=0}^{\infty}\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}\right]$$

$$= \frac{\partial f}{\partial t} - \frac{\partial}{\partial t}\sum_{n=0}^{\infty}\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\} - \frac{\partial}{\partial t}\sum_{n=0}^{\infty}\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}$$

$$= \frac{\partial f}{\partial t} - \sum_{n=0}^{\infty}\frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{R[u_n]\}}{v}\right\}\right] - \sum_{n=0}^{\infty}\frac{\partial}{\partial t}\left[\mathbb{T}^{-1}\left\{\frac{\mathbb{T}\{A_n\}}{v}\right\}\right] \quad (38)$$

By lemma (2), equation (38) becomes

$$\frac{\partial w}{\partial t} = 0 - \sum_{n=0}^{\infty} R[u_n] - \sum_{n=0}^{\infty} A_n \quad (39)$$

However, from equation (36) we have

$$\sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} u_k \frac{\partial u_{n-k}}{\partial z} = u_0\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_1}{\partial z} + u_1\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_2}{\partial z} + u_1\frac{\partial u_1}{\partial z} + u_2\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_3}{\partial z} + u_1\frac{\partial u_2}{\partial z} + u_2\frac{\partial u_1}{\partial z} + u_3\frac{\partial u_0}{\partial z} + \cdots$$

$$= u_0\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + u_1\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + u_2\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + u_3\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) + \cdots$$

$$= (u_0 + u_1 + u_2 + u_3 + \cdots)\left(\frac{\partial u_0}{\partial z} + \frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial z} + \cdots\right) = \left(\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty}\frac{\partial u_n}{\partial z}\right) = \left(\sum_{n=0}^{\infty} u_n\right)\left(\frac{\partial}{\partial z}\sum_{n=0}^{\infty} u_n\right) \quad (40)$$

Then substitute equation (40) in equation (39) to obtain

$$\frac{\partial w}{\partial t} = -R\left[\sum_{n=0}^{\infty} u_n\right] - \left(\sum_{n=0}^{\infty} u_n\right)\left(\frac{\partial}{\partial z}\sum_{n=0}^{\infty} u_n\right) = -R[w] - w\frac{\partial w}{\partial z}$$

$$\therefore \quad \frac{\partial w}{\partial t} = \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] - w\frac{\partial w}{\partial z}$$

Then $w(x, y, z, t)$ is satisfy equation (19). So, its exact solution.

Conclusion:

We employed the combination of new transform suggested by Luma and Alaa with HPM method to get a closed form solution of the three dimensional space PDE linear and nonlinear. The new method is free of unnecessary mathematical complexities. Although the problem considered has no exact solution, the accuracy, efficiency, and reliability of the new method are guaranteed. The convergence of obtained solution to the exact solution by using NTHPM is proved.

Conflicts of Interest: None.

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اسلوب جديد لحل معادلات تفاضلية جزئية ذات بعد فضاء ثالث

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الخلاصة:

هذا البحث يعرض طريقة تحويل جديدة لحل معادلات تفاضلية جزئية لإيجاد الحلول الدقيقة المناسبة في مجال اوسع و يمكن استخدامه لحل مسائل بدون اللجوء الى تقطيع و تردد المجال. يقرن التحويل الجديد مع طريقة الهوموتوبي المضطربة لحل معادلات تفاضلية جزئية من الرتبة الاولى ثلاثية الابعاد ذات شروط ابتدائية و من ثم اثبات تقارب تلك الحلول. تنفيذ الطريقة المقترحة اثبتت فائدتها في ايجاد الحلول المضبوطة. التطبيق العملي اثبت تأثير الاسلوب و سهولة التنفيذ في ايجاد الحل المضبوط. اخيرا جميع البرامج نفذت باستخدام الماتلاب اصدار 7.12

الكلمات المفتاحية: معادلات تفاضلية جزئية، طريقة الاضطراب هوموتوبي، تحويلات، تقارب، اقتران طريقتين