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Analytical Solutions for Advanced Functional Differential Equations with Discontinuous Forcing Terms and Studying Their Dynamical Properties

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Abstract:

This paper aims to find new analytical closed-forms to the solutions of the nonhomogeneous functional differential equations of the n^{th} order with finite and constants delays and various initial delay conditions in terms of elementary functions using Laplace transform method. As well as, the definition of dynamical systems for ordinary differential equations is used to introduce the definition of dynamical systems for delay differential equations which contain multiple delays with a discussion of their dynamical properties: The exponential stability and strong stability.

Keywords: Laplace transform , Linear functional-differential equations, Stability theory.

Introduction:

In electrical engineering, in which electronic components are often controlled by on-off switches, discontinuous forcing functions are the norm. Also, in physics, forces often change suddenly and are best described by discontinuous functions. A useful way for representing discontinuous functions is in terms of the unit step function (1). It is well known that the step function is discontinuous at the origin, but that is not necessary in the signal theory. The step function is an important tool for testing and introducing other signals such as multiplying shifted step functions by other different shifted step functions (2).

A bat hitting a ball and two billiard balls colliding are impulsive forces occur at nearly an instant of time. To deal with these types of forces mathematically, the impulse function (Dirac delta function or unit impulse) is defined (1). In engineering, the idea of an action occurring at a point is dealt with. Whether it is a force at a point in space or a signal at a point in time, it becomes a useful way to develop a quantitative definition for this phenomena. This leads us to the notion of a unit impulse, probably the second most essential function, in systems and signals (2). The delay differential equations (DDEs) are usually used in many mathematical, physical and engineering models. Several methods have been used to solve some of them by using numerical methods, others by using analytical methods. Many researchers used the Lambert W function to obtain solutions of the DDEs (3-6). Recently, Abdullah and et al. found analytical solutions of retarded dynamical systems of the third order and of the n^{th} order by using Lambert W function and a discussion of their stability in their two papers (7-8) respectively. Abdullah and et al. found approximate characteristic roots for DDEs with multiple delays via the method of spectral tau (9).

A few researchers have worked on finding analytical solutions without using Lambert W function such as Pospíšil and Jaroš who used the unilateral Laplace transform to introduce a closed-form formula for a solution of a system of nonhomogeneous linear delay differential equations with a finite number of constant delays (10). In this paper, many rules for finding analytical solutions of nonhomogeneous DDEs are obtained using Laplace transform without using the nonelementary Lambert W function. In addition, some of dynamical systems are constructed with a discussion of their stability.

Basic Concepts

Definition 1 (11) A dynamical system is a map $\sigma : G \times X \to X$, where X is an open set of

Euclidean space and writing $\sigma(t, \mathrm{x}) = \sigma_{\mathrm{t}}(x)$, the map $\sigma_t : \mathrm{X} \to \mathrm{X}$ satisfies:

(1) $\sigma_0 : \mathrm{X} \to \mathrm{X}$ is the identity; that is $\sigma_0(x) = x$ for all x in X.

(2) The composition $\sigma_t \circ \sigma_s = \sigma_{t+s}$ for all t, s in G.

In this work $\mathrm{G} = \mathrm{R}^+ \cup \{0\}$ and $X = R$.

In the case $G = R$ the dynamical system is called flow and in the case $G = R^+ \cup \{0\}$ the dynamical system is called semi flow.

Definition 2 (12) The semi group $(T_t)_{t\geq 0}$ is called strongly stable in X iff $\forall x \in X$, $\lim_{t\to\infty} T_t x = 0$ in X.

Definition 3 (12) The semi group $(T_t)_{t\geq 0}$ is exponentially stable iff their exists $D < \infty$ and $\omega > 0$ such that $\|T_t\| \leq D\ \mathrm{e}^{-\omega t}$, for $t \geq 0$.

Proposed Rules for Nonhomogeneous Delay Differential Equations of Advanced Type With Multiple Delays by Laplace Transform Method

In this section, many rules for solving nonhomogeneous DDEs are driven with different initial delay conditions and multiple delays T_{ij}, $i = 0,1,...,n$, $j = 1,2,...,m$. Let T_{ij}'s, be constant delays $\forall i, j$ such that "i" denotes the number of derivative of y and "j" denotes the j^{th} delay of $y^{(i)}$ and T_{ij}'s are arranged in increasing order. Consider that $T_{i0} = 0, \forall i$, and the coefficients a_{ij}'s are constants.

Recall that Laplace transform for the functions $y^{(i)}(t-T), i = 0,1,2,...,n$, with the initial delay condition:

$y(t) = \varphi(t),\ -T \leq t \leq 0$,

where T is a constant delay $T > 0$, is:

$$L\{y^{(i)}(t-T)\} = \int_0^\infty y^{(i)}(t-T)e^{-st}dt.$$

Using the assumption $z = t - T$, yields:

$$L[y^{(i)}(t)] = \int_{-T}^\infty y^{(i)}(z)e^{-s(z+T)}dz$$

$$= e^{-sT}\int_{-T}^0 y^{(i)}(t)e^{-st}dt + e^{-sT}\int_0^\infty y^{(i)}(t)e^{-st}dt$$

$$= e^{-sT}\int_{-T}^0 \varphi^{(i)}(t)e^{-st}dt + e^{-sT}(s^iY(s) - \sum_{k=0}^{i-1} s^{i-(k+1)}y^{(k)}(0)).$$

Theorem 3.1: Consider the DDE:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}y^{(i)}(t-T_{ij}) = g(t),\ \ T_{ij} > 0, \qquad (1)$$

with the initial delay condition:

$y(t) = \varphi(t)$, for $-T \leq t \leq 0$, and $T = \max\{T_{ij}\}$, $i = 0, 1, 2, ..., n$, $j = 0, 1, 2, ..., m$, and $L\{g(t)\} = \dfrac{k(s)}{h(s)}$ is the Laplace transform of $g(t)$, $h(s) \neq 0$. Then, the solution of the equation (1) is:

$$y(t) = L^{-1}[\{-h(s)\sum_{i=0}^{n}\sum_{j=1}^{m} a_{ij}e^{-sT_{ij}}\int_{-T_{ij}}^0 \varphi^{(i)}(t)e^{-st}dt +$$

$$h(s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1} a_{ij}s^{i-(k+1)}e^{-sT_{ij}}y^{(k)}(0) + k(s)\}$$

$$\cdot\frac{1}{h(s)\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}s^ie^{-sT_{ij}}}].$$

Proof: Equation (1) can be written as:

$$a_{00}y(t) + \sum_{i=1}^{n} a_{i0}y^{(i)}(t) + \sum_{j=1}^{m} a_{0j}y(t-T_{0j})$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}y^{(i)}(t-T_{ij}) = g(t).$$

Taking the Laplace transform to the both sides, gives:

$$a_{00}Y(s) + \sum_{i=1}^{n} a_{i0}[s^iY(s) - \sum_{k=0}^{i-1} s^{i-(k+1)}y^{(k)}(0)]$$

$$+ \sum_{j=1}^{m} a_{0j}[e^{-sT_{0j}}\int_{-T_{0j}}^0 \varphi(t)e^{-st}dt + e^{-sT_{0j}}Y(s)]$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}[e^{-sT_{ij}}\int_{-T_{ij}}^0 \varphi^{(i)}(t)e^{-st}dt + e^{-sT_{ij}}(s^iY(s)$$

$$- \sum_{k=0}^{i-1} s^{i-(k+1)}y^{(k)}(0))] = \frac{k(s)}{h(s)}.$$

Then:

$$h(s)(\sum_{i=0}^{n}\sum_{j=0}^{m} a_{ij}s^ie^{-sT_{ij}})\ Y\ (s) = -h(s)\sum_{i=0}^{n}\sum_{j=1}^{m} a_{ij}.$$

$$e^{-sT_{ij}}\int_{-T_{ij}}^0 \varphi^{(i)}(t)e^{-st}dt + h(s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1} a_{ij}s^{i-(k+1)}.$$

$$e^{-sT_{ij}}y^{(k)}(0) + k(s).$$

Therefore, the solution is:

$$y(t)=L^{-1}[\{-h(s)\sum_{i=0}^{n}\sum_{j=1}^{m}a_{ij}e^{-sT_{ij}}\int_{-T_{ij}}^{0}\varphi^{(i)}(t).e^{-st}dt+h(s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1}a_{ij}s^{i-(k+1)}e^{-sT_{ij}}y^{(k)}(0)+k(s)\}.\frac{1}{h(s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}}}].$$

Corollary 3.2: Let $-T\le t\le 0$ and $T=\max\{T_{ij}\}$, $i=0,1,2,\ldots,n$, $j=0,1,2,\cdots,m$. Then:

1. If $\varphi(t)=c$, the solution of the DDE (1) is:

$$y(t)=L^{-1}[\{-c\,h(s)(\sum_{j=1}^{m}a_{0j}-\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}})+c\,s\,h(s)\sum_{i=1}^{n}\sum_{j=0}^{m}a_{ij}s^{i-1}e^{-sT_{ij}}+s\,k(s)\}\cdot\frac{1}{s\,h(s)(\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}})}] \quad (2)$$

where c is constant.

2. If $\varphi(t)=t$, the solution of the DDE (1) is:

$$y(t)=L^{-1}[\{-h(s)(\sum_{j=1}^{m}a_{0j}-\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}})+s\,h(s)(\sum_{j=1}^{m}a_{0j}T_{0j}-\sum_{j=1}^{m}a_{1j}+\sum_{j=1}^{m}a_{1j}e^{-sT_{1j}})+s^{2}h(s)\sum_{i=2}^{n}\sum_{j=0}^{m}a_{ij}s^{i-2}e^{-sT_{ij}}+s^{2}\,k(s)\}\cdot\frac{1}{s^{2}\,h(s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}}}]. \quad (3)$$

3. If $\varphi(t)=t+T$, the solution of the delay differential equation (1) is:

$$y(t)=L^{-1}[\{-h(s)(\sum_{j=1}^{m}a_{0j}-\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}})+s\,h(s).(T\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}}-\sum_{j=1}^{m}a_{0j}(T-T_{0j})-\sum_{j=1}^{m}a_{1j}+\sum_{j=1}^{m}a_{1j}e^{-sT_{1j}})+s^{2}\,h(s)(T\sum_{i=1}^{n}\sum_{j=0}^{m}a_{ij}s^{i-1}e^{-sT_{ij}}+\sum_{i=2}^{n}\sum_{j=0}^{m}a_{ij}s^{i-2}e^{-sT_{ij}})+s^{2}\,k(s)\}.\frac{1}{s^{2}\,h(s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}}}]. \quad (4)$$

4. If $\varphi(t)=e^{ct}$, the solution of the delay differential equation (1) is:

$$y(t)=L^{-1}[\{-h(s)\sum_{i=0}^{n}\sum_{j=1}^{m}a_{ij}\,c^{\,i}(e^{-sT_{ij}}-e^{-cT_{ij}})+h(s).(c\text{-}s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1}a_{ij}s^{i-(k+1)}e^{-sT_{ij}}c^{k}+(c\text{-}s)k(s)\}.\frac{1}{h(s)\,(c\text{-}s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}}}], \quad (5)$$

where c is constant.

Proof 3. Let $\varphi(t)=t+T$. Since $\varphi^{(i)}(t)=0$ for $i=2,3,...,n$, then:

$$\sum_{i=0}^{n}\sum_{j=1}^{m}a_{ij}e^{-sT_{ij}}\int_{-T_{ij}}^{0}\varphi^{(i)}(t)e^{-st}dt=\sum_{j=1}^{m}a_{0j}(\frac{1}{s^{2}}-\frac{e^{-sT_{0j}}}{s^{2}}-\frac{T\,e^{-sT_{0j}}}{s}+\frac{T-T_{0j}}{s})+\sum_{j=1}^{m}a_{1j}(\frac{1}{s}-\frac{e^{-sT_{1j}}}{s}).$$

By Theorem 3.1,

$$y(t)=L^{-1}[\{-\frac{h(s)}{s^{2}}(\sum_{j=1}^{m}a_{0j}-\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}})+\frac{h(s)}{s}(\sum_{j=1}^{m}a_{0j}Te^{-sT_{0j}}-\sum_{j=1}^{m}a_{0j}(T-T_{0j})-\sum_{j=1}^{m}a_{1j}+\sum_{j=1}^{m}a_{1j}e^{-sT_{1j}})+h(s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1}a_{ij}.s^{i-(k+1)}e^{-sT_{ij}}y^{(k)}(0)+k(s)\}.\frac{1}{h(s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^{i}e^{-sT_{ij}}}]$$

$$=L^{-1}[\{-h(s)(\sum_{j=1}^{m}a_{0j}-\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}})+s\,h(s).(T\sum_{j=1}^{m}a_{0j}e^{-sT_{0j}}-\sum_{j=1}^{m}a_{0j}(T-T_{0j})-\sum_{j=1}^{m}a_{1j}+\sum_{j=1}^{m}a_{1j}e^{-sT_{1j}})+s^{2}\,h(s)(T\sum_{i=1}^{n}\sum_{j=0}^{m}a_{ij}s^{i-1}e^{-sT_{ij}}+\sum_{i=2}^{n}\sum_{j=0}^{m}a_{ij}s^{i-2}e^{-sT_{ij}})+s^{2}\,k(s)\}.$$

$$\cdot\frac{1}{s^2\, h(s)\sum_{i=0}^{n}\sum_{j=0}^{m}a_{ij}s^i e^{-sT_{ij}}}]$$

The proofs of 1, 2 and 4 are similar to the proof of 3, so they are omitted.

Analytical Solutions of New Forms of Advanced Differential Equations With Multiple Delays

In this section, analytical solutions for many new forms of nonhomogeneous DDEs have been found with Heaviside functions and Dirac delta functions and multiple delays $T_j, j=1,2,...,m$ such that $t \neq T_j,$. In other words, a real valued function is sought which is continuous on $(-T,\infty), T=\max\{T_j\},$ and is differentiable on $[0,T_1)\bigcup_{j=1}^{m-1}(T_j,T_{j+1})\cup(T_m,\infty).$

Theorem 1: Suppose that u is the Heaviside function and δ is the Dirac delta function. Then:

1. The solution of the DDE:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}a_i y^{(i)}(t-T_j)=m\,c\,a_0-c\,a_0\sum_{j=1}^{m}u(t-T_j)+b\sum_{j=0}^{m}\delta(t-T_j) \quad (6)$$

with the initial delay condition:

$\varphi(t)=c,$ for $t\in[-T,0]$ and $T=\max\{T_j\}, j=0,1,2,\ldots,m,$ is:

$$y(t)=L^{-1}\{\frac{c\sum_{i=1}^{n}a_i\, s^{i-1}+b}{\sum_{i=0}^{n}a_i\, s^i}\},$$

where b and c are constants.

2. The solution of the DDE:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}a_i y^{(i)}(t-T_j)=m\,a_0 t-a_0\sum_{j=1}^{m}u(t-T_j)\,(t-T_j)-a_0\sum_{j=1}^{m}T_j-a_1\sum_{j=1}^{m}u(t-T_j)+ma_1+b\sum_{j=0}^{m}\delta(t-T_j), \quad (7)$$

with the initial delay condition:

$\varphi(t)=t,$ for $t\in[-T,0]$ and $T=\max\{T_j\}$, $j=0,1,2,\cdots,m,$ is:

$$y(t)=L^{-1}\{\frac{\sum_{i=2}^{n}a_i\, s^{i-2}+b}{\sum_{i=0}^{n}a_i\, s^i}\},$$

where b is a constant.

3. The solution of the DDE:

$$\sum_{i=0}^{n}\sum_{j=0}^{m}a_i y^{(i)}(t-T_j)=m\,a_0 t-a_0\sum_{j=1}^{m}u(t-T_j)\,(t-T_j)-(Ta_0+a_1)\sum_{j=1}^{m}u(t-T_j)+a_0\sum_{j=1}^{m}(T-T_j)+ma_1+b\sum_{j=0}^{m}\delta(t-T_j), \quad (8)$$

with the initial delay condition:

$\varphi(t)=t+T,$ for $t\in[-T,0]$ and $T=\max\{T_j\},$ is:

$$y(t)=L^{-1}\{\frac{T\sum_{i=1}^{n}a_i\, s^{i-1}+\sum_{i=2}^{n}a_i\, s^{i-2}+b}{\sum_{i=0}^{n}a_i\, s^i}\},$$

where b is constant.

4. The solution of the DDE:

$$\sum_{i=1}^{n}\sum_{j=0}^{m}a_i y^{(i)}(t-T_j)=b\sum_{j=0}^{m}u(t-(T_j+a))+d\sum_{j=0}^{m}\delta(t-(T_j+a)) \quad (9)$$

with the initial delay condition:

$\varphi(t)=c,$ for $t\in[-T,0]$ and $T=\max\{T_j\},$ $j=0,\,1,\,2,\,\cdots,m,$ is:

$$y(t)=c+L^{-1}\{\frac{(b+d\,s)\,e^{-as}}{s\sum_{i=1}^{n}a_i\, s^i}\},$$

where b,c and d are constants and $a\geq 0.$

5. The solution of the DDE:

$$\sum_{i=2}^{n}\sum_{j=0}^{m}a_i y^{(i)}(t-T_j)=b\sum_{j=0}^{m}u(t-(T_j+a))+d\sum_{j=0}^{m}\delta(t-(T_j+a)) \quad (10)$$

with the initial delay condition:

$\varphi(t)=t,$ for $t\in[-T,0]$ and $T=\max\{T_j\},$ $j=0,\,1,\,2,\,\cdots,m,$ is:

$$y(t) = t + L^{-1}\{\frac{(b + d\ s)\, e^{-as}}{s\sum_{i=2}^{n} a_i\ \mathrm{s}^{\mathrm{i}}}\},$$

where b and d are constants and $a \geq 0$.

6. The solution of the DDE:

$$\sum_{i=2}^{n}\sum_{j=0}^{m} a_i y^{(i)}(t-T_j) = b\sum_{j=0}^{m} u(t-(T_j + a)) + d\sum_{j=0}^{m}\delta(t-(T_j + a)), \quad (11)$$

with the initial delay condition:

$\varphi(t) = t + T$, for $t \in [-T, 0]$ and $T = \max\{T_j\}$, $j = 0, 1, 2, \cdots, m$, is:

$$y(t) = T + t + L^{-1}\{\frac{(b + d\ s)\, e^{-as}}{s\sum_{i=2}^{n} a_i\ \mathrm{s}^{\mathrm{i}}}\}.$$

where b and d are constants and $a \geq 0$.

7. Suppose that n is an odd number and $a_{2(i)+1} = -a_{2(i)}$, $i = 0, 1, \ldots, \frac{n-1}{2}$. Then, the solution of the DDE is:

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_i y^{(i)}(t-T_j) = b\sum_{j=0}^{m} u(t-(T_j + a)) + d\sum_{j=0}^{m} u(t-(T_j + a)) \quad (12)$$

with the initial delay condition $\varphi(t) = e^t$, for $t \in [-T, 0]$ and $T = \max\{T_j\}$, $j = 0, 1, \ldots, m$, is:

$$y(t) = L^{-1}\{\frac{\sum_{i=1}^{n}\sum_{k=0}^{i-1} a_i\ \mathrm{s}^{\mathrm{i-(k+1)}}}{\sum_{i=0}^{n} a_i\ \mathrm{s}^{\mathrm{i}}} + \frac{(b + d\ s)\, e^{-as}}{s\sum_{i=0}^{n} a_i\ \mathrm{s}^{\mathrm{i}}}\},$$

where b and d are constants and $a \geq 0$.

Proof of 5: Let $\varphi(t) = t$, for $t \in [-T, 0]$ and $T = \max\{T_j\}$, $j = 0, 1, 2, \cdots, m$. Therefore, by equation (3) yields:

$$y(t) = L^{-1}\{\frac{s^3\sum_{i=2}^{n}\sum_{j=0}^{m} a_i s^{i-2} e^{-sT_j} + s^2(b\, e^{-as}\sum_{j=0}^{m} e^{-sT_j} + d\, s\, e^{-as}\sum_{j=0}^{m} e^{-sT_j})}{s^3\sum_{i=2}^{n}\sum_{j=0}^{m} a_i s^i\, e^{-sT_j}}\}$$

$$= t + L^{-1}\{\frac{(b + d\ s)\, e^{-as}}{s\sum_{i=2}^{n} a_i\ \mathrm{s}^{\mathrm{i}}}\}.$$

Proof of 7: Let $\varphi(t) = e^t$, for $t \in [-T, 0]$ and $T = \max\{T_j\}$, $j = 0, 1, 2, \cdots, m$, and $a_{2(i)+1} = -a_{2(i)}$ for $i = 0, 1, \ldots, \frac{n-1}{2}$. Therefore, by equation (5) yields:

$$y(t) = L^{-1}[\{-s\sum_{i=0}^{n}\sum_{j=1}^{m} a_i(e^{-sT_j} - e^{-T_j}) + s\,(1-s) \cdot\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1} a_i s^{i-(k+1)} e^{-sT_j} + (1-s)(b + d\ s) \cdot e^{-as}\sum_{j=0}^{m} e^{-sT_j}\}.\frac{1}{s(1-s)\sum_{i=0}^{n}\sum_{j=0}^{m} a_i\ \mathrm{s}^{\mathrm{i}} e^{-sT_j}}]$$

$$= L^{-1}[\{s(1-s)\sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i-1} a_i s^{i-(k+1)} e^{-sT_j} + (1-s)(b + d\ s)e^{-as}\sum_{j=0}^{m} e^{-sT_j}\} \cdot\frac{1}{s(1-s)\sum_{i=0}^{n}\sum_{j=0}^{m} a_i\ \mathrm{s}^{\mathrm{i}} e^{-sT_j}}],$$

since $a_{2(i)+1} = -a_{2(i)}$.

Therefore:

$$y(t) = L^{-1}\{\frac{\sum_{i=1}^{n}\sum_{k=0}^{i-1} a_i\ \mathrm{s}^{\mathrm{i-(k+1)}}}{\sum_{i=0}^{n} a_i\ \mathrm{s}^{\mathrm{i}}} + \frac{(b + d\ s)\, e^{-as}}{s\sum_{i=0}^{n} a_i\ \mathrm{s}^{\mathrm{i}}}\}.$$

The other proofs are similar.

Illustrative Examples

Example 1: Consider the fourth order DDE:

$$y^{(4)}(t) + 2y^{(4)}(t-1) - 5y''(t-2) + y''(t-4) = 1 + 2u(t-1) - \frac{5}{2}u(t-2)(t\text{-}2)^2 + \frac{1}{2}u(t-4)(t\text{-}4)^2 \quad (13)$$

with the initial delay condition:

$y(t) = \varphi(t) = t,\ -4 \leq t \leq 0$.

By equation (3) with $k(s) = s^2 + 2s^2 e^{-s} - 5e^{-2s} + e^{-4s}$, $h(s) = s^3$, the solution of the DDE (13) is given as follows:

$$y(t) = L^{-1}[\frac{(s^5 + s^2)(s^2 + 2s^2 e^{-s} - 5e^{-2s} + e^{-4s})}{s^7(s^2 + 2s^2 e^{-s} - 5e^{-2s} + e^{-4s})}]$$

$$= L^{-1}[\frac{1}{s^2} + \frac{1}{s^5}]$$

$$= t + \frac{1}{4!}t^4 .$$

Example 2: Consider the DDE:

$$y'''(t) + y'''(t-\frac{1}{2}) + y'''(t-1) + y'''(t-\frac{3}{2}) + y'''(t-\frac{5}{2}) + y'''(t-4) + 3y''(t) + 3y''(t-\frac{1}{2}) + 3y''(t-1) + 3y''(t-\frac{3}{2}) + 3y''(t-\frac{5}{2}) + 3y''(t-4) + 3y'(t) + 3y'(t-\frac{1}{2}) + 3y'(t-1) + 3y'(t-\frac{3}{2}) + 3y'(t-\frac{5}{2}) + 3y'(t-4) + y(t) + y(t-\frac{1}{2}) + y(t-1) + y(t-\frac{3}{2}) + y(t-\frac{5}{2}) + y(t-4) = 5t - u(t-\frac{1}{2})(t-\frac{1}{2}) - u(t-1).(t-1) - u(t-\frac{3}{2})(t-\frac{3}{2}) - u(t-\frac{5}{2})(t-\frac{5}{2}) - u(t-4)(t-4) - 7[u(t-\frac{1}{2}) + u(t-1) + u(t-\frac{3}{2}) + u(t-\frac{5}{2}) + u(t-4)] + \frac{51}{2} + 2(\delta(t) + \delta(t-\frac{1}{2}) + \delta(t-1) + \delta(t-\frac{3}{2}) + \delta(t-\frac{5}{2}) + \delta(t-4)),$$ (14)

with the initial delay condition $\varphi(t) = t + 4, \ t \in [-4,0]$. Since the DDE (14) is in the form of (8), the solution of the DDE (14) is given as follows:

$$y(t) = L^{-1}\{\frac{4(s^2+3s+3)+s+5}{s^3+3s^2+3s+1}\}$$
$$= L^{-1}\{\frac{4s^2+13s+17}{(s+1)^3}\}$$
$$= 4e^{-t} + 5te^{-t} + 4t^2e^{-t}.$$

In a similar manner, many DDEs and their analytical solutions can be introduced as shown in the following Table 1:

Table 1. DDEs and the corresponding closed-forms formulas.

1	$y'(t) + y'(t-2) - y(t) - y(t-2) = u(t-1) + u(t-3),\ \varphi(t) = e^t,\ -2 \le t \le 0,$ $y(t) = e^t - u(t-1) + e^{t-1}u(t-1),\ t \ge 0.$
2	$y^{(4)}(t) + y'''(t) - y''(t) + y^{(4)}(t-1) + y^{(4)}(t-2) + y'''(t-1) + y'''(t-2) - y''(t-1) - y''(t-2)$ $= \delta(t) + \delta(t-1) + \delta(t-2),\ y(t) = \varphi(t) = t + 2,\ -2 \le t \le 0,$ $y(t) = 1 + e^{-\frac{1}{2}t}\cosh\frac{\sqrt{5}}{2}t + \frac{3}{\sqrt{5}}e^{-\frac{1}{2}t}\sinh\frac{\sqrt{5}}{2}t,\ t \ge 0.$
3	$y^{(4)}(t) + 2y^{(4)}(t-1) - 5y''(t-2) + y''(t-4) = u(t-2) - \frac{5}{2}u(t-4)(t-4)^2 + \frac{1}{2}u(t-6)(t-6)^2$ $+ 2u(t-3),\ y(t) = \varphi(t) = t,\ -4 \le t \le 0,$ $y(t) = t + \frac{1}{4!}u(t-2)(t-2)^4,\ t \ge 0.$
4	$y^{(4)}(t) + y''(t) + y^{(4)}(t-2) + y^{(4)}(t-4) + y''(t-2) + y''(t-4) = u(t-1) + u(t-3) + u(t-5),$ $y(t) = \varphi(t) = t,\ -4 \le t \le 0,$ $y(t) = t + u(t-1)[\frac{1}{2}(t-1)^2 - 1 + \cos(t-1)],\ t \ge 0.$
5	$y^{(5)}(t) - y''(t) + y^{(5)}(t-1) + y^{(5)}(t-2) - y''(t-1) - y''(t-2) = 3\delta(t-2) + 3\delta(t-3) + 3\delta(t-4),$ $y(t) = \varphi(t) = t + 2,\ -2 \le t \le 0,$ $y(t) = 2 + t + u(t-2)[-3(t-2) + e^{t-2} - e^{-\frac{1}{2}(t-2)}\cos\frac{\sqrt{3}}{2}(t-2) + \sqrt{3}\,e^{-\frac{1}{2}(t-2)}\sin\frac{\sqrt{3}}{2}(t-2)],\ t \ge 0.$

$$y^{(4)}(t)+y'''(t)-y''(t)+y^{(4)}(t-1)+y^{(4)}(t-2)+y'''(t-1)+y'''(t-2)-y''(t-1)-y''(t-2)$$
6 $$=\delta(t-3)+\delta(t-4)+\delta(t-5),\ y(t)=\varphi(t)=t,\ -2\ \le\ t\ \le\ 0,$$
$$y(t)=t-u(t-3)[1+(t-3)-\mathrm{e}^{-\frac{1}{2}(t-3)}\cosh\tfrac{\sqrt{5}}{2}(t-3)-\frac{3}{\sqrt{5}}\ \mathrm{e}^{-\frac{1}{2}(t-3)}\sinh\tfrac{\sqrt{5}}{2}(t-3)],\ t\ge 0.$$

$$y^{(6)}(t)-y''(t)+y^{(6)}(t-5)-y''(t-5)=u(t-4)+u(t-9),\ y(t)=\varphi(t)=c,\ -5\le t\le 0,$$
7 $$y(t)\ =c+u(t-4)\,[-\frac{1}{2}(t-4)^2+\frac{1}{2}\cosh(t-4)-\frac{1}{2}\cos(t-4)],\ t\ge 0.$$

$$2y''(t)+2y''(t-2)+6y'(t)+6y'(t-2)+4y(t)+4y(t-2)=12-12u(t-2)+2\delta(t)+2\delta(t-2),$$
8 $$\varphi(t)=3,\ \ t\in[-2,0],$$
$$\boldsymbol{y(t)=-4e^{-2t}+7e^{-t},\ t\ge 0.}$$

$$y'''(t)+y'''(t-1)+y'''(t-2)-2y''(t)-2y''(t-1)-2y''(t-2)+y'(t)+y'(t-1)+y'(t-2)$$
9 $$=2-u(t-1)-u(t-2)+2\delta(t)+2\delta(t-1)+2\delta(t-2),\varphi(t)=t,\ \ t\in[-2,0],$$
$$y(t)=te^{t},\ t\ge 0.$$

$$y''(t)+y''(t-1)+y''(t-2)+y'(t)+y'(t-1)+y'(t-2)=5u(t-2)+5u(t-3)+5u(t-4),$$
10 $$\varphi(t)=2,\ t\in[-2,0],$$
$$y(t)=2+u(t-2)\,[5e^{-(t-2)}+5(t-2)-5],\ t\ge 0.$$

$$y^{(4)}(t)+y^{(4)}(t-2)+y^{(4)}(t-3)-2y'''(t)-2y'''(t-2)-2y'''(t-3)+4y''(t)+4y''(t-2)$$
11 $$+4y''(t-3)=u(t-1)+u(t-3)+u(t-4),\ \varphi(t)=t,\ t\in[-3,0],$$
$$y(t)=t+u(t-1)\,[\frac{1}{8}(t-1)+\frac{1}{8}(t-1)^2-\frac{1}{8\sqrt{3}}e^{t-1}\sin\sqrt{3}(t-1)],\ t\ge 0.$$

$$y^{(6)}(t)-y''(t)+y^{(6)}(t-2)+y^{(6)}(t-5)-y''(t-2)-y''(t-5)=\delta(t-4)+\delta(t-6)+\delta(t-9),$$
12 $$y(t)=\varphi(t)=t,\ -5\ \le\ t\ \le\ 0,$$
$$y(t)=t+u(t-4)\,[-(t-4)+\frac{1}{2}\sin(t-4)+\frac{1}{2}\sinh(t-4)],\ t\ge 0.$$

$$y^{(4)}(t)+y^{(4)}(t-2)+y^{(4)}(t-3)-2y'''(t)-2y'''(t-2)-2y'''(t-3)+4y''(t)+4y''(t-2)$$
13 $$+4y''(t-3)=u(t-1)+u(t-3)+u(t-4),\ \ \varphi(t)=t,\ \ \ t\in[-3,0],$$
$$y(t)=t+u(t-1)\,[\frac{1}{8}(t-1)+\frac{1}{8}(t-1)^2-\frac{1}{8\sqrt{3}}e^{t-1}\sin\sqrt{3}(t-1)],\ t\ge 0.$$

$$y^{(6)}(t)-y''(t)+y^{(6)}(t-2)+y^{(6)}(t-5)-y''(t-2)-y''(t-5)=\delta(t-4)+\delta(t-6)+\delta(t-9),$$
14 $$y(t)=\varphi(t)=t,\ -5\ \le\ t\ \le\ 0,$$
$$y(t)=t+u(t-4)\,[-(t-4)+\frac{1}{2}\sin(t-4)+\frac{1}{2}\sinh(t-4)],\ t\ge 0.$$

$$y^{(4)}(t)+y^{(4)}(t-1)+y^{(4)}(t-2)+4y'''(t)+4y'''(t-1)+4y'''(t-2)+4y''(t)+4y''(t-1)$$
15 $$+4y''(t-2)=2u(t-2)+2u(t-3)+2u(t-4),\ \varphi(t)=t+2,\ \ \ t\in[-2,0],$$
$$y(t)=2+t+u(t-2)\,[\frac{3}{8}-\frac{1}{2}(t-2)+\frac{1}{4}(t-2)^2-\frac{3}{8}e^{-2(t-2)}-\frac{1}{4}(t-2)e^{-2(t-2)}],\ t\ge 0.$$

$$y'(t)+y'(t-2)+y'(t-5)-y(t)-y(t-5)-y(t-2)=u(t-5)+u(t-7)+u(t-10),$$
16 $$\varphi(t)=e^{t},t\in[-5,0],$$
$$y(t)=e^{t}+u(t-5)\,[e^{t-5}-1],\ t\ge 0.$$

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Exponentially Stable and Strongly Stable of Advanced Differential Equations in Normed Space L_P

In this section, the previous results are used to construct dynamical systems resulting from nonhomogeneous DDEs with multiple delays and discussing some of their properties (exponentially stable and strongly stable) with the following initial delay conditions:

$y(t) = x(\theta) = c$, $y(t) = x(\theta) = \theta$ and $y(t) = x(\theta) = \theta + T$ for $\theta \in [-T, 0]$, $\theta > T$, $T = \max\{T_j\}$ in normed space L_P.

Let us formulate our problem by considering the first order DDE:

$$\sum_{i=0}^{1}\sum_{j=0}^{m} a_i y^{(i)}(t - T_j) = m\, c\, a_0 - c\, a_0 \sum_{j=1}^{m} u(t\text{-}T_j) \qquad (15)$$

where $T_j > 0$, with the initial delay condition:

$$y(t) = x(\theta) = c, \qquad (16)$$

for $\theta \in [-T, 0]$ such that $T = \max\{T_j\}$, where $x \in X$ such that X is a normed vector space of functions on $[0, T]$.

A semi dynamical system is defined as:

$$\breve{T}_t : X \to X, \qquad (17)$$

which can be found by the solution of the DDE (6) as follows:

$$(\breve{T}_t x)(\theta) = \breve{y}(t, \theta) = c\, e^{-\frac{a_0}{a_1}t},\ t \geq 0,\ \frac{a_0}{a_1} > 0,$$

$$\forall \theta \in [0, T], \qquad (18)$$

where $\breve{y}$ is the solution of (15) - (16).

Also, the following second order DDE is considered:

Case i) The DDE:

$$\sum_{i=0}^{2}\sum_{j=0}^{m} a_i y^{(i)}(t - T_j) = m\, a_0 t - a_0 \sum_{j=1}^{m} u(t\text{-}T_j)\ (t\text{-}T_j)$$

$$- a_0 \sum_{j=1}^{m} T_j - a_1 \sum_{j=1}^{m} u(t - T_j) + m a_1,\ T_j > 0 \qquad (19)$$

with the initial delay condition:

$$y(t) = x(\theta) = \theta, \qquad (20)$$

for $\theta \in [-T, 0]$ such that $T = \max\{T_j\}$ and $a_1 = 2\sqrt{a_2 a_0} > 0$ where $x \in X$ such that X is a normed vector space of functions on $[0, T]$.

A semi dynamical system is defined as:

$$\tilde{T}_t : X \to X, \qquad (21)$$

which can be found by the solution of the DDE (7) as:

$$(\tilde{T}_t x)(\theta) = \tilde{y}(t, \theta) = (\theta - T)\, e^{-\sqrt{\frac{a_0}{a_2}}t},\ \frac{a_0}{a_1} > 0,\ t \geq 0,$$

$$\forall \theta \in [0, T], \qquad (22)$$

where $\tilde{y}$ is solution of (19) - (20).

Case ii) The DDE:

$$\sum_{i=0}^{2}\sum_{j=0}^{m} a_i y^{(i)}(t - T_j) = m\, a_0 t - a_0 \sum_{j=1}^{m} u(t\text{-}T_j)\ (t\text{-}T_j)$$

$$- (a_0 T + a_1)\sum_{j=1}^{m} u(t - T_j) + a_0 \sum_{j=1}^{m} (T - T_j) +$$

$$m a_1 + b \sum_{j=0}^{m} \delta(t\text{-}T_j), \qquad (23)$$

with the initial delay condition:

$$y(t) = x(\theta) = \theta + T, \qquad (24)$$

for $\theta \in [-T, 0]$ such that $T = \max\{T_j\}$, $a_1 = 2\sqrt{a_2 a_0} > 0$ and $b = -\frac{a_1}{2}T$ where $x \in X$ such that X is a normed vector space of functions on $[0, T]$.

Also, A semi dynamical system is defined as:

$$\hat{T}_t : X \to X, \qquad (25)$$

which can be found by the solution of the DDE (8) as:

$$(\hat{T}_t x)(\theta) = \hat{y}(t, \theta) = \theta\, e^{-\sqrt{\frac{a_0}{a_2}}t},\ \frac{a_0}{a_2} > 0,\ t \geq 0,$$

$$\theta \in [0, T], \qquad (26)$$

where $\hat{y}$ is solution of (23) - (24).

Theorem 1: The semi group $(\breve{T}_t)_{t \geq 0}$ is exponentially stable in L_p if for every $T > 0$, $\frac{a_0}{a_1} > 0$, there exists $0 < D < \infty$ such that $c T^{\frac{1}{p}} \leq D$, $c > 0$.

Proof: Suppose that $x \in L_p$, then:

$$\left\| \breve{T}_t x \right\|_{L_p} = \left(\int_0^T \left| \breve{y}(t,\ \theta) \right|^p d\theta \right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \left| x(\theta - T)\, e^{-\frac{a_0}{a_1}t} \right|^p d\theta \right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \left| c\, \mathrm{e}^{-\frac{a_0}{a_1}t} \right|^p d\theta \right)^{\frac{1}{p}}$$

$$= c\mathrm{e}^{-\frac{a_0}{a_1}t}\, \theta^{\frac{1}{p}} \Big|_0^T$$

$$= cT^{\frac{1}{p}}\, \mathrm{e}^{-\frac{a_0}{a_1}t}.$$

Since there exists $0 < D < \infty$ such that $cT^{\frac{1}{p}} \le D$, then $cT^{\frac{1}{p}}\, \mathrm{e}^{-\frac{a_0}{a_1}t} \le D\, \mathrm{e}^{-\frac{a_0}{a_1}t}$. Therefore, the semi group $(\breve{T}_t)_{t\ge 0}$ is exponentially stable.

Theorem 2: The semi group $(\breve{T}_t)_{t\ge 0}$ is strongly stable in L_p if $T > 0$, $\frac{a_0}{a_1} > 0$.

Proof: Suppose that $x \in L_p$,

$\breve{T}_t x = c\, \mathrm{e}^{-\frac{a_0}{a_1}t}$.

Therefore:

$\lim_{t\to\infty} \breve{T}_t x = c \lim_{t\to\infty} \mathrm{e}^{-\frac{a_0}{a_1}t} = 0.$

Thus, the semi group $(\breve{T}_t)_{t\ge 0}$ is strongly stable in X.

Theorem 3: The semi group $(\tilde{T}_t)_{t\ge 0}$ is strongly stable in L_p if $T > 0$, $\frac{a_0}{a_2} > 0$.

Proof: Suppose that $x \in L_p$

$\tilde{T}_t x = (\theta - T)\, \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t}$.

Therefore:

$\lim_{t\to\infty} \tilde{T}_t x = (\theta - T) \lim_{t\to\infty} \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t} = 0.$

Thus, the semi group $(\tilde{T}_t)_{t\ge 0}$ is strongly stable in X.

Theorem 4: The semi group $(\hat{T}_t)_{t\ge 0}$ is exponentially stable in L_p if for every $T > 0$, $\frac{a_0}{a_2} > 0$ there exists $0 < D < \infty$ such that $\left(\frac{T^{p+1}}{p+1} \right)^{\frac{1}{p}} \le D$.

Proof: Suppose that $x \in L_p$, then:

$$\left\| \hat{T}_t x \right\|_{L_p} = \left(\int_0^T |\hat{y}(t,\theta)|^p d\theta \right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \left| x(\theta - T)\, \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t} \right|^p d\theta \right)^{\frac{1}{p}}$$

$$= \left(\int_0^T \left| \theta\, \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t} \right|^p d\theta \right)^{\frac{1}{p}}$$

$$= e^{-\sqrt{\frac{a_0}{a_2}}t} \left(\frac{\theta^{p+1}}{p+1} \Big|_0^T \right)^{\frac{1}{p}}$$

$$= e^{-\sqrt{\frac{a_0}{a_2}}t} \left(\frac{T^{p+1}}{p+1} \right)^{\frac{1}{p}}.$$

Since there exists $0 < D < \infty$ such that $\left(\frac{T^{p+1}}{p+1} \right)^{\frac{1}{p}} \le D$.

Therefore:

$\left(\frac{T^{p+1}}{p+1} \right)^{\frac{1}{p}} \mathrm{e}^{-\frac{a_0}{a_2}t} \le D\, \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t}$.

Thus, the semi group $(\hat{T}_t)_{t\ge 0}$ is exponentially stable.

Theorem 5: The semi group $(\hat{T}_t)_{t\ge 0}$ is strongly stable in L_p for $T > 0$, $\frac{a_0}{a_2} > 0$.

Proof: Suppose that $x \in L_p$

$\hat{T}_t x = \theta\, \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t}$.

Therefore:

$\lim_{t\to\infty} \hat{T}_t x = \theta \lim_{t\to\infty} \mathrm{e}^{-\sqrt{\frac{a_0}{a_2}}t} = 0.$

Thus, the semi group $(\hat{T}_t)_{t\ge 0}$ is strongly stable in X.

Conclusions:

This paper is concerned with finding analytical solutions of advanced differential equations because very few researchers have worked in this direction, most of them focus on finding numerical solutions. Solving delay differential equations as in the ordinary differential equations is adopted in this work, especially when nonhomogeneous delay differential equations have multiple delays contain discontinuous forcing functions (Heaviside

functions and Dirac delta functions). One can find many elementary solutions rather than nonelementary solutions such as the solutions which depend on the Lambert W function which is classified as a nonelementary and very difficult function. Also, the exponential stability and the strong stability for delay differential equations which contain multiple delays are discussed.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
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حلول تحليلية لمعادلات تفاضلية دالية متقدمة ذات حدود إرغام غير مستمرة ودراسة خصائصها الديناميكية

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الخلاصة:

يهدف هذا البحث الى ايجاد صيغ مغلقة تحليلية جديدة لحلول معادلات تفاضلية دالية غير متجانسة من الرتبة النونية بتباطؤات ثابته ومنتهية وشروط تباطؤية ابتدائية متنوعة بدلالة دوال ابتدائية باستخدام طريقة تحويل لابلاس. بالاضافة الى استخدام تعريف الانظمة الديناميكية للمعادلات التفاضلية الاعتيادية لتقديم تعريف الانظمة الديناميكية للمعادلات التفاضلية التباطؤية بتباطؤات متعددة ومناقشة خصائصها الديناميكية: الاستقرارية الاسية والاستقرارية القوية.

الكلمات المفتاحية: تحويل لابلاس، المعادلات التفاضلية- الدالية الخطية، نظرية الاستقرارية.