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A Proposed Analytical Method for Solving Fuzzy Linear Initial Value Problems

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Abstract:

In this article, we aim to define a universal set consisting of the subscripts of the fuzzy differential equation (5) except the two elements 0 and n, subsets of that universal set are defined according to certain conditions. Then, we use the constructed universal set with its subsets for suggesting an analytical method which facilitates solving fuzzy initial value problems of any order by using the strongly generalized H-differentiability. Also, valid sets with graphs for solutions of fuzzy initial value problems of higher orders are found.

Key words: Fuzzy differential equations, Fuzzy Laplace transform, Fuzzy numbers, Generalized H-differentiability.

Introduction:

The concept of the fuzzy derivative was first introduced by Chang and Zadeh (1), it was followed up by Dubios and Prade (2), and Puri and Ralescu (3). The fuzzy Laplace transform (FLT) is proposed to solve first order fuzzy differential equations (FDEs) by using the strongly generalized differentiability concept (4), and then some of well-known properties of the fuzzy Laplace transform were investigated. In addition, an existence theorem was given for fuzzy-valued function which possesses the fuzzy Laplace transform (5). A formula of the fuzzy Laplace transform of the nth-order derivative was initially introduced in terms of the number of derivatives in form (ii) by Mohammad Ali (6), and Haydar and Mohammad Ali (7), it was followed by introducing another formula for the fuzzy Laplace transform on fuzzy nth-order derivative by concept of the strongly generalized differentiability (8). In the direction of solving n-th order FDEs numerically, many efforts have been introduced by a number of authors (9-11). So far, a few number of works have been introduced in the subject of finding the analytical solutions of FDEs for example (12- 14). Also, some analytical methods for solving fuzzy differential equations are introduced in (15). Recently, approximated solutions of fuzzy initial value problems have been studied such as (16) and (17). In this paper, we extend the proposed method by Mohammed (18), for solving nth-order classical differential equations by the classical Laplace transform to solve nth-order FDEs by FLT under the strongly generalized H-differentiability. Also, we introduce theorem and some corollaries that help us in solving nth-order FDEs.

This paper is organized as follows: Basic concepts are given in Section 2. In Section 3, a new method for solving FIVPs of nth-order is introduced with some results. In Section 4, examples of several FIVPs are solved to show the activity of the method. In Section 5, discussion and conclusions are given.

Basic concepts

In this section, we are going to recall some basic concepts that we need in this paper.

Definition 1. (5) A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$, $0 \le \alpha \le 1$ which satisfy the following requirements:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(\alpha) \le \overline{u}(\alpha)$, $0 \le \alpha \le 1$.

Definition 2. (5) Let $x, y \in E$. If there exists $z \in E$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \Theta y$. In this paper, the sign "Θ" always stands for H-difference, and also note that $x \Theta y \neq x + (-1)y$.

Definition 3. (5) Let $f:(a,b) \to E$ and $x_0 \in (a,b)$. We say that f is strongly generalized differential at x_0 if there exists an element $f'(x_0) \in E$, such that

i. For all $h > 0$ sufficiently small, $\exists f(x_0 + h) \Theta f(x_0)$, $\exists f(x_0) \Theta f(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\Theta f(x_0))/h] = \lim_{h\to 0}[(f(x_0)\Theta f(x_0-h))/h] = f'(x_0)$$

or

ii. For all $h > 0$ sufficiently small, $\exists f(x_0) \Theta f(x_0 + h)$, $\exists f(x_0 - h) \Theta f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\Theta f(x_0+h))/(-h)] = \lim_{h\to 0}[(f(x_0-h)\Theta f(x_0))/(-h)] = f'(x_0)$$

or

iii. For all $h > 0$ sufficiently small, $\exists f(x_0 + h) \Theta f(x_0)$, $\exists f(x_0 - h) \Theta f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\Theta f(x_0))/h] = \lim_{h\to 0}[(f(x_0-h)\Theta f(x_0))/(-h)] = f'(x_0)$$

or

iv. For all $h > 0$ sufficiently small, $\exists f(x_0) \Theta f(x_0 + h)$, $\exists f(x_0) \Theta f(x_0 - h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\Theta f(x_0+h))/(-h)] = \lim_{h\to 0}[(f(x_0)\Theta f(x_0-h))/h] = f'(x_0)$$

Definition 4. (4) Let $f(x)$ be continuous fuzzy-valued function. Suppose that $f(x)\, e^{-px}$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^\infty f(x) e^{-px} dx$ is called fuzzy Laplace transform and is denoted as:

$L[f(x)] = \int_0^\infty f(x) e^{-px} dx$, ($p > 0$ and integer)

We have:

$$\int_0^\infty f(x)\, e^{-px} dx = \left(\int_0^\infty \underline{f}(x;\alpha)\, e^{-px} dx,\ \int_0^\infty \overline{f}(x;\alpha)\, e^{-px} dx\right),$$

also by using the definition of classical Laplace transform:

$$\ell\left[\underline{f}(x;\alpha)\right] = \int_0^\infty \underline{f}(x;\alpha)\, e^{-px} dx$$

and

$$\ell\left[\overline{f}(x;\alpha)\right] = \int_0^\infty \overline{f}(x;\alpha)\, e^{-px} dx,$$

then, we follow:

$$L\left[f(x)\right] = \left(\ell\left[\underline{f}(x;\alpha)\right], \ell\left[\overline{f}(x;\alpha)\right]\right).$$

Theorem 1. (6, 7) Suppose that $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are differentiable fuzzy valued functions such that $f^{(i_1)}(t), f^{(i_2)}(t), \ldots, f^{(i_m)}(t)$ are (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ when $0 \le m \le n$ and $f^{(p)}(t)$ is (i)-differentiable for $p \ne i_j, j = 1, 2, \ldots, m$, and if α-cut representation of fuzzy- valued function $f(t)$ is denoted by $f(t) = [\underline{f}(t;\alpha), \overline{f}(t;\alpha)]$, then:

(a) If m is an even number then

$$f^{(n)}(t) = [\underline{f}^{(n)}(t;\alpha), \overline{f}^{(n)}(t;\alpha)].$$

(b) If m is an odd number then

$$f^{(n)}(t) = [\overline{f}^{(n)}(t;\alpha), \underline{f}^{(n)}(t;\alpha)].$$

Theorem 2. (6, 7) Suppose that $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of an exponential order and that $f^{(n)}(t)$ is a piecewise continuous fuzzy-valued function on $[0,\infty)$. Let $f^{(i_1)}(t), f^{(i_2)}(t), \ldots, f^{(i_m)}(t)$ be (ii)-differentiable functions for $0 \le i_1 < i_2 < \ldots < i_m \le n-1$ and $f^{(p)}$ be (i)-differentiable function for $p \ne i_j, j = 1, 2, \ldots, m$ and $f(t) = (\underline{f}(t;\alpha), \overline{f}(t;\alpha))$; then

(1)If m is an even number, we have:

$$L[f^{(n)}(t)] = p^n L[f(t)] \Theta p^{n-1} f(0) \otimes \sum_{k=1}^{n-1} p^{n-(k+1)} f^{(k)}(0),$$

such that

$$\otimes = \begin{cases} \Theta, \text{ if } q \text{ is an even number,} \\ -, \text{ if } q \text{ is an odd number,} \end{cases}$$

where q is the number of (ii)-differentiable functions $f^{(i)}$ when $i < k$.

(2) If m is an odd number, we have:

$$L[f^{(n)}(t)] = -p^{n-1}f(0) \ \Theta \ (-p^n)L[f(t)] \otimes \sum_{k=1}^{n-1} p^{n-(k+1)} f^{(k)}(0),$$

such that

$$\otimes = \begin{cases} \Theta, \text{ if } q \text{ is an odd number,} \\ -, \text{ if } q \text{ is an even number,} \end{cases}$$

where q is defined as in (1) above.

A suggested method for solving nth-order fuzzy linear initial value problems

In this section, we are going to introduce a theorem and some corollaries which can be used for solving FIVPs of the nth-order.

Notations: First of all, we shall define the symbols $\mp_{i,f}$, $\mp_{i,o}$, $\mp_{n,f}$ and $\mp_{n,o}$ which are essential in this article. If n is the order of the fuzzy differential equation which is to be solved and m is the number of derivatives in form (ii) among $y(t), y'(t), \dots, y^{(n-1)}(t)$. Then

1- Let the symbol Θ_i be Θ or $+$ (i.e. $\oplus$).

2- Let $\mp_{i,f}$ be defined as:

$$\mp_{i,f} = \begin{cases} + \text{ (i.e. } \oplus\text{), if } q \text{ is an even number,} \\ (-1)\ \Theta, \text{ if } q \text{ is an odd number,} \end{cases} \qquad (1)$$

where q is the number of (ii)-differentiable functions $f^{(l)}$ when $l < i$ and $"f"$ in the above subscript refers to the word fuzzy for $i = 1, 2, ..., n.$

3- Let $\mp_{i,o}$ be defined as:

$$\mp_{i,o} = \begin{cases} +, \text{ if } q \text{ is an even number,} \\ -, \text{ if } q \text{ is an odd number,} \end{cases} \qquad (2)$$

where q is defined as in equation (1) above and $"o"$ in the above subscript refers to the word ordinary for $i = 1, 2, ..., n.$ Therefore, we note that:

$$\mp_{n,f} = \begin{cases} + \text{ (i.e. } \oplus\text{), if } m \text{ is an even number,} \\ (-1)\ \Theta, \text{ if } m \text{ is an odd number,} \end{cases} \qquad (3)$$

and

$$\mp_{n,o} = \begin{cases} +, \text{ if } m \text{ is an even number,} \\ -, \text{ if } m \text{ is an odd number,} \end{cases} \qquad (4)$$

Theorem 3. Suppose that we have the following FDE:

$$a_n y^{(n)} + \Theta_{n-1} a_{n-1} y^{(n-1)} + \cdots + \Theta_1 a_1 y' + \Theta_0 a_0 y = f(t), \ a_i \in R, \ i = 0, 1, \dots, n, \qquad (5)$$

with given fuzzy initial conditions $y(0), y'(0), \dots, y^{(n-1)}(0)$ and $y^{(i_1)}(t), y^{(i_2)}(t), \dots, y^{(i_m)}(t)$ are functions in form (ii) for $0 \le i_1 < i_2 < \dots < i_m \le n-1$, $0 \le m \le n$. Then the solution $y(t)$ satisfies the following relation:

$$\left(\mp_{n,f}\, a_n p^n + \Theta_0 a_0 + \sum_{i=1}^{n-1} \Theta_i \mp_{i,f}\, a_i p^i \right) L[y(t)] = L[f(t)] + u(p), \qquad (6)$$

where $\mp_{i,f}$ and $\mp_{n,f}$ are defined in equations (1) and (3) respectively and $u(p)$ is a polynomial of p whose degree is less than or equal to $n-1$.

Proof. The FDE (5) can be written as:

$$\Theta_0 a_0 y + a_n y^{(n)} + \sum_{i=1}^{i_1} \Theta_i a_i y^{(i)} + \sum_{i=i_1+1}^{i_2} \Theta_i a_i y^{(i)} + \cdots + \sum_{i=i_{m-1}+1}^{i_m} \Theta_i a_i y^{(i)} + \sum_{i=i_m+1}^{n-1} \Theta_i a_i y^{(i)} = f(t). \quad (7)$$

We suppose that n_0 is the number of derivatives in form (ii) which is less than the ith derivative for $1 \le i \le i_1$, n_k is the number of derivatives in form (ii) which is less than the ith derivative for $i_k + 1 \le i \le i_{k+1}$ when $k = 1, 2 ... m-1$ and n_m is the number of derivatives in form (ii) which is less than the ith derivative for $i_m + 1 \le i \le n-1$.

It is clear that $n_i = i$, $i = 0, 1, ..., m$.

Now, Let m be an even number then $m-1$ is an odd number and $m-2$ is an even number. Therefore equation (7) can be written as follows:

$$\Theta_0 a_0 y + \sum_{i=1}^{i_1} \Theta_i a_i y^{(i)} + \sum_{j=1}^{m/2} \sum_{i=i_{2j-1}+1}^{i_{2j}} \Theta_i a_i y^{(i)} + \sum_{j=1}^{(m-2)/2} \sum_{i=i_{2j}+1}^{i_{2j+1}} \Theta_i a_i y^{(i)} + \sum_{i=i_m+1}^{n-1} \Theta_i a_i y^{(i)} + a_n y^{(n)} = f(t). \quad (8)$$

We take FLT to both sides of equation (8):

$$\Theta_0 a_0 L[y] + a_n L[y^{(n)}] + \sum_{i=1}^{i_1} \Theta_i a_i L[y^{(i)}] + \sum_{j=1}^{m/2} \sum_{i=i_{2j-1}+1}^{i_{2j}} \Theta_i a_i L[y^{(i)}] + \sum_{j=1}^{(m-2)/2} \sum_{i=i_{2j}+1}^{i_{2j+1}} \Theta_i a_i . L[y^{(i)}] + \sum_{i=i_m+1}^{n-1} \Theta_i a_i L[y^{(i)}] = L[f(t)]. \quad (9)$$

Since $n_0 = 0$ (even), $n_{2j-1} = 2j-1$ (odd) for $j = 1, \ldots, \frac{m}{2}$, $n_{2j} = 2j$ (even) for $i = 1, \ldots, \frac{m-2}{2}$ and $n_m = m$ (even), then by using Theorem 2, equation (9) becomes:

$$\Theta_0 a_0 L[y] + \sum_{i=1}^{i_1} \Theta_i a_i [p^i L[y(t)] \ominus p^{i-1} y(0) \otimes_e \sum_{k=1}^{i-1} p^{i-(k+1)} y^{(k)}(0)] + \sum_{j=1}^{m/2} \sum_{i=i_{2j-1}+1}^{i_{2j}} \Theta_i a_i [-p^{i-1} y(0) \ominus (-p^i) L[y(t)] \otimes_o \sum_{k=1}^{i-1} p^{i-(k+1)} y^{(k)}(0)] +$$

$$\sum_{j=1}^{(m-2)/2} \sum_{i=i_{2j}+1}^{i_{2j+1}} \Theta_i a_i [p^i L[y(t)] \ominus p^{i-1} y(0) \otimes_e \sum_{k=1}^{i-1} p^{i-(k+1)} y^{(k)}(0)] + \sum_{i=i_m+1}^{n-1} \Theta_i a_i [p^i L[y(t)] \ominus p^{i-1} y(0) \otimes_e \sum_{k=1}^{i-1} p^{i-(k+1)} y^{(k)}(0)] + a_n [p^n . L[y(t)] \ominus p^{n-1} y(0) \otimes_e \sum_{k=1}^{n-1} p^{n-(k+1)} y^{(k)}(0)]$$

$$= L[f(t)],$$

where

$$\otimes_e = \begin{cases} \ominus, & \text{if } q \text{ is an even number}, \\ -, & \text{if } q \text{ is an odd number}, \end{cases} \quad (10)$$

and

$$\otimes_o = \begin{cases} \ominus, & \text{if } q \text{ is an odd number}, \\ -, & \text{if } q \text{ is an even number}, \end{cases} \quad (11)$$

where q is the number of (ii)-differentiable functions $f^{(l)}$ when $l < k$.
Then, we can get:

$$[\sum_{i=0}^{i_1} \Theta_i a_i p^i + \sum_{j=1}^{\frac{m-2}{2}} \sum_{i=i_{2j}+1}^{i_{2j+1}} \Theta_i a_i p^i + \sum_{i=i_m+1}^{n-1} \Theta_i a_i p^i + a_n p^n] L[y(t)] + [\sum_{j=1}^{m/2} \sum_{i=i_{2j-1}+1}^{i_{2j}} \Theta_i (-1) \ominus a_i p^i].$$

$$L[y(t)] = L[f(t)] + u_1(p), \quad (12)$$

where $u_1(p)$ is a polynomial of p its degree is less than or equal to $n-1$. It is clear that equation (12) can be written as:

$$\left(a_n p^n + \Theta_0 a_0 + \sum_{i=1}^{n-1} \Theta_i \mp_{i,f} a_i p^i \right) L[y(t)] = L[f(t)] + u(p),\ u(p) = u_1(p), \quad (13)$$

where $\mp_{i,f}$ is defined as in equation (1).

Similarly, if m is an odd number, (i.e., $m-1$ is an even number and $m-2$ is an odd number). Therefore, equation (7) can be written as follows:

$$\Theta_0 a_0 y + \sum_{i=1}^{i_1} \Theta_i a_i y^{(i)} + \sum_{j=1}^{(m-1)/2} \sum_{i=i_{2j-1}+1}^{i_{2j}} \Theta_i a_i y^{(i)} + \sum_{j=1}^{(m-1)/2} \sum_{i=i_{2j}+1}^{i_{2j+1}} \Theta_i a_i y^{(i)} + \sum_{i=i_m+1}^{n-1} \Theta_i a_i y^{(i)} + a_n y^{(n)} = f(t).$$

By taking FLT to both sides of the above equation and making some simplifications, we get:

$$[(-1) \ominus a_n p^n + \Theta_0 a_0 + \sum_{i=1}^{n-1} \Theta_i \mp_{i,f} a_i p^i] L[y(t)] = L[f(t)] + u(p),\ u(p) = v_1(p), \quad (14)$$

where $\mp_{i,f}$ is defined as in equation (1).
It is clear that equations (13) and (14) can be written as in equation (6).
By equation (6) and determining Θ_i, $\mp_{i,f}$ and $\mp_{n,f}$, we can get $l[\underline{y}(t;\alpha)]$ and $l[\overline{y}(t;\alpha)]$, then taking l^{-1} gives $\underline{y}(t;\alpha)$ and $\overline{y}(t;\alpha)$.

Corollary 1. Suppose that $l[\underline{f}(t;\alpha)] = \dfrac{k_1(p;\alpha)}{h_1(p;\alpha)}$ and $l[\bar{f}(t;\alpha)] = \dfrac{k_2(p;\alpha)}{h_2(p;\alpha)}$. If $\Theta_i = +$ and $a_i \geq 0$ for $i = 0,1,\ldots,n-1$ and $a_n > 0$. Then the $\alpha-$ cut representation $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ of the solution $y(t)$ are given as follows:

$$\underline{y}(t;\alpha) = l^{-1}[\frac{q_1(p;\alpha)}{h_1 h_2 \sum_{i=0}^{n} a_i p^i .(a_0 + \sum_{i=1}^{n} \mp_{i,o} a_i p^i)}], \tag{15}$$

$$\bar{y}(t;\alpha) = l^{-1}[\frac{q_2(p;\alpha)}{h_1 h_2 \sum_{i=0}^{n} a_i p^i .(a_0 + \sum_{i=1}^{n} \mp_{i,o} a_i p^i)}],$$

where $\mp_{i,o}$ is defined as in equation (2), q_1 and q_2 are polynomials of p such that:

$\deg(q_i) < 2n + \deg(h_1(p;\alpha)) + \deg(h_2(p;\alpha)),$.
$i = 1,2.$

Proof. The proof can be made by considering that $\Theta_i = + \ \forall i$ and $a_i \geq 0 \ \forall i$ in equation (6). Then:

$$\Theta_i \mp_{i,f} = \begin{cases} + \text{ (i.e. } \oplus\text{), if } q \text{ is an even number,} \\ \text{(-1) } \Theta\text{, if } q \text{ is an odd number,} \end{cases}$$

$$= \mp_{i,f},$$

where q is defined as in equation (1).
Also, we note that if m is an even number then $\mp_{n,f} = +$, and if m is an odd number then $\mp_{n,f} = (-1)\ \Theta.$

Corollary 2. Suppose that $l[\underline{f}(t;\alpha)] = \dfrac{k_1(p;\alpha)}{h_1(p;\alpha)}$ and $l[\bar{f}(t;\alpha)] = \dfrac{k_2(p;\alpha)}{h_2(p;\alpha)}$. If $\Theta_i = +$ and $a_i \leq 0$ for $i = 0,1,\ldots,n-1$ and $a_n > 0$. Then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are given as follows:

$$\underline{y}(t;\alpha) = l^{-1}[\frac{q_1(p;\alpha)}{h_1 h_2 \sum_{i=0}^{n} a_i p^i (\mp_{n,o} a_n p^n - a_0 - \sum_{i=1}^{n-1} \mp_{i,o} a_i p^i)}],$$

$$\bar{y}(t;\alpha) = l^{-1}[\frac{q_2(p;\alpha)}{h_1 h_2 \sum_{i=0}^{n} a_i p^i (\mp_{n,o} a_n p^n - a_0 - \sum_{i=1}^{n-1} \mp_{i,o} a_i p^i)}],$$

where $\mp_{n,o}$ and $\mp_{i,o}$ are defined as in equations (4) and (2) respectively, q_1 and q_2 are polynomials of p such that:

$\deg(q_i) < 2n + \deg(h_1(p;\alpha)) + \deg(h_2(p;\alpha)),$
$i = 1,2.$

Proof. We can prove by achieving similar steps in the proof of Corollary 1 with regarding that $\Theta_i = + \ \forall i$ and $a_i \leq 0 \ \forall i$.

Corollary 3. Suppose that $l[\underline{f}(t;\alpha)] = \dfrac{k_1(p;\alpha)}{h_1(p;\alpha)}$ and $l[\bar{f}(t;\alpha)] = \dfrac{k_2(p;\alpha)}{h_2(p;\alpha)}$. If $\Theta_i = \Theta$ and $a_i \geq 0$ for $i = 0,1,\ldots,n-1$ and $a_n > 0$ .Then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are given as follows:

$$\underline{y}(t;\alpha) = l^{-1}[\frac{q_1(p;\alpha)}{h_1 h_2 (a_n p^n - \sum_{i=0}^{n-1} a_i p^i)} . \frac{1}{(\mp_{n,o} a_n p^n - a_0 - \sum_{i=1}^{n-1} \mp_{i,o} a_i p^i)}], \tag{16}$$

$$\bar{y}(t;\alpha) = l^{-1}[\frac{q_2(p;\alpha)}{h_1 h_2 (a_n p^n - \sum_{i=0}^{n-1} a_i p^i)} . \frac{1}{(\mp_{n,o} a_n p^n - a_0 - \sum_{i=1}^{n-1} \mp_{i,o} a_i p^i)}],$$

where $\mp_{n,o}$ and $\mp_{i,o}$ are defined as in equations (4) and (2) respectively, q_1 and q_2 are polynomials of p such that:

$\deg(q_i) < 2n + \deg(h_1(p;\alpha)) + \deg(h_2(p;\alpha)),$
$i = 1,2.$

Proof. The proof can be achieved by considering that $\Theta_i = \Theta$ and $a_i \geq 0$,$\forall i$ in equation (6). Then:

$$\Theta_i \mp_{i,f} = \Theta \mp_{i,f}$$

$$= \begin{cases} \Theta, \text{ if } q \text{ is an even number}, \\ -, \text{ if } q \text{ is an odd number}, \end{cases} \qquad (17)$$

where q is defined as in equation (1).

Also, we note that if m is an even number then $\mp_{n,f} = +$, and if m is an odd number then $\mp_{n,f} = (-1)\ \Theta$.

Corollary 4. Suppose that $l[\underline{f}(t;\alpha)] = \dfrac{k_1(p;\alpha)}{h_1(p;\alpha)}$ and $l[\bar{f}(t;\alpha)] = \dfrac{k_2(p;\alpha)}{h_2(p;\alpha)}$. If $\Theta_i = \Theta$ and $a_i \le 0$ for $i = 0,1,\ldots,n-1$ and $a_n > 0$. Then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are given as follows:

$$\underline{y}(t;\alpha) = l^{-1}\left[\frac{q_1(p;\alpha)}{h_1 h_2 (a_n p^n - \sum_{i=0}^{n-1} a_i p^i)(a_0 + \sum_{i=1}^{n} \mp_{i,o} a_i p^i)}\right],$$

$$\bar{y}(t;\alpha) = l^{-1}\left[\frac{q_2(p;\alpha)}{h_1 h_2 (a_n p^n - \sum_{i=0}^{n-1} a_i p^i)(a_0 + \sum_{i=1}^{n} \mp_{i,o} a_i p^i)}\right],$$

where $\mp_{i,o}$ is defined in equation (2), q_1 and q_2 are polynomials of p such that:
$\deg(q_i) < 2n + \deg(h_1(p;\alpha)) + \deg(h_2(p;\alpha))$, $i = 1,2$.

Proof. We can prove by regarding that $\Theta_i = \Theta$ $\forall i$ and $a_i \le 0$ $\forall i$.

The main result in this paper is the following Corollary. In Corollary 5, we shall find the solutions of the FDE (5) when $\Theta_i = \Theta$ for some i and $\Theta_i = +(i.e.\ \oplus)$ for the others, as follows:

Corollary 5. Suppose that $l[\underline{f}(t;\alpha)] = \dfrac{k_1(p;\alpha)}{h_1(p;\alpha)}$ and $l[\bar{f}(t;\alpha)] = \dfrac{k_2(p;\alpha)}{h_2(p;\alpha)}$, $a_i \ge 0$ for $i = 0, 1, \ldots, n-1$ and $a_n > 0$. For the FDE (5), we define a universal set S of the subscripts that appear in the FDE (5) except $i = 0$ and $i = n$ i.e., $S = \{1, 2, \ldots, n-1\}$ and two disjoint subsets S_1 and S_2 of S such that $i \in S_1 \cup S_2$ if $\Theta_i = \Theta$ and $i \in (S_1 \cup S_2)^c$ if $\Theta_i = +$, $i \in S$. Also, we suppose that two subsets S_e and S_o of S form a partition for S such that $S_e\,(S_o)$ contains all $i \in S$ such that the number of derivatives in form (ii)- which is less than the ith derivative is an even (odd) number. Then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are given as follows:

$$\underline{y}(t;\alpha) = l^{-1}\left[\frac{q_1(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in S-(s_1 \cup s_2)} a_i p^i)} \cdot \frac{1}{\mp_{n,o} a_n p^n - a_0 - \sum_{i \in (s_1 \cup s_0) - s_2} a_i p^i + \sum_{i \in (s_2 \cup s_e) - s_1} a_i p^i}\right], \qquad (18)$$

$$\bar{y}(t;\alpha) = l^{-1}\left[\frac{q_2(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in S-(s_1 \cup s_2)} a_i p^i)} \cdot \frac{1}{\mp_{n,o} a_n p^n - a_0 - \sum_{i \in (s_1 \cup s_0) - s_2} a_i p^i + \sum_{i \in (s_2 \cup s_e) - s_1} a_i p^i}\right],$$

where $\mp_{n,o}$ is defined as in equation (4), q_1 and q_2 are polynomials of p such that:
$\deg(q_i) < 2n + \deg(h_1(p;\alpha)) + \deg(h_2(p;\alpha))$, $i = 1,2$.

Proof. We shall consider that $\Theta_0 = \Theta$. We take the subscripts i, $i = 1, 2, \ldots, n-1$ which appear in the FDE (5) to be the universal set $S = \{1, 2, \ldots, n-1\}$, and define two subsets $S_e \subseteq S$ and $S_o \subseteq S$ as follows:

$S_e = \{i \in S$, if the number of derivatives of type (ii)-differentiable which is less than the ith derivative is an even number$\}$,

$S_o = \{i \in S$, if the number of derivatives of type (ii)-differentiable which is less than the ith derivative is an odd number$\}$,

where "e" and "o" in the above subscripts refer to an even and odd respectively. Now, we determine the elements of S_1 and S_2 as follows:

For $i \in S_e$, we suppose that $i \in S_1$ if $\Theta_i = \Theta$ and $i \in S_e - S_1$ if $\Theta_i = +$, and for $i \in S_o$, we suppose that $i \in S_2$ if $\Theta_i = \Theta$ and $i \in S_o - S_2$ if $\Theta_i = +$.

It is clear that $S_1 \subseteq S_e$ and $S_2 \subseteq S_o$.

By using equation (1) and definitions of S_e, S_o, S_1 and S_2, then $\Theta_i \mp_{i,f}$ can be written as:

$$\Theta_i \mp_{i,f} = \begin{cases} \Theta, \text{ if } i \in S_1, \\ +, \text{ if } i \in S_e - S_1. \end{cases} \quad (19)$$

or

$$\Theta_i \mp_{i,f} = \begin{cases} -1, \text{ if } i \in S_2, \\ (-1)\,\Theta, \text{ if } i \in S_o - S_2. \end{cases} \quad (20)$$

Now, let m be an even number, then $\mp_{n,f} = +$. Then equation (6) gives:

$$[a_n p^n + \Theta a_0 + \sum_{i=1}^{n-1} \Theta_i \mp_{i,f}\, a_i p^i] L[y(t)] = L[f(t)] + u(p). \quad (21)$$

Since m is an even number, then S_e and S_o can be written as follows:

$$S_e = \{i_{2j} + k,\ k = 1,2,\ldots,i_{2j+1} - i_{2j},\ j = 0,1,\ldots,\frac{m-2}{2}\} \cup \{i_m + k,\ k = 1,2,\ldots,n-1-i_m\},\quad i_0 = 0,$$

and

$$S_o = \{i_{2j-1} + k,\ k = 1,2,\ldots,i_{2j} - i_{2j-1},\ j = 1,2,\ldots,\frac{m}{2}\}.$$

Since $a_i \geq 0$ for $i = 0,1,\ldots,n-1$ and by using (19) and (20), equation (21) becomes:

$$[a_n p^n - a_0 - \sum_{i \in S_1} a_i p^i + \sum_{i \in S_e - S_1} a_i p^i] l[\underline{y}(t;\alpha)] + [-\sum_{i \in S_2} a_i p^i + \sum_{i \in S_o - S_2} a_i p^i] l[\overline{y}(t;\alpha)] = \frac{u_2(p;\alpha)}{h_1(p;\alpha)},$$

$$[-\sum_{i \in S_2} a_i p^i + \sum_{i \in S_o - S_2} a_i p^i] l[\underline{y}(t;\alpha)] + [a_n p^n - a_0 - \sum_{i \in S_1} a_i p^i + \sum_{i \in S_e - S_1} a_i p^i] l[\overline{y}(t;\alpha)] = \frac{u_3(p;\alpha)}{h_2(p;\alpha)},$$

where

$$u_2(p;\alpha) = k_1(p;\alpha) + h_1(p;\alpha)\underline{u}(p;\alpha),$$
$$u_3(p;\alpha) = k_2(p;\alpha) + h_2(p;\alpha)\overline{u}(p;\alpha).$$

By solving the above system, we get:

$$l[\underline{y}(t;\alpha)] = \frac{q_1(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in (s_e - s_1) \cup (s_0 - s_2)} a_i p^i)} \cdot \frac{1}{(a_n p^n - a_0 - \sum_{i \in s_1 \cup (s_0 - s_2)} a_i p^i + \sum_{i \in s_2 \cup (s_e - s_1)} a_i p^i)},$$

$$l[\overline{y}(t;\alpha)] = \frac{q_2(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in (s_e - s_1) \cup (s_0 - s_2)} a_i p^i)} \cdot \frac{1}{(a_n p^n - a_0 - \sum_{i \in s_1 \cup (s_0 - s_2)} a_i p^i + \sum_{i \in s_2 \cup (s_e - s_1)} a_i p^i)},$$

such that

$$\deg(q_i) < 2n + \deg(h_1;\alpha) + \deg(h_2;\alpha),\ i = 1,2.$$

Since

$$(S_e - S_1) \cup (S_0 - S_2) = S - (S_1 \cup S_2),$$
$$S_1 \cup (S_0 - S_2) = (S_1 \cup S_0) - S_2,$$
$$S_2 \cup (S_e - S_1) = (S_2 \cup S_e) - S_1,$$

then, after taking l^{-1}, we get:

$$\underline{y}(t;\alpha) = l^{-1}\Big[\frac{q_1(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in S - (s_1 \cup s_2)} a_i p^i)} \cdot \frac{1}{(a_n p^n - a_0 - \sum_{i \in (s_1 \cup s_0) - s_2} a_i p^i + \sum_{i \in (s_2 \cup s_e) - s_1} a_i p^i)}\Big],$$

$$\overline{y}(t;\alpha) = l^{-1}\Big[\frac{q_2(p;\alpha)}{h_1 h_2 (a_n p^n - a_0 - \sum_{i \in s_1 \cup s_2} a_i p^i + \sum_{i \in S - (s_1 \cup s_2)} a_i p^i)} \cdot \frac{1}{(a_n p^n - a_0 - \sum_{i \in (s_1 \cup s_0) - s_2} a_i p^i + \sum_{i \in (s_2 \cup s_e) - s_1} a_i p^i)}\Big]. \quad (22)$$

If m is an odd number, then $\mp_{n,f} = (-1)\ \Theta$ and S_e and S_o can be written as follows:

$$S_e = \{i_{2j} + k,\ k = 1,2,\ldots,i_{2j+1} - i_{2j},\ j = 0,1,\ldots,\frac{m-1}{2}\},\ i_0 = 0,$$

and

$$S_o = \{i_{2j-1}+k,\ k=1,2,\ldots,i_{2j}-i_{2j-1},\ j=1,2,\ \ldots,\frac{m-1}{2}\} \cup \{i_m+k,\ k=1,2,\ldots,n-1-i_m\}.$$

In a similar manner, we can get:

$$\underline{y}(t;\alpha) = l^{-1}[\frac{q_1(p;\alpha)}{h_1h_2(a_np^n-a_0-\sum_{i\in s_1\cup s_2}a_ip^i+\sum_{i\in S-(s_1\cup s_2)}a_ip^i)} \cdot \frac{1}{(-a_np^n-a_0-\sum_{i\in(s_1\cup s_0)-s_2}a_ip^i+\sum_{i\in(s_2\cup s_e)-s_1}a_ip^i)}],$$

$$\bar{y}(t;\alpha) = l^{-1}[\frac{q_2(p;\alpha)}{h_1h_2(a_np^n-a_0-\sum_{i\in s_1\cup s_2}a_ip^i+\sum_{i\in S-(s_1\cup s_2)}a_ip^i)} \cdot \frac{1}{(-a_np^n-a_0-\sum_{i\in(s_1\cup s_0)-s_2}a_ip^i+\sum_{i\in(s_2\cup s_e)-s_1}a_ip^i)}]. \quad (23)$$

It is clear that (22) and (23) can be written as in(18). We note that if we have $\Theta_0=+$ in Corollary 5, then we must put $+a_0$ instead of $-a_0$ in (18).

Illustrative examples

In this section, we shall introduce several examples by using the proposed method.

Example 1. Consider the following FIVP:

$$y''(t)+2y'(t)+4y(t)=2,\ t\geq 0, \quad (24)$$
$$y(0)=(2\alpha,3-\alpha),\ y'(0)=(3\alpha,4-\alpha).$$

We note that $n=2$, $a_2=1$, $a_1=2$, $a_0=4$. Since $l[\underline{f}(t;\alpha)]=l[\bar{f}(t;\alpha)]=\frac{2}{p}$ then $h_1=h_2=p$.

By Corollary 1, we get

$$(\sum_{i=0}^{2}a_ip^i)(a_0+\sum_{i=1}^{2}\mp_{i,o}a_ip^i)=(p^2+2p+4).[\mp_{2,o}p^2+(\mp_{1,o}2)p+4]. \quad (25)$$

Now, we shall introduce the following case:

Let $y(t)$ be in (i) form and $y'(t)$ be in (ii) form. Since $\mp_{1,o}=+$ and $\mp_{2,o}=-$. Therefore by using (25), equation (15) becomes:

$$\underline{y}(t;\alpha)=l^{-1}[\frac{q_1(p;\alpha)}{p^2[(p+1)^2+3][(p-1)^2-5]}],$$
$$\bar{y}(t;\alpha)=l^{-1}[\frac{q_2(p;\alpha)}{p^2[(p+1)^2+3][(p-1)^2-5]}],$$
$$q_1=-q_1,\ q_2=-q_2.$$

Then, we can write

$$\underline{y}(t;\alpha)=A_1+A_2t+A_3e^{-t}\cos\sqrt{3}t+A_4e^{-t}.\sin\sqrt{3}t+A_5e^{t}\cosh\sqrt{5}t+A_6e^{t}\sinh\sqrt{5}t, \quad (26)$$

$$\bar{y}(t;\alpha)=B_1+B_2t+B_3e^{-t}\cos\sqrt{3}t+B_4e^{-t}.\sin\sqrt{3}t+B_5e^{t}\cosh\sqrt{5}t+B_6e^{t}\sinh\sqrt{5}t.$$

Since

$$y'(t)=(\underline{y}'(t;\alpha),\bar{y}'(t;\alpha)),$$
$$y''(t)=(\bar{y}''(t;\alpha),\underline{y}''(t;\alpha)). \quad (27)$$

Then by (26) and (27), equation (24) and the initial conditions gives $A_1=B_1=\frac{1}{2}$, $A_2=B_2=0$, and the following system:

$$-2B_3-2\sqrt{3}B_4+2A_3+2\sqrt{3}A_4=0,$$
$$2\sqrt{3}B_3-2B_4-2\sqrt{3}A_3+2A_4=0,$$
$$6B_5+2\sqrt{5}B_6+6A_5+2\sqrt{5}A_6=0,$$
$$2\sqrt{5}B_5+6B_6+2\sqrt{5}A_5+6A_6=0,$$
$$A_1+A_3+A_5=2\alpha,$$
$$B_1+B_3+B_5=3-\alpha,$$
$$-A_3+\sqrt{3}A_4+A_5+\sqrt{5}A_6=3\alpha,$$
$$-B_3+\sqrt{3}B_4+B_5+\sqrt{5}B_6=4-\alpha.$$

Solving the above system yields:

$$A_3=B_3=\frac{\alpha}{2}+1,\ A_4=B_4=\sqrt{3}(\frac{\alpha}{2}+1),$$
$$A_5=\frac{3\alpha-3}{2},\ B_5=\frac{3-3\alpha}{2},\ A_6=\frac{\alpha-1}{2\sqrt{5}},$$
$$B_6=\frac{1-\alpha}{2\sqrt{5}}.$$

Substituting values of A_i's and B_i's in (26) and (27) gives:

$$\underline{y}(t;\alpha)=\frac{1}{2}+(\frac{\alpha}{2}+1)e^{-t}\cos\sqrt{3}t+\sqrt{3}(\frac{\alpha}{2}+1)e^{-t}.\sin\sqrt{3}t+\frac{3\alpha-3}{2}e^{t}\cosh\sqrt{5}t+\frac{\alpha-1}{2\sqrt{5}}e^{t}\sinh\sqrt{5}t,$$

$$\bar{y}(t;\alpha)=\frac{1}{2}+(\frac{\alpha}{2}+1)e^{-t}\cos\sqrt{3}t+\sqrt{3}(\frac{\alpha}{2}+1)e^{-t}$$
$$\sin\sqrt{3}t-\frac{3\alpha-3}{2}e^{t}\cosh\sqrt{5}t-\frac{\alpha-1}{2\sqrt{5}}e^{t}\sinh\sqrt{5}t.$$

Example 2. Consider the following FIVP:

$$y'''(t)+2y''(t)\,\Theta\,y'(t)\,\Theta\,2y(t)=1,\ \ t\geq 0, \qquad (28)$$
$$y(0)=y'(0)=y''(0)=(\alpha,2-\alpha).$$

We note that:

$n=3,\ S=\{1,2\},\ a_3=1,\ a_2=2,\ a_1=1,\ a_0=2,$

$\Theta_2=+,\ \Theta_1=\Theta\ ,\ \Theta_0=\Theta.$

$S_1\cup S_2=\{1\},\ S-(S_1\cup S_2)=\{2\}.$

Since $l[\underline{f}(t;\alpha)]=l[\bar{f}(t;\alpha)]=L[1]=\frac{1}{p}$ then $h_1=h_2=p.$

Now, we shall introduce two cases as follows:

Case 1. Let $y(t)$ and $y''(t)$ be in form (i) and $y'(t)$ be in form (ii). Since $m=1$(odd), then $\mp_{3,o}=-$ and

$$S_e=\{i_{2j}+k,\ k=1,2,\ldots,i_{2j+1}-i_{2j},\ j=0,1,\ldots,\frac{m-1}{2}\}$$
$$=\{1\},$$
$$S_o=\{i_{2j-1}+k,\ k=1,2,\ldots,i_{2j}-i_{2j-1},\ j=1,2,\ldots,\frac{m-1}{2}\}\cup\{i_m+k,\ k=1,2,\ldots,n-1-i_m\}$$
$$=\{2\},$$

and we have $S_1=\{1\}$ and $S_2=\phi$. Then, from Corollary 5, we get:

$$\underline{y}(t;\alpha)=l^{-1}[\frac{q_1(p;\alpha)}{p^2(p-1)(p+1)(p+2)^2(p^2+1)}],$$
$$\bar{y}(t;\alpha)=l^{-1}[\frac{q_2(p;\alpha)}{p^2(p-1)(p+1)(p+2)^2(p^2+1)}],$$
$$q_1=-q_1,\ q_2=-q_2.$$

By using the same manner given in example 1, we can find the level sets of the solution $y(t)$ as follows:

$$\underline{y}(t;\alpha)=\frac{-1}{2}+\frac{7}{6}e^{t}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-2t}+(\alpha-1).$$
$$(\cos t+\sin t),$$
$$\bar{y}(t;\alpha)=\frac{-1}{2}+\frac{7}{6}e^{t}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-2t}-(\alpha-1).$$
$$(\cos t+\sin t).$$

Case 2. Let $y'(t)$ be in form (i) and $y(t)$ and $y''(t)$ be in form (ii). Since $m=2$ (even), then $\mp_{3,o}=+$ and

$$S_e=\{i_{2j}+k,\ k=1,2,\ldots,i_{2j+1}-i_{2j},\ j=0,1,\ldots,\frac{m-2}{2}\}\cup\{i_m+k,\ k=1,2,\ldots,n-1-i_m\}$$
$$=\phi,$$
$$S_o=\{i_{2j-1}+k,\ k=1,2,\ldots,i_{2j}-i_{2j-1},\ j=1,2,\ldots,\frac{m}{2}\}$$
$$=\{1,2\},$$

and we have $S_1=\phi$ and $S_2=\{1\}$. Then, from Corollary 5, we get:

$$\underline{y}(t;\alpha)=l^{-1}[\frac{q_1(p;\alpha)}{p^2(p-1)(p+1)(p+2)(p-2)(p^2+1)}],$$
$$\bar{y}(t;\alpha)=l^{-1}[\frac{q_2(p;\alpha)}{p^2(p-1)(p+1)(p+2)(p-2)(p^2+1)}],$$

Therefore, we can get:

$$\underline{y}(t;\alpha)=\frac{-1}{2}+\frac{7}{6}e^{t}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-2t}+(\alpha-1).$$
$$(\cos t-\sin t),$$
$$\bar{y}(t;\alpha)=\frac{-1}{2}+\frac{7}{6}e^{t}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-2t}-(\alpha-1).$$
$$(\cos t-\sin t).$$

Remark. If we have $a_i\leq 0$ for some $i\in\{0,1,\ldots,n-1\}$ in the FDE (5), or any FDE is not in the formulas given in the Corollaries 1, 2, 3, 4 and 5, we can use Theorem 3 to solve this FDE, as in the following example:

Example 3. Consider the following FIVP:

$$y''(t)-y'(t)+2y(t)=t,\qquad t\geq 0, \qquad (29)$$
$$y(0)=(1+\alpha,3-\alpha),\ y'(0)=(1+\alpha,5-3\alpha).$$

We note that:

$n=2,\ a_2=1,\ a_1=-1,\ a_0=2,\ \Theta_1=+,\ \Theta_0=+.$

Since $l[\underline{f}(t;\alpha)] = l[\overline{f}(t;\alpha)] = l[t] = \frac{1}{p^2}$ then

$h_1 = h_2 = p^2$. By Theorem 3, we get:

$$[\mp_{2,f}\, p^2 - (\mp_{1,f})p + 2]L[y(t)] = L[f(t)] + u(p). \qquad (30)$$

Now, we suppose that $y'(t)$ is in form (i) and $y(t)$ is in form (ii). Since $m=1$ (odd), then $\mp_{2,f} = (-1)\ \Theta$. We have $\mp_{1,f} = (-1)\ \Theta$, then equation (30), becomes:

$$[(-1)\ \Theta\ p^2 - (-1)\ \Theta\ p + 2]L[y(t)] = L[f(t)] + u(p),$$

where $u(p)$ is a polynomial of p its degree is less than or equal to 1.

This equation gives:

$$(2p^2 - p^3)l[\underline{y}(t;\alpha)] + p^4 l[\overline{y}(t;\alpha)] = u_2(p),$$

$$p^4 l[\underline{y}(t;\alpha)] + (2p^2 - p^3)l[\overline{y}(t;\alpha)] = u_3(p),$$

where

$$u_2(p) = 1 + p^2\underline{u}(p;\alpha),$$

$$u_3(p) = 1 + p^2\overline{u}(p;\alpha).$$

Solving the above system gives:

$$l[\underline{y}(t;\alpha)] = \frac{q_1(p;\alpha)}{p^4(p+2)(p-1)[(p-\frac{1}{2})^2 + \frac{7}{4}]},$$

$$l[\overline{y}(t;\alpha)] = \frac{q_2(p;\alpha)}{p^4(p+2)(p-1)[(p-\frac{1}{2})^2 + \frac{7}{4}]}.$$

Then, we can get:

$$\underline{y}(t;\alpha) = \frac{1}{4} + \frac{1}{2}t + (\alpha - 1)e^{-2t} + (\frac{13}{4\sqrt{7}} - \frac{2\alpha}{\sqrt{7}}).\, e^{\frac{t}{2}}\sin\frac{\sqrt{7}}{2}t + \frac{7}{4}e^{\frac{t}{2}}\cos\frac{\sqrt{7}}{2}t,$$

$$\overline{y}(t;\alpha) = \frac{1}{4} + \frac{1}{2}t - (\alpha - 1)e^{-2t} + (\frac{13}{4\sqrt{7}} - \frac{2\alpha}{\sqrt{7}}).\, e^{\frac{t}{2}}\sin\frac{\sqrt{7}}{2}t + \frac{7}{4}e^{\frac{t}{2}}\cos\frac{\sqrt{7}}{2}t.$$

Discussion

In this paper we suggest a method for finding analytical solutions for FIVPs of higher orders by using fuzzy Laplace transform by the concept of strongly generalized H-differentiability. This method depends on introducing the subscripts which appear in the fuzzy differential equation as a universal set , and defining other subsets which form a partition of that universal set. Several rules have been given for solving FIVPs directly by obtaining easy algebraic systems. For a FDE, multiple exact solutions can be found by the concept of strongly generalized H-differentiability, but the level sets are not necessarily identical for all the exact solutions even for the same FIVP as follows:

For example 1, the level sets of the solution $y(t)$ for the FIVP (24) are valid for $t \in [0,\infty)$. The graph of the solution of the FIVP (24) has been shown in Fig. 1.

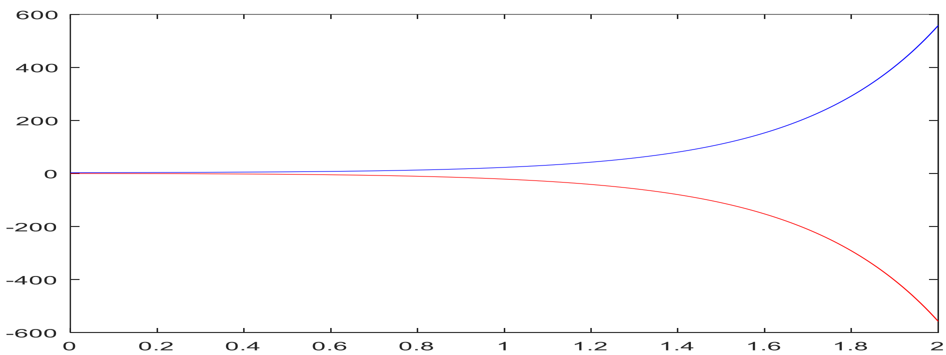

Figure 1. The level sets of the solution $y(t)$ for example 1.

For example 2 case1, the level sets of the solution $y(t)$ of the FIVP (28) are valid for $t \in [0, \frac{3\pi}{4}]$. The graph of the solution in this case is given in Fig. 2a while for example 2 case 2, the level sets of the solution of the FIVP (28) are valid for $t \in [0, \frac{\pi}{4}]$. The graph of the solution in this case has been shown in Fig. 2b.

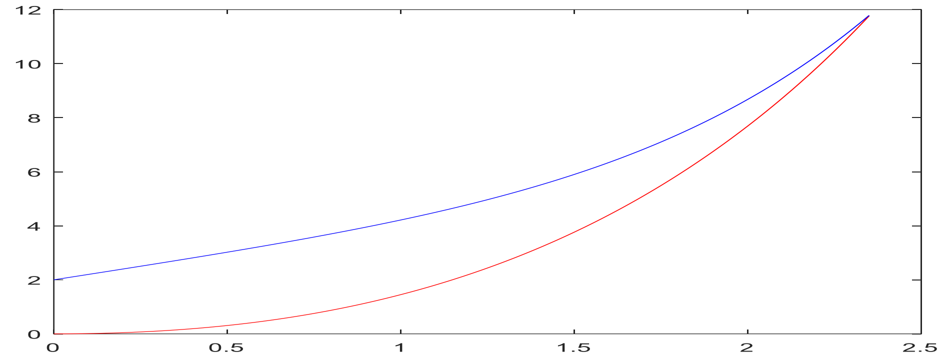

Figure 2a. The level sets of the solution $y(t)$ for example 2 case 1.

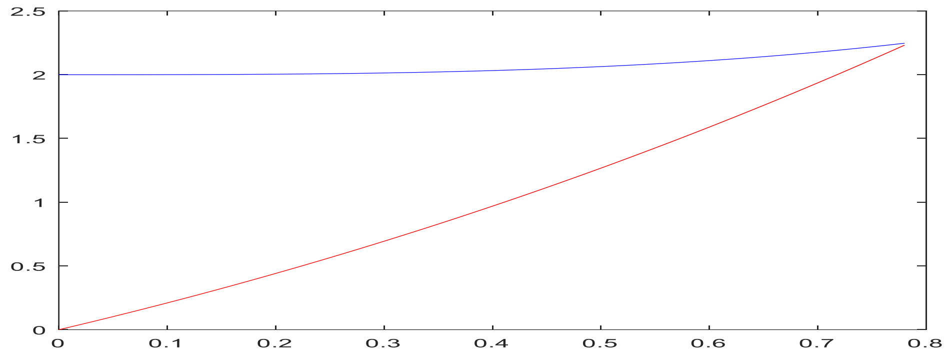

Figure 2b. The level sets of the solution $y(t)$ for example 2 case 2.

The level sets of the solution $y(t)$ for the FIVP (29) are valid for $t \in [0,0.392357254287259]$.

The graph of the solution of the FIVP (29) is given in Fig. 3.

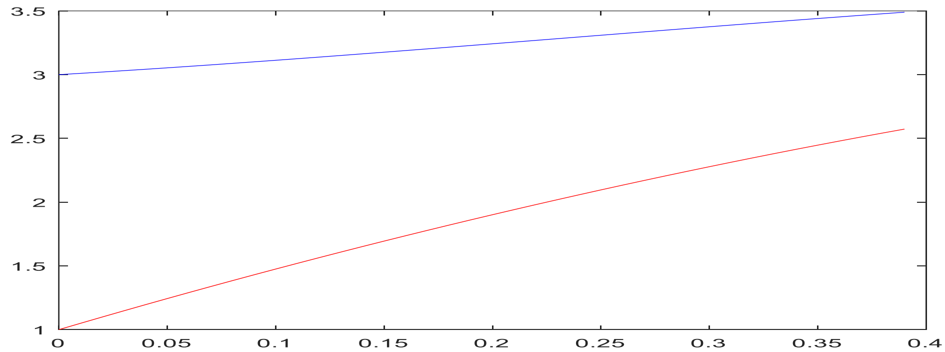

Figure 3. The level sets of the solution $y(t)$ for example 3.

Conclusion:

An analytical method for solving FIVPs is introduced by using fuzzy Laplace transform and the concept of strongly generalized H-differentiability. The suggested method depends on defining a universal set and some certain subsets of that universal set for each FIVP. Also, multiple exact solutions can be found such that the level sets of solutions may be various for the same FIVP.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Kufa.

References:

1. Chang SSL, Zadeh L. On fuzzy mapping and control. IEEE Trans Syst Cybern. 1972; 2: 30-34.

2. Dubios D, Prade H. Towards fuzzy differential calculus. Fuzzy Set Syst. 1982; 8: 225-233.
3. Puri ML, Ralescu D. Differential for fuzzy function. J. Math Anal Appl. 1983; 91: 552-558.
4. Allahviranloo T, Ahmadi MB. Fuzzy Laplace transforms. Soft Comput. 2010; 14: 235-243. doi: 10.1007/s00500-008-0397-6.
5. Salahshour S, Allahviranloo T. Applications of fuzzy Laplace transforms. Soft Comput. 2013; 17: 145-158. doi: 10.1007/s00500-012-0907-4.
6. Mohammad Ali HF. A method for solving n-th order fuzzy linear differential equations by using Laplace transforms. MSc [thesis]. Najaf: University of Kufa; 2013.
7. Haydar AK, Mohammad Ali HF. Generalization of fuzzy Laplace transforms for fuzzy derivatives. J. Kerbala Univ. 2015; 13(2): 120-137.
8. Ahmadi MB, Kiani NA, Mikaeilvand N. Laplace transform formula on fuzzy nth-order derivative and its application in fuzzy ordinary differential equations. Soft Comput. 2014; 18: 2461–2469. doi: 10.1007/s00500-014-1224-x.
9. Rabiei F, Hamid FA, Rashidi MM, Ismail F. Numerical simulation of fuzzy differential equations using general linear method and B-series. Adv. Mech. Eng. 2017; 9(9): 1–16. doi: 10.1177/1687814017715419.
10. Jameel A, Anakira NR, Alomari AK, Hashim I, Shakhatreh MA. Numerical solution of n'th order fuzzy initial value problems by six stages. J. Nonlinear Sci. Appl. 2016; 9: 627-640.
11. Bildik N, Açık A, Deniz S. A new approach to fuzzy differential equations using logarithmic mean. Journal of Fuzzy Set Valued Analysis. 2018; 2018(1): 10-21.
12. Haydar AK, Mohammad Ali HF. Fuzzy Laplace transforms for derivatives of higher orders. Math. Theor Model. 2014; 4(2): 24-31.
13. Haydar AK, Mohammad Ali HF. Solving fuzzy initial value problems of the fifth order via fuzzy Laplace transforms. International Journal of Engineering and Information Systems. 2019; 3(6): 24-31.
14. Haydar AK. Fuzzy Sumudu transform for fuzzy nth-order derivative and solving fuzzy ordinary differential equations. Int. J. Sci. Res. 2015; 4(12): 1372-1378.
15. Chakraverty S, Tapaswini S, Behera D. Fuzzy differential equations and applications for engineers and scientists. Boca Raton, New York: CRC Press, Taylor & Francis Group, LLC; 2017. Chapter 3, Analytical methods of fuzzy differential equations; p. 17-49.
16. Jameel AF, Saaban A, Altaie SA, Anakira NR, Alomari AK, Ahmad N. Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. Int. J. Pure Appl. Math. 2018; 118(1): 49-64.
17. Biswas S, Banerjee S, Roy TK. Solving intuitionistic fuzzy differential equations with linear differential operator by Adomian decomposition method. 3rd Int. IFS Conf., 29 Aug – 1 Sep 2016, Mersin, Turkey. Notes on Intuitionistic Fuzzy Sets. 2016; 22(4): 25–41.
18. Mohammed AH. A Suggested method of Laplace transformations without using any initial condition. Al-Qadisiya University Magazine for Completely Sciences. 2001; 6(2):18-23.

طريقة تحليلية مقترحة لحل مسائل القيمة الابتدائية الخطية الضبابية

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الخلاصة

يهدف هذا البحث الى تعريف مجموعة شاملة عناصرها هي أدلة المعادلة التفاضلية الضبابية (5) عدا العنصرين 0 و n، وايضا مجاميع جزئية من تلك المجموعة الشاملة تم تعريفها وفقا الى شروط معينة. بعد ذلك نستخدم المجموعة الشاملة التي تم إنشاءها مع مجاميعها الجزئية لاقتراح طريقة تحليلية تعمل على تسهيل حل مسائل القيمة الابتدائية الضبابية لأي رتبة باستخدام قابلية الاشتقاق -H المعممة بقوة. ايضا تم ايجاد مجاميع القبول مع الرسوم البيانية لحلول مسائل القيمة الابتدائية الضبابية من الرتب العليا.

الكلمات المفتاحية: المعادلات التفاضلية الضبابية، تحويل لابلاس الضبابي، الاعداد الضبابية، قابلية الاشتقاق -H المعممة.