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A Generalization of t-Practical Numbers

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Abstract:

This paper generalizes and improves the results of Margenstren, by proving that the number of t-practical numbers $n, n \le x, (t \ge 1)$, which is defined by $N(x)$ has a lower bound in terms of t. This bound is more sharper than Mangenstern bound when $t = 1$. Further general results are given for the existence of t-practical numbers, by proving that the interval $\left(x, x + 2\left(\frac{x}{t}\right)^{1/2}\right)$, $t \ge 1$ contains a t-practical for all $x \ge \frac{t}{3}$.

Keywords: Bound for the t-practical numbers, Existence of t-practical numbers in an interval, Practical numbers, t-practical numbers.

Introduction:

The t-practical numbers $n, (t \ge 1)$is a generalization of practical numbers when $t = 1$which is defined in (1). Nicholas Schwab and Lola Thompson (2) adopted the multiplicative function $f(d)$, where $d's$ the divisors of n and referring to each n as f-practical proved series of results related to the distribution of f-practical numbers. P. Leonetti and C.Sanna (3) proved that the most of the binomial coefficients$\binom{n}{k}$, $0 \le k \le n$ are practical numbers and

$$f(n) < n^{1-(\log 2-\delta)/\log\log n}$$

when$f(n)$ denotes the number of coefficients $\binom{n}{k}$ that are not practical for all $n \in [3, x]$, $x > 3, 0 < \delta < \log 2$. Further results proved by Wang. L-Y and Sun. Z-W (4) showing that $n^2 + bn + c$ is practical for some integer $n > 1, b \ge 0$and $c > 0$. They proved that there are infinitely many practical numbers of the form $q^4 + 2$ with q practical number.

Shapiro (5), Saias (6) prove that

$$p(x) \ge \frac{cx}{(\log x)}$$

which is analogous with the asymptotic behavior of primes. In (7) Weingartner gave non explicit bound by proving that

$$p(x) = \frac{cx}{\log x}\left[1 + O\left(\frac{\log\log x}{\log x}\right)\right].$$

Margenstren (1) noted that the number of practical number $n, n \le x$ is

$$p(x) \ge \frac{Ax}{\exp.\left[\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right]},$$

$$A = \frac{2^{5/2}}{5}$$

In this paper the bound above is generalized and improved in Theorem (3), for all $t \ge 1$.

Finally, in this paper the bound given by Theorem (3) sharper than the bound mentioned above. Further general result proved for the t- practical number is by showing that there exists a t-practical number in an interval $\left(x, x + 2\left(\frac{x}{t}\right)^{1/2}\right), t \ge 1$, where the case $t = 1$ represent the result of (5).

Preliminary Results and Definitions:

Definition (1) (1): Let $n \ge 1$. Then n be called a practical number if for every integer $m, 1 \le m < n$ having the form

$$m = \sum_{d|n} c_d d, \quad c_d = 0, 1$$

Definition (2): The number $n, (n \ge 1)$ is called a t-practical number if every integer $m, 1 \le m \le tn, (t \ge 1)$ is of the form

$$m = \sum_{d|n} c_d d, \quad 1 \le c_d \le t.$$

Definition (3): Define $N(x)$ to be the number of t-practical numbers $n, n \leq x$.

The following lemmas will be required

Lemma (1): For any t-practical,$t \geq 1$,

$$(t+1)n \leq t\sigma(n) + 1 \quad \dots (1)$$

where some $\sigma(n)$ is the sum of positive divisors of n.

Proof: If $n = 1$, then (1) follows. Let $n \geq 2$ and $d_1, d_2, \dots, d_r$ and all positive divisors of n, then from Theorem (1)

$$d_r < t\sigma_{r-1} + 1,$$
$$d_r + td_r \leq t\sigma_{r-1} + td_r + 1$$
$$(t+1)d_r \leq t\sigma_r + 1$$

where$d_r = n, \sigma_r = d_1 + d_2 + \dots + d_r$

$$(t+1)n \leq t\sigma(n) + 1$$

Lemma (2): Let $\ell \in \mathbb{N}$. Then $(t+1)^{\ell}, (t \geq 1)$ is a t-practical number.

Proof: Let $m \in \mathbb{N}, 1 \leq m \leq (t+1)^{\ell}$, then we can write

$$m = \sum_{i=1}^{\ell-1} c(t+1)^i, \qquad (1 \leq c_i \leq t)$$

since$(t+1)^i$ are distinct divisors of $(t+1)^{\ell}$, then $(t+1)^{\ell}$ is a t-practical.

Lemma (3): The number $s^{\ell}, 1 \leq s \leq t+1$ and $\ell \in \mathbb{N}$ is t-practical.

Proof: For any integer $m, 1 \leq m < s^{\ell}$, we can write

$$m = \sum_{i=1}^{\ell-1} c_i s^i, \qquad 1 \leq c_i \leq t.$$

thens^i are distinct divisors of s^{ℓ} and therefore s^{ℓ} is t-practical.

The following theorems are required.

Theorem (1) (4): Let $p_1 < p_2 < \dots < p_k$ be distinct primes and let $a_1, a_2, \dots a_k \in \mathbb{N}$. Then $n = {p_1}^{a_1}.{p_2}^{a_2} \dots {p_k}^{a_k}$is practical number if and only if $p_1 = 2$ and

$$p_j < \sigma\left({p_1}^{a_1}{p_2}^{a_2} \dots p_{j-1}\right) + 1, \qquad (1 \leq j \leq k),$$

where$\sigma(n)$ is the sum of all positive divisors of n.

Robinson (8) prove the following

Theorem (2) (8): Let $d_1, d_2, \dots, d_r$ be the positive divisors of . Then n is a practical number if and only if

$$d_{k+1} \leq \sigma_k + 1, \qquad (1 \leq k \leq r-1)$$

where$\sigma_k = d_1 + d_2 + \dots + d_k$.

A generalization of Robinson's results (8) given by the following:

Theorem (3):n is t-practical if and only if

$$d_{k+1} \leq t\sigma_k + 1, \qquad (0 \leq k \leq r-1)$$

Proof: Suppose that

$$d_{k+1} \leq t\sigma_k + 1.$$

In fact, by proving that for any k, $0 \leq k \leq (r-1)$every integer m such that $t\sigma_k < m \leq t\sigma_{k+1}$can be express as:

$$\sum_{i=1}^{k+1} c_i d_i, \qquad 0 \leq c_i \leq t \qquad \dots (1)$$

where$t\sigma_0 = 0$ and $t\sigma_r = t(d_1 + d_2 + \dots + d_r), d_r = n$, this shows that every integer $m, 1 \leq m \leq n$ has the required representation for n to be t-practical number. The proof of (1) is by induction on k. If $= 0$, then (1) implies that every integer $m, 0 < m \leq t$is of the form $c_1 d_1, 0 \leq c_1 \leq t$, where $d_1 = 1$ and hence m is t-practical. Assume that (1) is true for $k = \mathrm{K}$, with $\mathrm{K} < r-1$, then we will show that (1) is true for$k = \mathrm{K} + 1$. Therefore, let

$$t\sigma_{\mathrm{K}+1} < m \leq \sigma_{\mathrm{K}+2}$$

since$t\sigma_{\mathrm{K}+2} < m \leq t\sigma_{\mathrm{K}+1} + 1$, then

$$d_{\mathrm{K}+2} \leq m \leq t\sigma_{K+2},$$

and if$m < (t+1)d_{\mathrm{K}+2}, m$ can be written as

$$m = ud_{\mathrm{K}+2} + v$$

with$0 \leq v \leq d_{\mathrm{K}+2}$ and $0 \leq u \leq t$. Therefore,

$$0 \leq v \leq d_{\mathrm{K}+2} \leq t\sigma_{\mathrm{K}+1} + 1$$
$$v \leq t\sigma_{\mathrm{K}+1}$$

where by induction hypothesis,

$$v = \sum_{i=1}^{\mathrm{K}+1} c_i d_i, \qquad 0 \leq c_i \leq t$$

and

$$m = ud_{\mathrm{K}+2} + \sum_{i=1}^{\mathrm{K}+1} c_i d_i$$
$$\therefore m = \sum_{i=1}^{\mathrm{K}+2} c_i d_i$$

with$c_{\mathrm{K}+2} = u$ which is the required form. If $m \geq (t+1)d_{\mathrm{K}+2}$, then

$$d_{\mathrm{K}+2} \leq m - td_{\mathrm{K}+2} \leq t\sigma_{\mathrm{K}+2} = t\sigma_{\mathrm{K}+1}$$

Hence, induction hypothesis, m is t-practical and

$$m = \sum_{i=1}^{\mathrm{K}+2} c_i d_i, \qquad 0 \leq c_i \leq t.$$

Conversely, if n t-practical, then for any k, $0 \leq k \leq (r-1)$, it follows that

$$d_{\mathrm{K}+1} - 1 = \sum_{i=1}^{r} c_i d_i, \qquad 0 \leq c_i \leq t$$
$$d_{\mathrm{K}+1} - 1 = \sum_{i=1}^{\mathrm{K}} c_i d_i \leq t \sum_{i=1}^{\mathrm{K}} d_i = t\sigma_{\mathrm{K}}$$
$$\therefore d_{\mathrm{K}+1} \leq t\sigma_{\mathrm{K}} + 1.$$

The Number of t-Practical Numbers:

Margenstren (1) proved the following.

Theorem (4) (1): The number of practical numbers $n, n \leq x$ and x large real number is at most,

$$\frac{Ax}{\exp.[\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x]}$$

where $= \frac{2^{5/2}}{5}$.

a generalization of Theorem (4) (1) for the t-practical is given by the following.

Theorem (5): Let x be a large real number. Then

$$N(x) \geq (2\log 2)^{1/2}.\frac{x}{\exp.[\frac{1}{2\log 2}(\log\log x)^2 + (\log\log x) + \log(t+1)]}$$

Proof: Let $x \geq (t+1)^3$. Then we can take x such that

$$(t+1)^{2^r+1} \leq x < (t+1)^{2^{r+1}+1}, \quad \dots (1)$$

with$r \geq 1$ and $t \geq 1$. Let $y = \frac{x}{t+1}$, then by writing (1) as

$$(t+1)^{2^r} \leq y \leq (t+1)^{2^{r+1}}, \quad \dots (2)$$

from Theorem (1), if $n = p_1, \dots p_r$, where $p_1.p_2 \dots p_r$ are distinct primes then

$$p_{j+1} \leq t\sigma(p_1 p_2 \dots p_j) + 1, \quad 1 \leq j \leq r-1 \quad \dots (3)$$

where$1 < p_1 < t+1 < p_2 < \cdots < p_r$ is a t-practical number. Therefore by writing

$$\left.\begin{array}{c} 1 < p_1 < t+1 \\ t+1 < p_2 < 2(t+1) \\ (t+1)^2 < p_3 < 2(t+1)^2 \\ \vdots \\ (t+1)^{2^r} < p_r < 2(t+1)^{2^r} \end{array}\right\} \dots (4)$$

from (4), since $1 < p_1 < t+1$, then by Lemma (3) p_1 is a t-practical and by Lemma (1), it follows that

$$t\sigma(p_1) + 1 > (t+1)p_1, \quad p_1 \geq 2$$

then$p_1.p_2$ is also t-practical. Now, as induction hypothesis, assuming that $p_1.p_2 \dots p_i, (2 \leq i \leq r)$is a t-practical and

$$(t+1)^{2^k} \leq p_i \leq 2(t+1)^k, \quad 0 \leq k \leq r$$

then from the L.H.S of (4), it follows that

$$p_1.p_2 \dots p_i \geq 2(t+1)^{1+2+2^2+\cdots+2^k} = 2(t+1)^{2^{k+1}-1}$$

therefore,

$$p_1.p_2 \dots p_i \geq 2(t+1)^{2^{k+1}-1} \quad \dots (5)$$

and by Lemma (1)

$$t\sigma(p_1.p_2 \dots p_i) + 1 > p_1.p_2 \dots p_i$$

using (5) to get

$$t\sigma(p_1.p_2 \dots p_i) \geq 2(t+1)^{2^{k+1}} > p_{i+1}.$$

Hence, $p_1.p_2 \dots p_{i+1}$ is a t-practical. From the R.H.S of (4) we have also

$$n = p_1.p_2 \dots p_r \leq 2^{r-1}.(t+1)^{2^{r+1}}$$

$$\therefore n < (t+1)^{2^{r+1}+1}$$

Since the number of p_1, $1 < p_1 \leq (t+1)$ is estimated to be at least

$$\frac{(t+1)}{\log(t+1)}$$

and the number of $p_i, 2 \leq i \leq r$ is at least

$$\frac{(t+1)^{2^m}}{2^m.\log(t+1)}, \quad m = 0, 1, \dots, r.$$

Then,

$$N((t+1)^{2^{r+1}+1}) \geq \frac{(t+1).(t+1).(t+1)^2 \dots (t+1)^{2^r}}{\log(t+1).\log(t+1).2^2\log(t+1)\dots 2^r\log(t+1)}$$

That is

$$N((t+1)^{2^{r+1}+1}) \geq \frac{(t+1)^{2^{r+1}}}{2^{r(r+1)/2}.[\log(t+1)]^r} \quad \dots (6)$$

from inequality (2),

$$\log(t+1) \leq \frac{\log y}{2^r}$$

and this implies that

$$\frac{1}{[\log(t+1)]^r} \geq \frac{2^{r^2}}{(\log y)^r} \quad \dots (7)$$

where from (2)

$$y < (t+1)^{2^{r+1}} \quad \dots (8)$$

using (7), (8) in (6) to get

$$N((t+1)^{2^{r+1}+1}) \geq 2^{r(r-1)/2}.\frac{y}{(\log y)^r} \dots (9)$$

by using (2), then

$$2^r\log(t+1) \leq \log y < 2^r\log(t+1)^2 \dots (10)$$

since$x = (t+1)y$, then

$$\log y \leq \log x \leq 2\log y \quad \dots (11)$$

and both (9), (11) implies that the number of t-practical numbers $n, n \leq x$ is

$$N(x) \geq 2^{r(r-1)/2}.\frac{x}{(t+1)(\log x)^r} \quad \dots (12)$$

from (10) above, by taking

$$2^r = \frac{\log y}{\log 2}$$

and from R.H.S of (11)

$$2^r \geq \frac{\log x}{2\log 2} \quad \dots (13)$$

using (13) in (12), then

$$N(x) \geq (2\log 2)^{1/2}.\frac{x}{(2\log 2)^{r/2}.(t+1)(\log x)^{(r+1)/2}} \dots (14)$$

from the L.H.S of (10),

$$2^r \leq \frac{\log y}{\log(t+1)}$$

where$\log(t+1) \geq \log 2$, using L.H.S of inequality (11), then

$$2^r \leq \frac{\log y}{\log 2} \leq \frac{\log x}{\log 2},$$

$$\therefore r \leq \frac{1}{\log 2}[\log\log x - \log\log 2] \quad \dots (15)$$

using (15), and write

$$2^{r/2} = \exp.[\frac{r}{2}\log 2] \leq \exp[\frac{1}{2}\log\log x - \frac{1}{2}\log\log 2]$$

$$(\log 2)^{r/2} = \exp.[\frac{r}{2}\log\log 2] \leq \exp[\frac{\log\log 2}{2\log 2}.\log\log x - \frac{(\log\log 2)^2}{2\log 2}]$$

and since,

$$(\log x)^{r+1/2} = \exp.[\frac{r+1}{2}\log\log x]$$

then by (15),

$$(\log x)^{r+1/2} = \exp.[\frac{r+1}{2}\log\log x]$$
$$\leq \exp.[\frac{1}{2\log 2}(\log\log x)^2 - \frac{\log\log 2}{2\log 2}\log\log x + \frac{1}{2}\log\log x]$$
$$\therefore (2\log 2)^{r/2}.(\log x)^{r+1/2} \leq \exp.[\frac{1}{2\log 2}(\log\log x)^2 + \log\log x]$$
$$\therefore \frac{1}{(2\log 2)^{r/2}.(\log x)^{r+1/2}} \geq \frac{1}{\exp.[\frac{1}{2\log 2}.(\log\log x)^2 + \log\log x]} \quad ...(16)$$

Insert (16) in (14), then the number of t-practical numbers $n, n \leq x$ is $N(x)$ and

$$N(x) \geq (2\log 2)^{1/2}.\frac{x}{\exp.[\frac{1}{2\log 2}.(\log\log x)^2 + \log\log x + \log(t+1)]}$$

This end the proof. This bound is more sharper than Margenstren result in (1) even when $t = 1$, for $x \geq 64$.

t-Practical Numbers in an Interval

In (5) Hausman and Shapiro proved the following:

Theorem (6) (5): For all $x \geq \frac{1}{3}$ the interval$(x, x + 2x^{1/2})$ contains a practical number.

A generalization of Theorem (6) (5) given by the following:

Theorem (7): The interval $\left(x, x + 2\left(\frac{x}{t}\right)^{1/2}\right), (t \geq 1)$ contains a t-practical number for all $x \geq \frac{t}{3}$.

Proof: Considering $x > 4t, t \geq 1$ and $\frac{t}{3} \leq x \leq 4t$. For if $x > 4t$ and

$$2^a < (\frac{x}{t})^{1/2} \leq 2^{a+1}, \quad a \in \mathbb{N} \qquad \dots (1)$$

Then the interval $\left(x, x + 2\left(\frac{x}{t}\right)^{1/2}\right)$ is of the length $\left(\frac{x}{t}\right)^{1/2}$, therefore it contains |at least one multiple of 2^a such as $2^a m$ on the other word

$$x < 2^a m < x + (\frac{x}{t})^{1/2} \qquad \dots (2)$$

Thus, either m or $m+1$ is even integer, where since

$$2^a(m+1) = 2^a m + 2^a < x + 2\left(\frac{x}{t}\right)^{1/2},$$

then

$$x < 2^a m < 2^a(m+1) < x + 2\left(\frac{x}{t}\right)^{1/2}.$$

Now, by showing that one of these integers $2^a m, 2^a(m+1)$ is a t-practical, suppose that neither is a t-practical number and, without loss of generality, let m be an even integer. Then since $2^a m$ is not t-practical number.

then there exists a prime p_1 of $2^a m$ such thats

$$p_i > t\sigma(n_{i-1}) + 1 \qquad \dots (3)$$

with$2^a m = {p_1}^{k_1}.{p_2}^{k_2} \dots, {p_l}^{k_l}, p_1 = 2, n_{i-1} = {p_1}^{k_1}.{p_2}^{k_2} \dots {p_{i-1}}^{k_{i-1}}, i \geq 2$ and $n_0 = 1$. Particularly, (3) implies that

$$p_i > t + 1$$

andp_i is odd prime for $i \geq 2$. Therefore $2^{a+1}|n_{i-1}$ and

$$p_i > t\sigma(2^{a+1}) + 1 = t(2^{a+2} - 1) + 1,$$
$$p_i \geq t\sigma(2^{a+2} - 1) + 2,$$
$$2^a m \geq 2.2^{2(a+1)}.t - 2^{a+1}.t + 2^{a+2}, \qquad \dots (4)$$

and from (2),

$$x + \left(\frac{x}{t}\right)^{1/2} \geq 2x - 2^{a+1}.t + 2\left(\frac{x}{t}\right)^{1/2}$$
$$2^{a+1}.t \geq x + (\frac{x}{t})^{1/2} \qquad \dots (5)$$

It follows from the L.H.S of (1), that

$$2x^{1/2}.t > 2^{a+1}.t$$

therefore (5) implies that

$$2x^{1/2}.t^{1/2} > x + \left(\frac{x}{t}\right)^{1/2}$$
$$2t^{1/2} - (\frac{1}{t})^{1/2} > x^{1/2}$$
$$2t^{1/2} > x^{1/2}$$

and this implies that $x < 4t$ a contradiction. A similar procedure can be followed for $2^a(m+1)$ which also leads to a contradiction.

If $t < x \leq 4t$, then

$$2 \leq 2\left(\frac{x}{t}\right)^{1/2} \leq 4$$

and

$$x + 2 \leq x + 2\left(\frac{x}{t}\right)^{1/2} \leq 4 + x$$
$$1 < t + 1 < x + 2 \leq x + 2(\frac{x}{t})^{1/2} \leq 4(t+1) \qquad \dots (6)$$

By Lemma (2), the inequality (6) shows that the interval

$$\left(x, x + 2\left(\frac{x}{t}\right)^{1/2}\right)$$

contains the integer $(t+1)$ which is a t-practical. If $\frac{t}{3} \leq x < t$, then

$$2\left(\frac{1}{3}\right)^{1/2} < 2\left(\frac{x}{t}\right)^{1/2} < 2$$

i.e

$$x + 2\left(\frac{1}{3}\right)^{1/2} < x + 2\left(\frac{x}{t}\right)^{1/2} < x + 2 < t + 2,$$

since$x > \frac{t}{2}$, then

$$1 < \frac{t}{3} + 1 < x + 2\left(\frac{x}{t}\right)^{1/2} < t + 2$$

and

$$1 < \frac{t}{3} + 1 \leq t + 1.$$

Therefore, by Lemma (3) $m = (\frac{t}{3} + 1)$ is a t-practical number which end the proof.

Theorem (6) (5) follows immediately when $t = 1$.

Corollary: The interval$(x^2, (x + (\frac{1}{t})^2)$,$t \geq 1, x \geq 1$ contains a t-practical number.

Proof: From Theorem (7) we have $(x, x + 2(\frac{x}{t})^{1/2})$ that contains a t-practical number for $x \geq \frac{t}{3}$, then this implies $(x^2,\ x^2 + 2((\frac{x}{t})^2)^{1/2})$ that also contains a t-practical number for $x \geq 1$. Since

$$\left(x^2, x^2 + 2\left(\frac{x}{t}\right)^{1/2}\right) \subseteq (x^2, x^2 + 2\frac{x}{t} + \frac{1}{t^2}$$

$$= (x^2, \left(x + \frac{1}{t}\right)^2)$$

this shows that there is a t-practical number between x^2 and $(x + \frac{1}{t})^2$.

Conclusion:

This work gives a new lower bound for all t-practical numbers to be at most

$$(2\log 2)^{1/2}.\frac{x}{exp.[\frac{1}{2\log 2}(\log\log x)^2 + \log\log x + \log(t+1)]}$$

and this lower bound is the sharper bound in which one can determine these numbers $n, n \leq x$ for any real number x. Further results in this paper show that the interval $(x, x + 2\left(\frac{x}{t}\right)^{1/2})$, containing t-practical numbers for all $x \geq \frac{t}{3}$, $(t \geq 1)$.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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تعميم في الاعداد العملية ذات التكرار t

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الخلاصة :

في هذا البحث تم تعميم وتحسين نائجMargenstren باثبات ان N(x) التي تمثل عدد الاعداد العملية $n \leq x, n$ ذات التكرارt, $(t \geq 1)$ له الحد الادنى

$$N(x) \geq (2\log 2)^{1/2}.\frac{x}{exp.[\frac{1}{2\log 2}(\log\log x)^2 + \log\log x + \log(t+1)]}$$

وهذا الحد الادنى يعتبر افضل من الحد المعطى من قبل Margenstren عندما t=1.

وعلاوة على ذلك تم برهنت وجود اعداد عملية بتكرار t, $(t \geq 1)$ في الفترة $(x, x + 2(\frac{x}{t})^{1/2}$لعدد حقيقي x عندما $x \geq \frac{t}{3}$.

الكلمات المفتاحية: الحد الادني لعدد الاعداد العملية ذات تكرار t، الاعداد العملية، الاعداد العملية بتكرار t، وجود الاعداد العملية بتكرار t ض

من فترة.