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Darboux Integrability of a Generalized 3D Chaotic Sprott ET9 System

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Abstract:

In this paper, the first integrals of Darboux type of the generalized Sprott ET9 chaotic system will be studied. This study showed that the system has no polynomial, rational, analytic and Darboux first integrals for any value of a and b. All the Darboux polynomials for this system were derived together with its exponential factors. Using the weight homogenous polynomials helped us prove the process.

Key words: Analytic first integral, Darboux first integral, Darboux polynomial, Exponential factors.

Introduction:

In recent decades, there have been reports in the literature of several chaotic differential systems such as [1-4] and many others. Recently, even differential systems with only one equilibrium points to have chaotic behavior also have been demonstrated [5–8].

In 1994, Sprott [9] displayed 19 distinct simple three-dimensional autonomous ordinary differential equations of chaotic flows and quadratic non-linearities, based on two real parameters, and then described their properties. These examples are simpler than Lorenz and Rössler system. Among these 19 systems, Case A is the simplest one that [10,11] used to describe a one-dimensional Nose-Hoover mechanical system exhibiting chaos. Some systems attracted much attention see [12–14]. Twenty one years later, in 2015, Sprott [15] made generalization of Nose-Hoover oscillator, revealing 11 cases with strange attractors (hidden or self-excited) among these he introduced a chaotic system ET9 with only one non-hyperbolic equilibrium point. This equilibrium is nonlinearity unstable and strange attractor is self-excited. Self-excited attractor are examples such as van der Pol, Belousov-Zhabotinskii, Lorenz, Rössler, Chen, Chua, Lu, Jerk or Sprott's system (case B-S). Various studies of systems with self-excited attractors have been conducted in different science areas especially in engineering applications, like in the design electronic circuits, communications, control systems, and artificial intelligence [16–19], Nevertheless, in these systems with the self-excited attractors, there are still various issues that invite further research.

This research modifies the Sprott ET9 system [15] by considering two parameters in the nonlinear portion, which are prospective to a more chaotic system of behavior. More specifically, the following generalized system will be studied

$$\dot{x} = y,\quad \dot{y} = -x + yz,\quad \dot{z} = -z - axy - bxz, \qquad (1)$$

where a and b are nonzero real parameters. In [15], Sprott presented system (1) with $a = 4$ and $b = -1$ as ET9 (Fig. 1) among eleven different autonomous systems having chaotic conduct.

To the best of our knowledge, this rich dynamical system (1) has never been investigated from the integrability perspective. The key objective of this work involves the characterization of the rational and Darboux first integrals. For this, the invariant algebraic surfaces need to be fully characterized based on their parameters. To achieve invariant algebraic surfaces as such, the theory of Darboux integrability needs to be utilized, for further details on this theory see [20–27].

In 1878 Darboux [28] demonstrated how the first integrals of 2D differential system could have been formed with enough invariant algebraic curves. In particular, he had shown that if a polynomial autonomous system with degree n

possess a first integral if it has at least $\left[\frac{m(m+1)}{2}+1\right]$ invariant algebraic curves that has a simple expressions, according to its invariant algebraic curves.

This research provides the invariant of system (1) which consists of the rational and Darboux first integral. The analytic and polynomial first integrals are also provided. For our system, one first integral reduces the complexity of its dynamics, and the presence of two irreducible first integrals solves entirely the issue of determining its phase portraits.

1. Some Definitions and Preliminary Results

This section begins with a brief overview of the integrability problem, the Darboux method, and the auxiliary results. To prove the main results of this paper, few basic definitions and theorems are given as a background to this study.

Let $f = f(x, y, z)$ be a real polynomial defined as a *Darboux polynomial* for the system (1) if

$$y\frac{\partial f}{\partial x} + (yz - x)\frac{\partial f}{\partial y} - (z + axy + bxz)\frac{\partial f}{\partial z} = Kf, \quad (2)$$

for a real polynomial $K(x, y, z)$, that is a *cofactor* of f with a degree of almost one. As a result, the cofactor form can be assumed as follow

$$K(x, y, z) = k_0 + k_1x + k_2y + k_3z, \quad (3)$$

where $k_i \in \mathbb{C}$ for $i = 0, .., 3$ if $f(x, y, z)$ is a Darboux polynomial of the differential system (1), then the *invariant algebraic surface* in $\mathbb{R}^3$ is $f = 0$. It is called as such because of the fact that when a solution of system (1) includes a point on the invariant algebraic surface, then the entire solution is contained in it [20].

It is known if a Darboux polynomial H is with zero cofactor then it is defined as a *polynomial first integral* of system (1). i.e.

$$y\frac{\partial H}{\partial x} + (yz - x)\frac{\partial H}{\partial y} - (z + axy + bxz)\frac{\partial H}{\partial z} = 0. \quad (4)$$

Once the function H is satisfying Eq. (4) and is also rational (analytic) then it is a *rational (analytic) first integral*.

Definition [29] A nonconstant polynomial $f(x) \in F[x]$ is irreducible over the field F or is an irreducible polynomial in $F[x]$ if $f(x)$ cannot be expressed as a product $g(x) * h(x)$ of two polynomials $g(x)$ and $h(x)$ in $F[x]$ both of lower degree than the degree of $f(x)$. If $f(x) \in F[x]$ is a nonconstant polynomial that is not irreducible over F, then $f(x)$ is reducible over F.

First, depending on the following two Propositions that have been proved in [20,30,31]:

Proposition 1.1. *Let f be a polynomial and $f = \prod_{j=1}^{s} f_j^{\alpha_j}$ its decomposition into irreducible complex factors in $C[x, y, z]$. Next, f is a Darboux polynomial if and only if all the f_j are Darboux polynomials. Furthermore, if K* and *K_j denote the cofactors of f and f_j, then $K = \sum_{j=1}^{s} \alpha_j K_j$.*

It is noticeable that it is enough to search the irreducible Darboux polynomials of system (1), in view of Proposition 1.1.

Proposition 1.2. *The presence of a rational first integral for a polynomial autonomous system (1) indicates the occurrence of either a polynomial first integral or two Darboux polynomials with the identical non-zero cofactor.*

To validate system (1) that does not have analytic first integral in the neighborhood of the origin, the following result is necessary which is due to Li, Llibre and Zhang in [32].

Theorem 1.3. *Take the autonomous system*

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}), \quad \mathrm{x} = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n, \quad (5)$$

where the vector valued function f *of dimension n matches* $\mathrm{f}(0) = 0$.

If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the Jacobian matrix A of system (5) at $\mathrm{x} = 0$ *do not match any resonance situations like*

$$\sum_{i=1}^{n} k_i \lambda_i = 0, \quad k_i \in \mathbb{Z}^+, \quad \sum_{i=2}^{n} k_i \geq 1,$$

then system (5) does not admit any analytic first integral in a neighborhood of $\mathrm{x} = 0$.

It is obvious to us that an *exponential factor* E of system (1) is defined as an exponential function of the form $E = exp(g/h) \notin \mathbb{C}$ with $g, h \in \mathbb{C}[x, y, z]$ and let f and g are coprime in the ring $\mathbb{C}[x, y, z]$, and satisfying

$$y\frac{\partial E}{\partial x} + (yz - x)\frac{\partial E}{\partial y} - (z + axy + bxz)\frac{\partial E}{\partial z} = LE, \quad (6)$$

for certain polynomials $L = L(x, y, z)$ having degree at most 1, is said to be the *cofactor* of E.

The result below provides a geometrical meaning of the exponential factor concept, which can be found in [30] for the plane and [24,33] for higher dimension systems.

Proposition 1.4. The following statements hold.

(a*) If $E = exp(g/h)$ is an exponential factor for the polynomial differential system (1), then $h = 0$ is an invariant algebraic surface, where h is not a constant polynomial.*

(b) *Finally, $exp(g)$ coming from the multiplicity of the infinite invariant planes, could be an exponential factor.*

Proposition 1.5. *Assume that $exp\left(\frac{g_1}{h_1}\right), \dots, exp(g_r/h_r)$ are exponential factors of some polynomial differential system*

$x' = P(x,y,z),\ y' = Q(x,y,z),\ z' = R(x,y,z),$ *with* $P,Q,R \in \mathbb{C}[x,y,z]$ *with cofactors* L_j *for* $j = 1,\dots,r$. *Then* $exp(G) = exp(\frac{g_1}{h_1} + \cdots + \frac{g_r}{h_r})$ *is also an exponential factor of system (1) with cofactor* $L = \sum_{j=1}^{r} L_j$.

For the proof of the above result, see [5].

The first integral of system (1) can be considered a Darboux type when expressed as:

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} E_1^{\mu_1} \dots E_q^{\mu_q},$$

where $f_1,\dots,f_p$ are Darboux polynomials, $E_1,\dots,E_q$ are exponential factors, and λ_i and μ_j are complex numbers, for $i = 1,\dots,p$ and $j = 1,\dots,q$.

Theorem 1.6. *A polynomial system (1) of degree* m *is assumed to incorporate* p *invariant algebraic surfaces* $f_i = 0$ *combined with cofactors* k_i *for* $i = 1,\dots,p$ *and* q *exponential factors* $E = exp(g_j/h_j)$ *with cofactors* L_j *for* $j = 1,\dots,q$. *Then, there exist* λ_i *and* $\mu_j \in \mathbb{C}$ *not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_i L_i = 0, \qquad (7)$$

if and only if the function of Darboux type

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} E_1^{\mu_1} \dots E_q^{\mu_q}$$

is a first integral of system (1).

For proof of Theorem 1.6 and more information, refer to [20,33].

The weight homogeneous polynomials can now be defined, which is used in the proof of Theorem 2.3. The invariant algebraic surfaces of many popular systems, such as Lorenz system [34], Chen system [35], Moon-Rand system [5], and et al. have been commonly used in this procedure.

A polynomial $g(x)$ with $x \in \mathbb{R}^n$ is considered as weight homogeneous if there was any $s = (s_1,\dots,s_n) \in \mathbb{N}^n$ and $m \in \mathbb{N}$ such that for all $\alpha > 0$,

$$g(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \dots, \alpha^{s_n}x_n) = \alpha^m g(x),$$

where $\mathbb{N}$ signifies the set of positive integers. The variable s refers to the weight exponent of g, and m denotes the weight degree of g with the weight exponent s.

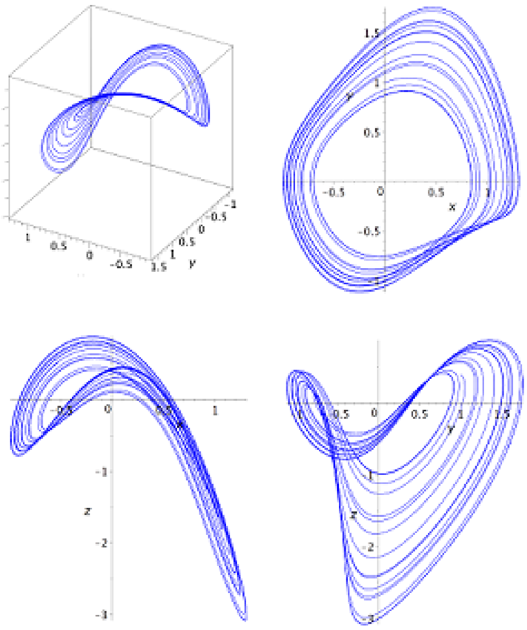

Figure 1. Phase portraits of system (1) when $a = 4$ and $b = -1$: 3D view and 2D projections on xy, xz, and yz planes.

2. Main Results and their Proofs of the Chaotic ET9 System

The study of Darboux integrability is presented in this section. These results are expected to prove that system (1) has only one irreducible Darboux polynomial, where the parameter a is zero. It is also anticipated to demonstrate that the system has neither a polynomial first integral nor a rational first integral. Subsequently, it can be proven that the system contains only one exponential factor when b is not zero. Finally, the system will be proven that is not Darboux integrable. 3D and 2D projections of system (1) were plotted for a given set of initial conditions by selecting the parameters $a = 4$ and $b = -1$ with initial condition $[0, 1, 0.4]$. These projections were subjected to detailed numerical and theoretical analysis (Fig. 1).

Firstly, the study begins with the following two lemmas:

Lemma 2.1. $k_2 = 0$.

Proof. $f = \sum_{i=0}^{n} f_i(x,z)y^i$ can be written, where each f_i is a polynomial of x and z only. Then from substituting into Eq. (2), it becomes

$$y\frac{\partial \sum_{i=0}^{n} f_i(x,z)y^i}{\partial x} + (yz - x)\frac{\partial \sum_{i=0}^{n} f_i(x,z)y^i}{\partial y} - (z + axy + bxz)\frac{\partial \sum_{i=0}^{n} f_i(x,z)y^i}{\partial z} = K\sum_{i=0}^{n} f_i(x,z)y^i, \qquad (8)$$

The terms of y^{n+1} in Eq. (8) match

$$\frac{\partial f_n}{\partial x} - a\,x\,\frac{\partial f_n}{\partial z} = k_2\,f_n.$$

Solving this partial differential equation results in $f_n = G\left(\frac{1}{2}\,a\,x^2 + z\right)e^{k_2\,z}$,
where G denotes any polynomial function of x and z.
One of either k_2 or G has to be zero, so f_n is a polynomial.
If $G\left(\frac{1}{2}\,a\,x^2 + z\right) = 0$ and $k_2 \neq 0$ then $f_n(x,z) = 0$ and consequently $f = f(x,z)$
Then, Eq. (8) leads to

$$y\,\frac{\partial f(x,z)}{\partial x} - (z + a\,xy + b\,x\,z)\frac{\partial f(x,z)}{\partial z} = (k_0 + k_1 x + k_2 y + k_3 z)\,f(x,z)$$

Taking the terms of y from the above partial differential equation, results in

$$y\,\frac{\partial f(x,z)}{\partial x} - a\,xy\,\frac{\partial f(x,z)}{\partial z} = k_2 y\,f(x,z)$$

Then $f = G\left(\frac{1}{2}\,a\,x^2 + z\right)e^{k_2\,x}$,
for some function G of the variables x and z. For f to be a polynomial, results in $k_2 = 0$.
Thus, completing the proof of this lemma. □

Lemma 2.2. $k_0 = -n$ and $k_1 = -n\,b$, $k_3, n \in \mathbb{N}$.

Proof. $f = \sum_{i=0}^{n} f_i(x,y)z^i$ can be written, where each f_i is a polynomial in x and y. Then substituting f into Eq. (2), it becomes

$$y\,\frac{\partial \sum_{i=0}^{n} f_i(x,y)z^i}{\partial x} + (yz - x)\frac{\partial \sum_{i=0}^{n} f_i(x,y)z^i}{\partial y} - (z + axy + bxz)\frac{\partial \sum_{i=0}^{n} f_i(x,y)z^i}{\partial z} = K\sum_{i=0}^{n} f_i(x,y)z^i, \quad (9)$$

The terms of z^{n+1} in Eq. (9) match

$$y\frac{\partial f_n}{\partial y} = k_3\,f_n,$$

Solving this partial differential equation results in $f_n = G_n(x)y^{k_3}$.
Since f_n is polynomial then $G_n(x)$ must be a polynomial, and $k_3 \in \mathbb{N} \cup \{0\}$.
The computation of the terms of z^n in Eq. (9) leads to

$$y\,\frac{\partial f_{n-1}}{\partial y} + y\,\frac{\partial f_n}{\partial x} - x\,\frac{\partial f_n}{\partial y} + (-1 - b\,x)\,n\,f_n = k_3\,f_{n-1} + (k_0 + k_1\,x)f_n.$$

Substituting f_n in the above equation it makes

$$y\,\frac{\partial f_{n-1}}{\partial y} + \left(y^{k_3+1}\right)\frac{dG_n(x)}{dx} + k_3(-x)G_n(x)y^{k_3-1} + (-1 - b\,x)\,n\,G_n(x)y^{k_3} = k_3\,f_{n-1} + (k_0 + k_1\,x)G_n(x)y^{k_3},$$

the solution of the above partial equation for f_{n-1} derives

$$f_{n-1} = \left(-y\,\frac{dG_n(x)}{dx} - \frac{k_3\,x\,G_n(x)}{y} + (k_0 + n + (b\,n + k_1)x)G_n(x)\ln y + G_{n-1}(x)\right)y^{k_3},$$

where $G_{n-1}(x)$ is a polynomial function of x.
Since f_{n-1} must be a polynomial then

$$(k_0 + n + (b\,n + k_1)x)G_n(x) \equiv 0.$$

If $G_n(x) = 0$, and $(k_0 + n + (b\,n + k_1)x) \neq 0$ then $f_n = 0$ so $f = f_0(x,y)$, then Eq. (2) leads to

$$y\,\frac{\partial f_0}{\partial x} + (yz - x)\frac{\partial f_0}{\partial y} = (k_0 + k_1 x + k_3 z)\,f_0$$

Compute the coefficients of z^i , $i = 0,1$ results in the following equations:

$$i = 1:\quad y\,\frac{\partial f_0}{\partial y} = k_3 z\,f_0 \quad (10)$$

$$i = 0:\quad y\,\frac{\partial f_0}{\partial x} - x\,\frac{\partial f_0}{\partial y} = (k_0 + k_1 x)f_0 \quad (11)$$

Eq. (10) derives $f_0 = G_0(x)\,y^{k_3}$, where G_0 denotes a polynomial of x only.
After substituting $f_0 = G_0(x)\,y^{k_3}$ from Eq. (11) and solving it results in

$$G_0(x) = c\,e^{\frac{x\,(2\,y\,k_0 + x\,y\,k_1 + x\,k_3)}{2\,y^2}}.$$

Which is a contradiction as $G_0(x)$ signifies a polynomial function of x only, then must be $(k_0 + n + (b\,n + k_1)x) \equiv 0$.
Hence the conclusion can be
$k_0 = -n$ and $k_1 = -n\,b$, $k_3, n \in \mathbb{N}$.
Thus, completing the proof of lemma. □

Theorem 2.3. *System (1) has only one irreducible Darboux polynomial expressed as z with the cofactor $-1 - bx$ only when $a = 0$.*

Proof. Lemmas 2.1 and 2.2, reaches to $k_2 = 0$, $k_0 = -n$ and $k_1 = -n\,b$, then the cofactor in Eq. (3) becomes

$$K = -n - n\,b\,x + k_3 z,\quad k_3, n \in \mathbb{N}. \quad (12)$$

Now presenting the weight change of variables for simplicity in the computation:
$x = X$, $y = Y$, $z = \lambda^{-1}Z$, $t = \lambda T$, and $\lambda \in \mathbb{R}\backslash\{0\}$, then system (1) becomes

$$\dot{X} = \lambda Y\,,\ \dot{Y} = -\lambda X + YZ,$$
$$\dot{Z} = -\lambda Z - a\lambda^2 XY - b\lambda XZ,\quad (13)$$

where the primes denote the derivative relating to T.
Based on the assumption that f is a Darboux polynomial of system (1) with cofactor K given in Eq. (12). By using the transformation (13) and setting
$F(X,Y,Z) = \lambda^n f(X,Y,\lambda^{-1}Z)$, where n is the degree of f and
$K = \lambda K(X,Y,\lambda^{-1}Z) = -\lambda n - \lambda nbX + k_3 Z$.

The equation $F=\sum_{i=0}^{n}\lambda^{i}F_{n-i}$ is assumed, where F_i denotes a homogenous polynomial depends on the variables X, Y and Z with degree $n-i$ for $i=0,1,\dots,n$.
The definition of Darboux polynomial can derive

$$\lambda Y\sum_{i=0}^{n}\lambda^{i}\frac{\partial F_i}{\partial X}+(-\lambda X+YZ)\sum_{i=0}^{n}\lambda^{i}\frac{\partial F_i}{\partial Y}+(-\lambda Z-a\lambda^{2}XY-b\lambda XZ)\sum_{i=0}^{n}\lambda^{i}\frac{\partial F_i}{\partial Z}=(-\lambda n-\lambda nbX+k_3 Z)\sum_{i=0}^{n}\lambda^{i}F_i. \quad (14)$$

Equating the terms with λ^i for $i=0,1,\dots,n+2$.
Computing the coefficients of λ^0 in Eq. (14) leads to

$$(YZ)\frac{\partial F_0}{\partial Y}=(k_3 Z)F_0.$$

From solving the above differential equation results in

$$F_0=W_0(X,Z)Y^{k_3}, \quad (15)$$

where W_0 is a polynomial function.
Now computing the coefficients of λ in Eq. (14) results in

$$(YZ)\frac{\partial F_1}{\partial Y}+(Y)\frac{\partial F_0}{\partial X}+(-X)\frac{\partial F_0}{\partial Y}+(-Z-bXZ)\frac{\partial F_0}{\partial Z}=k_3 ZF_1+(-n-nbX)F_0.$$

Solving the above differential equation results in

$$F_1=\left(\left(Z\frac{\partial W_0(X,Z)}{\partial Z}-n\,W_0(X,Z)\right)\frac{(bX+1)}{Z}ln(Y)-\frac{Y}{Z}\frac{\partial W_0(X,Z)}{\partial X}-\frac{k_3\,X}{YZ}W_0(X,Z)+W_1(X,Z)\right)Y^{k_3},$$

where W_1 is a polynomial function of X and Z.
So, since $F_1(X,Y,Z)$ is a homogenous polynomial of degree $n-1$, then must be
$Z\,\frac{\partial W_0(X,Z)}{\partial Z}-n\,W_0(X,Z)=0$.
Solving the above differential equation results in $W_0(X,Z)=W_0(X)Z^n$.
Substitute $W_0(X,Z)=W_0(X)Z^n$ from Eq. (15) leads to

$$F_0=W_0(X)Z^nY^{k_3}.$$

But F_0 is of degree n then must be $k_3=0$ and $W_0(X)=c$, where c is a constant.
So $F_0=cZ^n$ and $F_1=W_1(X,Z)$.
Now computing the terms of λ^2 in Eq. (14) leads to

$$(YZ)\frac{\partial F_2}{\partial Y}+(Y)\frac{\partial F_1}{\partial X}+(-X)\frac{\partial F_1}{\partial Y}+(-Z-bXZ)\frac{\partial F_1}{\partial Z}-aXY\frac{\partial F_0}{\partial Z}=(-n-nbX)F_1.$$

So

$$F_2=\left(Z\,\frac{\partial F_1(X,Z)}{\partial Z}-n\,F_1(X,Z)\right)\frac{(bX+1)}{Z}ln(Y)-\frac{Y}{Z}\frac{\partial F_1(X,Z)}{\partial X}-c\,n\,aXYZ^{n-2}+W_2(X,Z).$$

Since F_2 is a polynomial of degree $n-2$, then must $a=0$ and $Z\,\frac{\partial F_1(X,Z)}{\partial Z}-n\,F_1(X,Z)=0$,
hence $F_1(X,Z)=W_1(X)Z^n$.
But $F_1(X,Z)$ is a polynomial of degree $n-1$ then must be $W_1(X)=0$.
Then $F_1(X,Z)=0$
By mathematical induction suppose that $F_{m-1}=F_{m-1}(X,Z)$, where $0<m-1<n$
Now computing the terms of λ^m in Eq. (14) leads to

$$(YZ)\frac{\partial F_m}{\partial Y}+(Y)\frac{\partial F_{m-1}}{\partial X}+(-X)\frac{\partial F_{m-1}}{\partial Y}+(-Z-bXZ)\frac{\partial F_{m-1}}{\partial Z}=(-n-nbX)F_{m-1}.$$

So

$$F_m=\left(Z\,\frac{\partial F_{m-1}(X,Z)}{\partial Z}-n\,F_{m-1}(X,Z)\right)\frac{(bX+1)}{Z}ln(Y)-\frac{Y}{Z}\frac{\partial F_{m-1}(X,Z)}{\partial X}+W_m(X,Z).$$

Since F_m is a polynomial of degree m, then must be

$$Z\,\frac{\partial F_{m-1}(X,Z)}{\partial Z}-n\,F_{m-1}(X,Z)=0,$$

hence $F_{m-1}(X,Z)=W_{m-1}(X)Z^n$.
But $F_{m-1}(X,Z)$ is a polynomial of degree $m-1$ then must be $W_{m-1}(X)=0$.
Then $F_{m-1}(X,Z)=0$.
Hence mathematical induction $F_i(X,Y,Z)=W_i(X)Z^n=0$ for $i=1,\dots,n$
Hence $F=cz^n$ with the cofactor $K=-n-nbx$.
The proof of Theorem 2.3 is completed here. □

Theorem 2.4. *System (1) has no first integrals of polynomials.*
Proof. Here, H is assumed to be a polynomial first integral of system (1). Without the loss of generality it can be supposed that it has no constant term. Then H satisfies Eq. (4). H can be written as $H(x,y,z)=\sum_{i=0}^{n}h_i(x,y)z^i$, where each h_i denotes a polynomial depends on x and y only.
From Eq. (4), the terms of z^{n+1} match

$y\frac{\partial h_n}{\partial y} = 0$, that is $h_n = h_n(x)$, where $h_n(x)$ denotes a polynomial depends on x.
Through computation of the terms of z^n in Eq. (4), the following can be generated:

$$y\,\frac{\partial h_{n-1}}{\partial y} + y\,\frac{\partial h_n}{\partial x} - x\frac{\partial h_n}{\partial y} - (1+bx)n\,h_n = 0,$$

Solving the above partial differential equation for h_{n-1} leads to

$$h_{n-1} = -y\,\frac{dh_n}{dx} + (1+bx)n\,h_n\,lny + G_{n-1}(x), \quad (16)$$

where $G_{n-1}(x)$ denotes a polynomial in the variable x.
Since h_{n-1} signifies a polynomial then from Eq. (16) must be $n\,h_n(x) = 0$.
The following cases can be considered.
Case 1. If $n = 0$, that is $H = h_0(x,y)$. Imposing that $H = h_0(x,y)$ satisfies Eq. (4) and computing the coefficients of z^i, for $i = 0,1$ in Eq. (4) results in
for i=1

$$y\,\frac{\partial h_0(x,y)}{\partial y} = 0.$$

Solving the above equation leads to $h_0(x,y) = h_0(x)$.
For i=0

$$y\,\frac{\partial h_0(x)}{\partial x} = 0.$$

Solving the above equation leads to $h_0(x) = c$, where c is a constant. This is contradiction.
This completes the proof.
Case 2. If $h_n(x) = 0$, then $h_n = 0$, this means that $H = h_0(x,y)$.
The arguments of the proof for this case is similar also to case 1.□

Theorem 2.5. *For System (1) there is no first integrals of rational type.*
Proof. Based on the Propositions 1.1 and 1.2. system (1) has a rational first integral if it has a polynomial first integral or it has two Darboux polynomials with the same cofactor. But by Theorems 2.3 and 2.4, system (1) does not have any polynomial first integrals and it has only one Darboux polynomial.
Thus system (1) has no rational first integral. This completes the proof.□

Theorem 2.6. *The next two statements hold for system (1).*

a) *For $a \neq 0$, the distinctive independent exponential factors of the autonomous system (1) is e^x, with the cofactor y, with an exception if $b = 0$ an extra exponential factor $e^{-\frac{a}{2}x^2-z}$ can be derived with cofactor z.*
b) *For $a = 0$, the exclusive independent exponential factors of system (1) is e^x, with the cofactor y, with the exception of $b = 0$ then an extra exponential factor e^{-z} can be derived with cofactor z.*

Proof. (a) Let $F = e^{\frac{g}{h}}$ be an exponential factor of system (1) with cofactor L, where $g, h \in \mathbb{C}[x,y,z]$ with $(g,h) = 1$.
Now, taking into consideration Theorem 2.3 and Proposition 2.4, h is a constant, put $h = 1$. Thus $F = e^g$ and g satisfies Eq. (6)

$$y\,\frac{\partial e^g}{\partial x} + (y\,z - x)\frac{\partial e^g}{\partial y} + (-z - a\,x\,y - b\,x\,z)\frac{\partial e^g}{\partial z} = Le^g.$$

The above equation becomes

$$y\,\frac{\partial g(x,y,z)}{\partial x} + (y\,z - x)\frac{\partial g(x,y,z)}{\partial y} - (z + a\,x\,y + b\,x\,z)\frac{\partial g(x,y,z)}{\partial z} = L, \quad (17)$$

where $L = d_0 + d_1x + d_2y + d_3z$ (18).
Here, g is written by way of a polynomial in the variable of z in the formula $g(x,y,z) = \sum_{i=0}^{n} g_i(x,y)z^i$, where each g_i denotes a polynomial only in x and y.
First, assume that $n > 1$.
Compute the coefficients in Eq. (17) of z^{n+1} leads to
$y\,\frac{\partial g_n}{\partial y} = 0$ that is $g_n(x,y) = G_n(x)$, where $G_n(x)$ denotes a polynomial of x.
Now compute the coefficients in Eq. (17) of z^n results in

$$y\,\frac{\partial g_{n-1}}{\partial y} + \frac{dg_n}{dx} + (-1 - b\,x)n\,g_n = 0.$$

The solution of above equation is

$$g_{n-1} = -y\frac{dg_n}{dx} + (1 + b\,x)\,n\,g_n lny + G_{n-1}(x),$$

where $G_{n-1}(x)$ denotes a polynomial of x.
Since g_{n-1} is a polynomial then must be $g_n = 0$.
Therefore $g(x,y,z) = g_0(x,y) + g_1(x,y)z$.
Equation (17) becomes

$$y\,\frac{\partial(g_0 + g_1z)}{\partial x} + (y\,z - x)\frac{\partial(g_0 + g_1z)}{\partial y} + (-z - axy - b\,x\,z)\,\frac{\partial(g_0 + g_1z)}{\partial z} = L$$

Compute the coefficients of $z^i, i = 0,1,2$, the following equations will obtain:

i=2: $y\,\frac{\partial g_1(x,y)}{\partial y} = 0,$ (19)

i=1: $y\,\frac{dg_1}{dx} + y\,\frac{\partial g_0}{\partial y} + (-1 - b\,x)\,g_1 = d_3,$ (20)

i=0: $y\,\frac{\partial g_0}{\partial x}+(-x)\frac{\partial g_0}{\partial y}+(-axy)\,\frac{\partial g_0}{\partial z}-b\,x\,g_1=d_0+d_1x+d_2y.$ (21)
From Eq. (19) results $g_1(x,y)=g_1(x)$, where $g_1(x)$ is a polynomial of x.
Solving Eq. (20) leads to
$g_0(x,y)=-y\,\frac{dg_1}{dx}+(g_1+b\,x\,g_1+d_3)lny+F(x)$, where $F(x)$ is a polynomial of x.
Since g_0 is a polynomial then must be $g_1+b\,x\,g_1+d_3=0$.
Then $g_1(x)=\frac{-d_3}{1+b\,x}$ (22)
Case 1. If $b\neq 0$ then from Eq. (22) must be $g_1(x)=d_3=0$, so $g_0(x,y)=g_0(x)$.
Now from solving the differential equation (21) leads to
$g_0(x)=\frac{d_1x^2}{2\,y}+d_2x+\frac{d_0x}{y}+c.$
Since g_0 must be a polynomial, then must be $d_0=d_1=0$.
Hence $g(x,y,z)=c+d_2x$ with the cofactor $L=d_2y$.
Case 2. For the case of $b=0$ then from Eq. (22), $g_1(x)=-d_3$, so $g_0(x,y)=g_0(x)$.
Solving Eq. (21) leads to
$g_0(x)=\frac{-1}{2}a\,d_3x^2+\frac{d_1x^2}{2\,y}+d_2x+\frac{d_0x}{y}+c.$
Since g_0 must be a polynomial, then must be $d_0=d_1=0$.
Hence $g(x,y,z)=\frac{-1}{2}a\,d_3x^2+d_2x-d_3z+c$ with the cofactor $L=d_2\,y+d_3\,z$.
This conclude the proof. □
Proof. (b) It can be noted that in view of Proposition 1.4 and Theorem 2.3, when $a=0$ the exponential factors of system (1) has the form $E=e^{\frac{g}{z^s}}$ for some non-negative integer s, and $g\in\mathbb{C}[x,y,z]$ is a polynomial of $x,y,$ and z, such that g and z^s are prime. Then from Theorem 2.3 and that E satisfies the following partial differential equations depending on the definition of exponential factor

$$y\,\frac{\partial}{\partial x}\left(e^{\frac{g}{z^s}}\right)+(y\,z-x)\frac{\partial}{\partial y}\left(e^{\frac{g}{z^s}}\right)+(-z-b\,x\,z)\frac{\partial}{\partial z}\left(e^{\frac{g}{z^s}}\right)=L\,e^{\frac{g}{z^s}},$$

where L defined in Eq. (18).
After simplifying the above equation, it becomes
$y\,\frac{\partial g(x,y,z)}{\partial x}+(y\,z-x)\frac{\partial g(x,y,z)}{\partial y}-z(1+b\,x)\frac{\partial g(x,y,z)}{\partial z}+s\,(1+b\,x)\,g(x,y,z)=L\,z^s$ (23)
Case 1. for $s\geq 1$, in this situation, denoting the restriction of g to $z=0$ by $\tilde{g}$ in Eq. (23), it can be derived that $\tilde{g}\neq 0$ (if not, g would become divisible by $1+bz$, which is impossible or illogical) and $\tilde{g}$ satisfies
$y\,\frac{\partial\tilde{g}(x,y)}{\partial x}+(-x)\frac{\partial\tilde{g}(x,y)}{\partial y}+s\,(1+b\,x)\,\tilde{g}(x,y)=0.$
Solving the above linear differential equation results in
$\tilde{g}(x,y)=F(x^2+y^2)e^{-s\left(-b\,y+\tan^{-1}\frac{x}{y}\right)}$
but $s\neq 0$ that is $\tilde{g}(x,y)=0$ which is contradiction and this case is illogical.
Case 2. For $s=0$, thus $E=e^g$, where $g\in\mathbb{C}[x,y,z]$ is a polynomial of degree $n\in\mathbb{N}$.
From Theorem (2.6.a) put $a=0$ results in $g(x,y,z)=(c+d_2x)-d_3z$ with the cofactor $L=d_2\,y+d_3z$.
This completes the proof. □

Theorem 2.7. *For System (1) there is no Darboux first integrals for any arbitrary values of a and b.*
Proof. Based on Theorem 1.6, the system (1) contains a Darboux first integral if and only if λ_i and $\mu_j\in\mathbb{C}$ exists and not all zero where Eq. (7) is satisfied, and where p represents the numbers of Darboux polynomials, and q represents the number of exponential factors.

It is consistent with Theorems 2.3 and 2.6 the following cases:

1) When $a,b\neq 0$, by Theorem 2.3 system (1) has no Darboux polynomial and by Theorem 2.6.a there is only one cofactor of the form $L_1=d_2\,y$. Thus Eq. (7) becomes $\mu_1y=0$. Solving this equation, $\mu_1=0$ was obtained.
2) When $a\neq 0,$ and $b=0$ by Theorem 2.3 system (1) has no Darboux polynomial and by Theorem 2.6.a there are two cofactors of the form $L_1=d_2\,y$ and $L_2=d_3\,z$. Thus Eq. (7) becomes $\mu_1y+\mu_2z=0$. Solving this equation leads to $\mu_1=\mu_2=0$.
3) When $a=0,$ and $b\neq 0$ by Theorem 2.3 system (1) has one Darboux polynomial with cofactor $K_1=-1-bx$ and by Theorem 2.6.b there is one cofactor of the form $L_1=d_2\,y$. Thus Eq. (7) becomes $\lambda_1(-1-bx)+\mu_1z=0$. Solving this equation leads to $\lambda_1=\mu_1=0$.
4) When $a=b=0$ by Theorem 2.3 system (1) has one Darboux polynomial with cofactor $K_1=-1$ and by Theorem 2.6.b there are two cofactors of the form $L_1=d_2\,y$ and $L_2=d_3\,z$. Thus Eq. (7) becomes $\lambda_1(-1)+\mu_1y+\mu_2z=0$. Solving this equation leads to $\lambda_1=\mu_1=\mu_2=0$.

This completes the proof. □

Theorem 2.8. *System (1) has no analytic first integrals for any arbitrary values of a and b in a neighborhood located at the origin.*

Proof. Since the origin is a unique equilibrium point of system (1), then the Jacobian matrix at (0,0,0) of system (1) is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

A characteristic equation of J is

$$p(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1 = 0.$$

Then the eigenvalues of J are $\lambda_1 = -1,\ \lambda_2 = i,$ and $\lambda_3 = -i$.

Since there is not exist $k_1, k_2,$ and k_3 positive integers such that

$$k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3 = 0 \text{ and } k_1 + k_2 + k_3 > 0,$$

hence by Theorem 1.3 it follows the result of Theorem 2.8. □

Conclusion:

By the end of this paper the following conclusions are achieved; first is that the system (1) has only one invariant algebraic surface $-1 - \mathrm{bx} = 0$ only when $\mathrm{a} = 0$ (refers to Theorem 2.3). Secondly, the system (1) does not have polynomial, rational and Darboux first integral (refer to Theorem 2.4, 2.5 & 2.7). Finally, it has been shown that it has no analytic first integral in the neighborhood of the origin (refers to Theorem 2.8).

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Salahaddin.

Authors' contributions statement:

Adnan A.J. conceived of the presented idea. Adnan A.J. and Azad I.A. developed the theory and verified the analytical methods. Azad I.A. and Nejmaddin A.S. supervised the findings of this work. Adnan A.J. wrote the manuscript. All authors discussed the results and contributed to the final manuscript.

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التكامل الداربوكس لتعميم النظام الفوضوي الثلاثي الأبعاد Sprott ET9

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الخلاصة:

في هذا البحث تم دراسة التكامل الاول من نوع داربوكس لتعميم النظام الفوضوي الثلاثي الابعاد Sprott ET9 . حيث وضحنا ان النظام لايمتلك متعددة حدود . دالة كسرية، تحليلية والداربوكس للتكامل الاول لاي قيمتين a و b . كما استطعنا ايجاد متعددة حدود داربوكس لهذا النظام بقرب المفكوك الاسي. باستخدام وزن متعددة الحدود المتجانسة التي ساعدتنا في برهان الطريقة.

الكلمات المفتاحية: التكامل الاول التحليلي ، التكامل الاول داربوكس ، المفكوك الاسي.