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Principally Quasi-Injective Semimodules

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Abstract:

In this work, the notion of principally quasi- injective semimodule is introduced, discussing the conditions needed to get properties and characterizations similar or related to the case in modules.

Let $\mathcal{B}$ be an $\mathcal{R}$-semimodule with endomorphism semiring $\mathcal{S}$. The semimodule $\mathcal{B}$ is called principally quasi-injective, if every $\mathcal{R}$ -homomorphism from any cyclic subsemimodule of $\mathcal{B}$ to $\mathcal{B}$ can be extended to an endomorphism of $\mathcal{B}$.

Key words: Principally quasi-injective semimodules, (injective, quasi-injective) semimodules, semimodules

Introduction:

The study of semimodules over semirings has a long history where the construction of semirings is useful generalizations of rings. Semirings are moved from rings but simultaneously there are important differences of them. A semiring is a nonempty set $\mathcal{R}$ together with two operations, addition and multiplication such that
(i) addition and multiplication are associative,(ii) addition is commutative,(iii) the distribution law holds, that is, if r, s, t $\in \mathcal{R}$ then $r(s + t) = r\ s + r\ t$ and $(r + s)t = r\ t + s\ t$, (iv) there is an additive identity element(denoted 0) and a multiplicative identity element(denoted l), (v)these two operations are associative, $\mathcal{R}$ is commutative if the second operation is commutative. For instance the set of natural number $\mathbb{N}$ is a commutative semiring under usual addition and multiplication, but it is not ring. A semimodule $\mathcal{B}$ over semiring $\mathcal{R}$ is defined similarly in module over ring. A subsemimodule $\mathcal{U}$ of an $\mathcal{R}$-semimodule $\mathcal{B}$ is a nonempty subset of $\mathcal{B}$, if b, $b' \in \mathcal{U}$ and $t \in \mathcal{R}$, then $b+b' \in \mathcal{U}$ and $t\ b \in \mathcal{U}$, This means $\mathcal{U}$ itself is an $\mathcal{R}$-semimodule. An $\mathcal{R}$-semimodule N is called ($\mathcal{B}$-injective), if for any subsemimodule $\mathcal{U}$ of $\mathcal{B}$, any homomorphism from $\mathcal{U}$ into N can be extended to an $\mathcal{R}$-homomorphism from $\mathcal{B}$ to N. And N is injective if it is injective relative to every $\mathcal{R}$-semimodule. It is quasi-injective semimodule if it is N-injective.

In the present work, we discuss new object ″ principally quasi-injectivity ″ for a unitary left $\mathcal{R}$-semimodule $\mathcal{B}$ over a commutative semiring with identity.

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Some remarks that needed in this work. were added. Nicholson, Park and Yousif (1) were studied principally quasi-injective modules, where $\mathcal{B}$ is called principally quasi-injective module if each $\mathcal{R}$-homomorphism from a principal submodule of $\mathcal{B}$ to $\mathcal{B}$ can be extended to an endomorphism of $\mathcal{B}$, an analogous, that concept for semimodules was introduced, studied the relationship between it and endomorphisms semiring. Further we examined their relations with other concepts like, principally- injective, self-generators, regular, Z-regular semimodules. Before that we added some remarks which we need in our work. Also we gave some characterizations of principally quasi-injective semimodules.

This paper is organized as follows

- In section 2: we discuss some definitions, properties and remarks that lead to the main results.
- In section 3: we study principally quasi-injective semimodules and other related concepts with some properties about those concepts.

Preliminaries

In this section some definitions were demonstrated, properties and remarks that derive the main results.

Definition 1 (2). A nonempty subset $\mathcal{I}$ of a semiring $\mathcal{R}$ is a right (resp. left) ideal of $\mathcal{R}$ if for $s, s' \in \mathcal{I}$ and $t \in \mathcal{R}$ then $s+s' \in \mathcal{I}$ and $s\ t$ (resp. ts)$\in \mathcal{I}$. $\mathcal{I}$ is (two-sided) ideal of $\mathcal{R}$ if it is both a left and a right ideal of $\mathcal{R}$.

The concept principal ideal in commutative semiring with an identity element can be defined on the similar as in commutative ring with an identity element. (3)

Definition 2. Let $\mathcal{R}$ be a semiring, then for any $a \in \mathcal{R}$,

$\mathcal{R}a = \{x: x = ta$, for some $t \in \mathcal{R}\}$ is left ideal of $\mathcal{R}$ called the **principal left ideal** generated by a.

Definition 3 (2). Let $\mathcal{R}$ be a semiring. A **left $\mathcal{R}$-semimodule** is a commutative monoid $(\mathcal{B}, +)$ which has a zero element, together with a mapping $\mathcal{R} \times \mathcal{B} \to \mathcal{B}$ (sending (s, b) to sb) such that the following conditions hold $\forall$ s, t of $\mathcal{R}$ and $\forall b, b'$ of $\mathcal{B}$:

(i) $(s\,t)\,b = s\,(t\,b)$

(ii) $s\,(b + b') = s\,b + s\,b'$

(iii) $(s + t)\,b = s\,b + t\,b$

(iv) $s\,0_{\mathcal{B}} = 0_{\mathcal{B}} = 0_{\mathcal{R}}b$

If the condition $1\,b = b$, for all b in $\mathcal{B}$ holds then the semimodule $\mathcal{B}$ is called **unitary**.

Definition 4 (2). A nonempty subset $Ụ$ of a left $\mathcal{R}$-semimodule $\mathcal{B}$ is called **subsemimodule** of $\mathcal{B}$ if $Ụ$ is closed under addition and scalar multiplication, and denoted by $Ụ \leq \mathcal{B}$.

Examples 5.

(i) Every semiring over itself is a semimodule.

(ii) let $\mathcal{R}=(\mathbb{Z}^+, +, .)$ where $\mathbb{Z}^+$ is a positive integers and $a'+a''= max\{a', a''\}$, $a'.a'' = min\{a', a''\}$, $\forall a', a'' \in \mathbb{Z}^+$

let $\mathcal{B}$ be a left $\mathcal{R}$- semimodule over itself, the proper subsemimodules of $\mathcal{B}$ are of the form $(K_n, +, .) = \{1, 2, \ldots, n\} \subseteq \mathbb{Z}^+$, for each n.

(iii) Let $\mathcal{B} = \mathbb{Z}_8$ be an $\mathcal{R}$- semimodule, where $\mathcal{R}$ is the semiring $\mathbb{Z}_8$, the proper subsemimodules of $\mathcal{B}$ are $\{\bar{0}\}, \{\bar{0}, \bar{4}\}, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ also $\mathbb{Z}_6$ as $\mathbb{Z}^+$-semimodule have proper subsemimodules $\{\bar{0}\}, \{\bar{0}, \bar{3}\}, \{\bar{0}, \bar{2}, \bar{4}\}$.

Definition 6 (2). A subsemimodule $Ụ$ of $\mathcal{B}$ is called a subtractive subsemimodule, if for each $b, b' \in \mathcal{B}$, that $b + b', b \in Ụ$ leads to $b' \in Ụ$. It is clear that $\{0\}$ and $\mathcal{B}$ are subtractive subsemimodules of $\mathcal{B}$. A semimodule $\mathcal{B}$ is called subtractive semimodule if it has only subtractive subsemimodules.

In Example (5(ii)) K_n is subtractive subsemimodule of $\mathcal{B}$. since for any element $x \in K_n$ and $z \in \mathbb{Z}^+$ such that $x+z = max\{x, z\} \in K_n$ implies that $z \in K_n$.

Definition 7 (4). A semimodule $\mathcal{B}$ is called a semisubtractive, if for any $b, b' \in \mathcal{B}$ there is always some $h \in \mathcal{B}$ satisfying $b+h = b'$ or some $k \in \mathcal{B}$ satisfying $b'+k = b$.

Definition 8 (2). An element a' of a left $\mathcal{R}$-semimodule $\mathcal{B}$ is cancellable if $a'+ n = a'+ k$ implies that $n=k$. The $\mathcal{R}$-semimodule $\mathcal{B}$ is cancellative if and only if every element of $\mathcal{B}$ is cancellable.

Definition 9 (5). An $\mathcal{R}$-semimodule $\mathcal{B}$ is said to be a **direct sum** of subsemimodules $Ụ_1, Ụ_2, \ldots, Ụ_k$ of $\mathcal{B}$, if each $b \in \mathcal{B}$ can be uniquely written as $b=u_1+u_2+\ldots+u_i$ where $u_i \in Ụ_i$, $1 \leq i \leq k$. It is denoted by

$\mathcal{B} = Ụ_1 \oplus Ụ_2 \oplus \ldots \oplus Ụ_k$. And $Ụ_i$ is called a direct summand of $\mathcal{B}$.

It is known that if a module $\mathcal{B}$ is a direct sum of submodules $Ụ$ and $Ụ_1$, then $\mathcal{B} = Ụ \oplus Ụ_1$ if and only if

$\mathcal{B} = Ụ + Ụ_1$ and $Ụ \cap Ụ_1 = \{0\}$ This is not true, in general for semimodule. We will prove this property under certain conditions on a semimodule.

The following remark proves the same property.

Remark 10. let $\mathcal{B}$ be a cancellative semisubtractive $\mathcal{R}$- semimodule and each subsemimodule of it is subtractive, then $\mathcal{B} = Ụ \oplus Ụ_1$ if and only if $\mathcal{B} = Ụ + Ụ_1$ and $Ụ \cap Ụ_1 = \{0\}$.

Proof: ($\Longrightarrow$) Assume that $\mathcal{B} = Ụ \oplus Ụ_1$ we must to prove that $\mathcal{B} = Ụ + Ụ_1$ and $Ụ \cap Ụ_1 = \{0\}$.

If $\mathcal{B} = Ụ \oplus Ụ_1$ this means, for each $b \in \mathcal{B} \Longrightarrow b = u+u'$, $u \in Ụ$, $u' \in Ụ_1 \Longrightarrow \mathcal{B} = Ụ + Ụ_1$

If $b \in Ụ \cap Ụ_1 \Longrightarrow (b = b + 0) \in Ụ$ and$(b = 0+b) \in Ụ_1 \Longrightarrow b=0$ and $0 = b$ (by uniqueness).

($\Longleftarrow$) Asssume $\mathcal{B} = Ụ + Ụ_1$ and $Ụ \cap Ụ_1 = \{0\}$. We will prove $\mathcal{B} = Ụ \oplus Ụ_1$.

Suppose that $b = u+ u' = v+ v'$ where $u, v \in Ụ$, $u', v' \in Ụ_1$ Since $\mathcal{B}$ is semisubtractive, then there is h in $\mathcal{B}$ and there is two cases:

Case1 $u=h+ v$ (since $Ụ$ is subtractive, then $h \in Ụ$) $\Longrightarrow h + v + u' = v + v'$ (by cancellative) $\Longrightarrow$ $h + u' = v'$. But $Ụ_1$ is subtractive subsemimodule of $\mathcal{B}$, then $h \in Ụ_1$. We have $Ụ \cap Ụ_1 = \{0\}$ which implies $h=0$ and hence $u' = v'$.

case2 $u+ h=v$ ($Ụ$ is subtractive, then $h \in Ụ$) $\Longrightarrow u + u' = u + h + v'$ (by cancillative) $\Longrightarrow u' = h + v' \in Ụ_1$(by subtractive), then $h \in Ụ_1$. We have $Ụ \cap Ụ_1 = \{0\}$ implies $h=0$ and hence $u'=v'$.

Similarly, we show that $u=v$. Therefor the representation is unique. ////

Definition 11 (4). Let $\mathcal{B}$ be a left $\mathcal{R}$-semimodule and $b \in \mathcal{B}$, the **left annihilator** of b in $\mathcal{R}$ is defined by $ann_{\mathcal{R}}(b) = \{t \in \mathcal{R} \mid tb = 0\}$, it is clear that $ann_{\mathcal{R}}(b)$ is a left ideal of $\mathcal{R}$. Also if $Ụ$ subsemimodule of $\mathcal{B}$, then $ann_{\mathcal{R}}(Ụ) = \{ t \in \mathcal{R} \mid tu = 0, \forall u \in Ụ\}$.

Definition 12 (5). If $\mathcal{R}$ is a semiring and $\mathcal{B}$, N are left $\mathcal{R}$-semimodules, then a map $\psi: \mathcal{B} \to N$ is called a **homomorphism** of $\mathcal{R}$-semimodules, if :

(i) $\psi(b + b') = \psi(b) + \psi(b')$

(ii) $\psi(t\,b) = t\,\psi(b)$, for all $b, b' \in \mathcal{B}$ and $t \in \mathcal{R}$.

The set of $\mathcal{R}$-homomorphisms of $\mathcal{B}$ into N is denoted by Hom $(\mathcal{B}, N)$. A homomorphism ψ is called an epimorphism if its onto, it is called a monomorphism if ψ is one-one and it is isomorphism if ψ is one-one and onto.

Remarks 13 (4).

For a homomorphism of $\mathcal{R}$-semimodules $\psi: \mathcal{B} \to N$ we define

(i) $ker(\psi) = \{ b \in \mathcal{B} \mid \psi(b) = 0 \}$

(ii) $\psi(\mathcal{B}) = \{ \psi(b) \mid b \in \mathcal{B} \}$

(iii) $Im(\psi) = \{n \in N \mid n + \psi(b) = \psi(b')$ for some $b, b' \in \mathcal{B}\}$

It is obvious that $ker(\psi)$ is a subtractive subsemimodule of $\mathcal{B}$, $Im(\psi)$ is a subtractive subsemimodule of N and $\psi(\mathcal{B})$ is a subsemimodule of N. In module theory $\psi(\mathcal{B}) = Im(\psi)$, in semimodule theory is not true always. It is clear that $\psi(\mathcal{B}) \subseteq Im(\psi)$, the equality is satisfied if $\psi(\mathcal{B})$is subtractive subsemimodule of N.

It is known that in module theory, a homomorphism $\psi: \mathcal{B} \to N$ of $\mathcal{R}$-modules is monomorphism (one-one) if and only if $ker(\psi) = 0$. But in semimodule theory that is not true always. For instance, see (6, p. 176).

The following remark explains the relationship between monomorphism and kernel of $\mathcal{R}$-semimodules.

Remark 14. Let $\psi: \mathcal{B} \to N$ be a homomorphism of $\mathcal{R}$-semimodules, then:

(i) If ψ is a monomorphism, then $ker(\psi) = 0$.

(ii) If $ker(\psi) = 0$, $\mathcal{B}$ is semisubtractive and N is cancellative, then ψ is a monomorphism.

Proof: (i) Let b be any element of $\mathcal{B}$, then $0 = 0\,\psi(b) = \psi(0b) = \psi(0)$. Hence $0 \in ker\,\psi$.

If $\psi(b') = 0$, then $\psi(b') = \psi(0)$. But ψ is one to one implies $b' = 0$. Therefor $ker(\psi) = \{0\}$.

(ii) Let $\psi(b_1) = \psi(b_2)$ since $\mathcal{B}$ is semisubtractive semimodule, then there is h in $\mathcal{B}$ such that $b_1+h=b_2$ or some k in $\mathcal{B}$ satisfying $b_2+k=b_1$, we have two cases

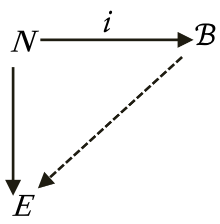
Case1 $b_2= b_1+h \Longrightarrow \psi(b_2) = \psi(b_1) + \psi(h) \Longrightarrow \psi(h) = 0$ (by cancellative) since $ker(\psi) = 0$, then $h = 0$ this implies $b_1= b_2$.

Case 2 $b_1 = b_2+k \Longrightarrow \psi(b_1) = \psi(b_2) + \psi(k) \Longrightarrow \psi(k) = 0$ (by cancellative), since $ker(\psi)=0$, then $k = 0$, hence $b_2= b_1$. Therefor ψ is a monomorphism. ////

Definition 15(7). Let $\{\mathcal{B}_i\}_{i\in I}$ be a family of left $\mathcal{R}$-semimodules then their **Cartesian product** $\prod_{i\in I}\mathcal{B}_i$ also has the structure of a left $\mathcal{R}$-semimodule under componentwise addition and scalar multiplication. It is called the direct product of $\{\mathcal{B}_i\}$. By the direct sum of $\{\mathcal{B}_i : i \in I\}$ denoted by $\oplus_{i\in I}\mathcal{B}_i$ we mean the subset of $\prod_{i\in I}\mathcal{B}_i$ consisting of all $(m_i) \in \prod_{i\in I}\mathcal{B}_i$ for which only finite number of $m_i \neq 0$. Then $\oplus_{i\in I}\mathcal{B}_i$ is a left $\mathcal{R}$-subsemimodule of $\prod_{i\in I}\mathcal{B}_i$.

Definition 16 (8). A left $\mathcal{R}$-semimodule $\mathcal{B}$ is called **cyclic** if $\mathcal{B}$ can be generated by a single element, that is $\mathcal{B} = \langle b \rangle = \mathcal{R}b = \{t\,b \mid t \in \mathcal{R}\}$for some $b \in \mathcal{B}$

Definition 17 (7). An $\mathcal{R}$-semimodule E is $\mathcal{B}$-**injective** (E is injective relative to $\mathcal{B}$) if, for each subsemimodule N of $\mathcal{B}$, any $\mathcal{R}$-homomorphism from N to E can be extended to an $\mathcal{R}$-homomorphism from $\mathcal{B}$ to E. (where i is the inclusion map)



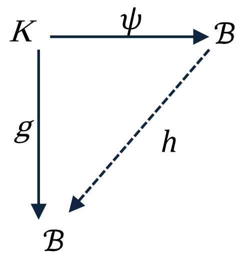
A left $\mathcal{R}$-semimodule E is **injective** if it is injective relative to every left $\mathcal{R}$-semimodule.

Proposition 18 (7). Let $(E_\alpha)_{\alpha\in\Omega}$ be an indexed set of a left $\mathcal{R}$-semimodules then $\prod_\Omega E_\alpha$ is injective if and only if each $E_\alpha$is injective for each α.

Definition 19 (9). A nonzero $\mathcal{R}$-semimodule $\mathcal{B}$ is called **simple** if $\mathcal{B}$ has no nonzero proper $\mathcal{R}$-subsemimodule.

Remark 20 (9). If $\mathcal{B}$ is simple, then every semimodule E is injective relative to $\mathcal{B}$.

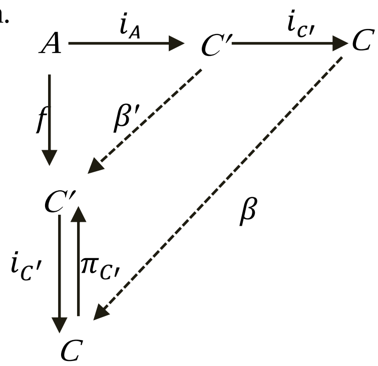
Remark 21 (7). A semimodule $\mathcal{B}$ is **quasi - injective** if it is $\mathcal{B}$-injective. As the following diagram. i.e., there exist h such that $h\psi = g$ (with ψ is a monomorphism).



To the best of our knowledge, the following proposition is not found in the literatures, we will give its proof for semimodules similar to in modules.

Proposition 22. A direct summand of quasi-injective semimodule is quasi-injective.

Proof: Let $C = C' \oplus C''$ be quasi-injective semimodule and let i_A and $i_{c'}$ be the inclusion maps of A into C' and C' into C respectively. Let $\pi_{c'}: C \to C'$ be the projection map. Consider the following diagram.



since C is quasi-injective semimodule, then there exists a homomorphism β:C→C such that $\beta i_{c'} i_A = i_{c'} f$

take $\beta' = \pi_{c'}\beta i_{c'}$

then $\beta' i_A = \pi_{c'}\beta i_{c'} i_A = \pi_{C'} i_{C'} f = 1_{c'} f = f$

this mean, β'extends to an endomorphism of C'. ////

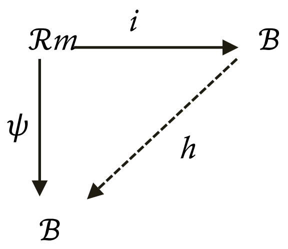
Remark 23. It is clear that every injective semimodule is quasi-injective.

Principally Quasi-Injective Semimodules

In this section we extend this work by studying principally quasi-injective semimodules, their endomorphism semirings, also we discuss some concepts which have relation to this notion, Most of the results of this section are shown (for modules) in (1) and (10). However, we discuss it for semimodule.

In (1) some results for injective modules were given. In the following, we state analogous to those results for semimodule.

Definition 1. An $\mathcal{R}$-semimodule is called **principally quasi-injective** if each $\mathcal{R}$-homomorphism from cyclic subsemimodule of $\mathcal{B}$ to $\mathcal{B}$ can be extended to an endomorphism of $\mathcal{B}$. In other words, the following diagram is commutative. i.e., $hi = \psi$.



Note. We will use the notation P.Q.-injective for principally quasi-injective.

Examples 2.

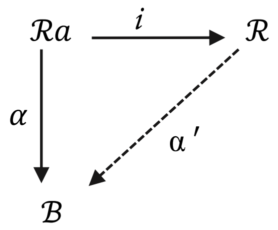
(i)Every injective semimodule is P.Q.-injective .

(ii)Every semi-simple semimodule is P.Q.-injective and hence every simple semimodule is P.Q.-injective.

(iii) $\mathbb{Z}_2$ as $\mathbb{N}$-semimodule is P.Q.-injective but not not injective.

Proposition 3. Every direct summand of P.Q.-injective semimodule is again P.Q.-injective. **proof:** Similar to Proposition (22). ////

In (10) principally injective module was introduced as follows: an $\mathcal{R}$-module $\mathcal{B}$ is called principally injective (p-injective) if each $\mathcal{R}$-homomorphism $\alpha : \mathcal{R}a \to \mathcal{B}$ such that $a \in \mathcal{R}$, extends to $\mathcal{R}$, i.e., the following diagram is commutative, $\alpha' i = \alpha$. Where i is inclusion map.



For instance $\mathbb{Z}$ as $\mathbb{Z}$-semimodule is not p-injective, let $f{:}\mathbb{Z}2 \to \mathbb{Z}$ be $\mathbb{Z}$-homomorphism such that $2x \mapsto 3x$ can not be extended to $g{:}\mathbb{Z} \to \mathbb{Z}$.($g$ from $\mathcal{R} = \mathbb{Z}$ to $\mathcal{B} = \mathbb{Z}$)since, if $g(1) = 3n$ then $g(2) = 6n$ but $f(2) = 3 \Longrightarrow f(2) \neq g(2)$ this contradiction, then g is not an extension of f, so $\mathbb{Z}$ as a $\mathbb{Z}$-semimodule is not p-injective.

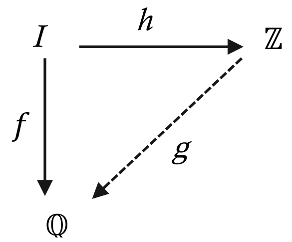
In (7) Ahsan, Shabir and Liu introduced P-injective semimodule as follows.

Definition 4 (7). An $\mathcal{R}$-semimodule $\mathcal{B}$ is called **P-injective** if for any principal ideal $\mathcal{U}$ of $\mathcal{R}$ and each $\mathcal{R}$-homomorphism f: $\mathcal{U} \to \mathcal{B}$, there exists an $\mathcal{R}$-homomorphism g: $\mathcal{R} \to \mathcal{B}$, which extends f.

Example 5. $\mathbb{Q}$ as a $\mathbb{Z}$ semimodule is P-injective.

Proof: Let $I = \mathbb{Z}n$ where $n \in \mathbb{Z}$ (principal ideal of $\mathbb{Z}$) and $f{:}I \to \mathbb{Q}$ be $\mathbb{Z}$- homomorphism such that $f(n) = q$ where, $n \in I$, $q \in \mathbb{Q}$, define a $\mathbb{Z}$-homomorphism $g{:}\mathbb{Z} \to \mathbb{Q}$

by $g(1) = \frac{q}{n}$, consider the following diagram:



Then $g(kn) = kn\ g(1) = (kn)\frac{q}{n} = kq = kf(n) = f(kn)$.

The concept "regular module" is defined by several forms see (11), (12) and (13). In this work we will choose the certain condition to define a regular semimodule. Also we investigate relation this concept with P.Q-injective semimodule where every regular semimodule is P.Q-injective.

Examples 6.

(i)It is clear that every injective semimodule is principally injective.

(ii) Every regular semimodule is P.Q.-injective semimodule. In fact, if $\mathcal{R}x \leq \mathcal{B}$, then $\mathcal{R}x$ is a direct summand of $\mathcal{B}$, there exists $B \leq \mathcal{B}$ such that $\mathcal{B} = \mathcal{R}x \oplus B$. Now let $\alpha{:}\mathcal{R}x \to \mathcal{B}$ be a homomorphism. Define α': $\mathcal{R}x \oplus B \to \mathcal{R}x \oplus B$ by $\alpha'(tx, y) = \alpha(tx)$; it is clear that α' is an extension of α.

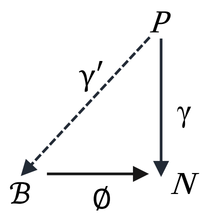
In(13), regular module was defined, where $\mathcal{B}$ is called regular if every cyclic submodule of $\mathcal{B}$ is a direct summand of $\mathcal{B}$.

Definition 7 (8). A semimodule $\mathcal{B}$ is called **regular** if every cyclic subsemimodule of $\mathcal{B}$ is a direct summand.

Example 8. $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is regular**.**

In (13) Z-regular module appeared, where an $\mathcal{R}$-module $\mathcal{B}$ is called Z-regular if every cyclic submodule of $\mathcal{B}$ is projective and direct summand of $\mathcal{B}$. Also in (1) principally self-generator module was studied, analogous concepts for semimodule are introduced. Before we define these concepts we need to define a projective semimodule and give its characteristic.

Definition 9 (6). A left $\mathcal{R}$-semimodule P is said to be $\mathcal{B}$**-projective** if for every an epimorphism $\phi: \mathcal{B} \rightarrow N$ and for every homomorphism γ:$P\rightarrow N$ there is a homomorphism γ': $P\rightarrow \mathcal{B}$ such that the diagram commutes.



A semimodule P is **projective** if it is projective relative to every left $\mathcal{R}$-semimodule.

Example 10. Every semiring over itself is projective.

proposition 11 (6). Let $P_{\iota\in\Gamma}$ be an indexed set of left $\mathcal{R}$-semimodules , then $\oplus P_\iota$ is projective if and only if each P_ι is projective for each ι.

Definition 12. A semimodule $\mathcal{B}$ is called ***Z*-regular** if every cyclic subsemimodule of $\mathcal{B}$ is projective and direct summand.

Remark 13. Note that any Z -regular semimodule is regular, hence it is P.Q.-injective by Examples (6(ii)).

Remark 14(8). For any $\mathcal{R}$-semimodule $\mathcal{B}$, $End_{\mathcal{R}}(\mathcal{B})$ is the set $Ş$ of **endomorphisms** of $\mathcal{B}$, it is a semiring with respect to addition and multiplication defined as follows: $\forall f, g$, $h \in End(\mathcal{B})$, $f+g=h$ where $h(b) = f(b) + g(b)$ for all $b\in \mathcal{B}$, $f\circ g = h$ where $h(b) = f(g(b))$ for all $b\in \mathcal{B}$. It easy to check that $Ş$ is a semiring called the endomorphism semiring of $\mathcal{B}$.

Remark 15. If $\mathcal{B}$ is left $\mathcal{R}$-semimodule then $\mathcal{B}$ can be made into a right $Ş$-semimodule as follows: define, $\Phi: \mathcal{B} \times Ş \rightarrow \mathcal{B}$ by $\Phi(b, f) = bf$, then

(i) $b(f_1+f_2) = bf_1 + bf_2$

(ii) $(b + b')f = bf + b'f$ where f, f_1, $f_2 \in Ş$ and b, $b' \in \mathcal{B}$.

Remarks 16.

(i) $ann_{\mathcal{B}}(t) = \{b\in \mathcal{B} | tb = 0\}$. We will use the notation $r(t)$=$ann_{\mathcal{B}}(t)$, where $t \in \mathcal{R}$.

(ii) $bŞ = \{bf \mid f \in Ş\} = \{bf = f(b) \mid f \in Ş\}$

(iii) $ann_{\mathcal{B}}(ann_{\mathcal{R}}(b))$=$\{x\in \mathcal{B} \mid tx = 0, \forall t \in ann_{\mathcal{R}}(b)\}$=$\{x\in \mathcal{B} \mid tx = 0$ whenever $tb = 0\}$.We will use the notation $r(l(b))$=$ann_{\mathcal{B}}(ann_{\mathcal{R}}(b))$

In (1) some characterizations of P.Q.-injective module were given. In the following, we state and prove analogous to these characterizations for semimodule.

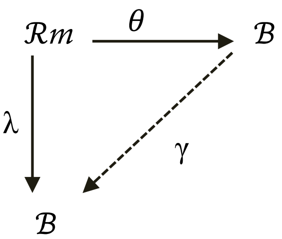
Proposition 17. Given a left $\mathcal{R}$-semimodule with $Ş = End_{\mathcal{R}}(\mathcal{B})$, where $\mathcal{B}$ is cancellative, the following are equivalent:

(i) $\forall m \in \mathcal{B}$, every $\mathcal{R}$-homomorphism $\mathcal{R}m \rightarrow \mathcal{B}$ can be extended to an endomorphism in $Ş$, i.e., $\mathcal{B}$ is P.Q.-injective semimodule.

(ii) $r(l(m)) = mŞ$, $\forall m \in \mathcal{B}$.

(iii) If $l(m)\subseteq l(n)$ where $m, n \in \mathcal{B}$, then $nŞ \subseteq mŞ$.

(iv) $\forall m \in \mathcal{B}$, if the $\mathcal{R}$-homomorphisms $\lambda, \theta: \mathcal{R}m \rightarrow \mathcal{B}$ are given with θ is a monomorphism, then there exists $\gamma: \mathcal{B} \rightarrow \mathcal{B}$ such that $\gamma\theta = \lambda$, i.e., the following diagram commutes:



Proof: (i)$\Longrightarrow$ (ii)

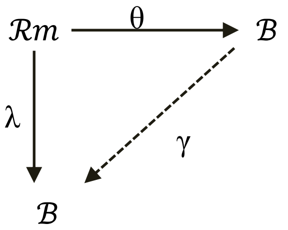
Let $\vartheta(m) \in mŞ$ where $\vartheta \in Ş$. If $t\,m$=0 then $0 = \vartheta(t\,m) = t\,\vartheta(m)$.This implies $\vartheta(m)\in r(l(m))$ hence, $mŞ\subseteq r(l(m)$. To show the opposite inclusion, let $n\in r(l(m)$. Define $\gamma: \mathcal{R}m \rightarrow \mathcal{B}$ by $\gamma(tm) = tn\ \forall t \in \mathcal{R}$. γ is well-defined.

By (i) γ extends to $\gamma' \in Ş$. Now $n = \gamma(m) = \gamma' i(m) = \gamma'(i(m)) = \gamma'(m) \in mŞ$. Hence $r(l(m) \subseteq mŞ$.

(ii)$\Longrightarrow$ (iii) From (ii) $nŞ = r(l(n)$, Since $l(m)\subseteq l(n)$, then $r(l(n)) \subseteq r(l(m))$.Therefore $nŞ$=$r(l(n)) \subseteq r(l(m))$=$mS$, means $nŞ\subseteq mŞ$.

(iii)$\Longrightarrow$ (iv)Since θ is monomorphism, we have $l(\theta(m)) \subseteq l(\lambda(m))$, in fact, let $t\in l(\theta(m))$, then $t\theta(m) = \theta(tm) = 0$. Thus $tm\in ker\ \theta$ hence $tm = 0$, so $\lambda(tm) = t\lambda(m) = 0$ which implies $t\in l(\lambda(m))$, so $l(\theta(m)) \subseteq l(\lambda(m))$. By (iii) $\lambda(m)Ş = \theta(m)Ş$. Then there exists $\gamma \in Ş$ such that $\lambda(m) = \gamma\theta(m))$ as required.

(iv) $\Longrightarrow$(i) Take $\theta: \mathcal{R}m \rightarrow \mathcal{B}$ be the inclusion homomorphism in (iv), then there exists $\gamma: \mathcal{B} \rightarrow \mathcal{B}$ such that the following diagram is commutative. Hence $\lambda: \mathcal{R}m \rightarrow \mathcal{B}$ extends to an endomorphism in $Ş$. This means proving(i). ////



In (1) principally self-generator module was given. In the following we give an analogous of that notion for semimodule.

Definition 18. An $\mathcal{R}$-semimodule $\mathcal{B}$ is said to be **principally self-generator** if for every element $b \in \mathcal{B}$, there exists an epimorphism $\alpha: \mathcal{B} \to \mathcal{R}b$, and then there exists $b' \in \mathcal{B}$ such that $\alpha(b') = b$.

Examples 19.

(i) Every cyclic semimodule is principally self-generator.

(ii) The semiring $\mathcal{R}$ is principally self-generator $\mathcal{R}$-semimodule.

(iii) Every regular semimodule is principally self-generator.

(iv) Every Z-regular semimodule is principally self-generator.

Proof: Clear. ////

Remarks 20. Let $\mathcal{R}$ be a semiring, A is a subset of $\mathcal{R}$, X is a subset of the left semimodule ${}_R\mathcal{R}$ ($\mathcal{R}$ over itself), $a \in \mathcal{R}$ and $x \in X$, then:

(i) $l(A) = \{x \in {}_R\mathcal{R} \mid xa = 0, \forall a \in A\}$.

(ii) $l(a) = l(\{a\}) = \{x \in {}_R\mathcal{R} \mid xa = 0\}$.

(iii) $r(X) = \{t \in \mathcal{R} \mid xt = 0, \forall x \in X\}$.

(iv) $r(x) = r(\{x\}) = \{t \in \mathcal{R} \mid xt = 0\}$.

In the following some properties for P-injective semimodule which introduced in(9) for modules. We dealt those properties by adding specific conditions for semimodule.

Proposition. 21. Let $\mathcal{R}$ be a semiring such that ${}_R\mathcal{R}$ is subtractive, semisubtractive and cancellative. Then, the following conditions are equivalent:

(i) ${}_R\mathcal{R}$ is P-injective as $\mathcal{R}$-semimodule.

(ii) $r(l(a)) = a\mathcal{R}$ for all a in $\mathcal{R}$.

(iii) $l(b) \subseteq l(a)$ where a, b in $\mathcal{R}$, implies $a\mathcal{R} \subseteq b\mathcal{R}$.

(iv) $r[\mathcal{R}b \cap l(a)] = r(b) + a\mathcal{R}$ for all a, b in $\mathcal{R}$. (we will add the conditions $r(b) + a\mathcal{R}$ is subtractive subsemimodule of ${}_R\mathcal{R}$ and ${}_R\mathcal{R}$ is semisubtractive semimodule).

Proof: (i)⟹(ii) $a\mathcal{R} \subseteq r(l(a))$, for $x \in a\mathcal{R} \Rightarrow x = at$ for some $t \in \mathcal{R}$ and so $sa = 0 \Rightarrow sx = s(at) = (sa)t = 0t = 0$ that is $x \in r(l(a))$. Now assume ${}_R\mathcal{R}$ is p-injective. To prove $r(l(a)) \subseteq a\mathcal{R}$. Let $x \in r(l(a))$, this means $sa = 0 \Rightarrow sx = 0$ for each $s \in \mathcal{R}$. So, the map $\mathcal{R}a \to {}_R\mathcal{R}$ by $sa \mapsto sx$, $s \in \mathcal{R}$ is well defined homomorphism which can be extended to a homomorphism, say f: $\mathcal{R} \longrightarrow {}_R\mathcal{R}$ But $x = 1x = f(1a) = f(a1) = af(1) \in a\mathcal{R}$. Therefor $r(l(a)) \subseteq a\mathcal{R}$.

(ii)⟹(iii) $l(b) \subseteq l(a)$ means [$sb = 0$ implies $sa = 0$], so $a \in r(l(b))$ and $r(l(b)) = b\mathcal{R}$ (by(ii)), hence $a\mathcal{R} \subseteq b\mathcal{R}$.

(iii)⟹(i) Let $\alpha: \mathcal{R}a \to {}_R\mathcal{R}$ be an $\mathcal{R}$-homomorphism and let $\alpha(a) = b$ then it is clear that $l(a) \subseteq l(b)$, so by (iii) we have $b\mathcal{R} \subseteq a\mathcal{R}$, let $b = at$. Define $\alpha': \mathcal{R} \to {}_R\mathcal{R}$, by $x \mapsto xt$ for each $x \in \mathcal{R}$ then $\alpha'(a) = at = b = \alpha(a)$, that is α' is an extension of α to ${}_R\mathcal{R}$. Therefore ${}_R\mathcal{R}$ is p-injective.

(iv)⟹(ii) $r[\mathcal{R}b \cap l(a)] = r(b) + a\mathcal{R}$ for all a, b in $\mathcal{R}$. If $b=1$ then $\mathcal{R}b = \mathcal{R}$, $\mathcal{R}b \cap l(a) = l(a)$ and $r(b)=0$ so we get, $r(l(a)) = a\mathcal{R}$.

(iii) ⟹ (iv)Let $x \in r[\mathcal{R}b \cap l(a)]$, then $l(ba) \subseteq l(bx)$ [$t \in l(ba) \Rightarrow t(ba) = 0 \Rightarrow (tb)a = 0 \Rightarrow tb \in \mathcal{R}b \cap l(a) \Rightarrow (tb)x = 0 \Rightarrow t(bx) = 0$ that is $t \in l(bx)$] then, by (iii), it follows $bx\mathcal{R} \subseteq ba\mathcal{R}$ and there is s in $\mathcal{R}$ such that $bx = bas$. Now, since ${}_R\mathcal{R}$ is semisubtractive there is two cases:

Case1 there exists h in ${}_R\mathcal{R}$ such that $x = h + as$, then $bx = bh + bas \Rightarrow bh = 0$ (by cancellative) $\Rightarrow h \in r(b)$, that is $x \in r(b) + a\mathcal{R}$.

Case2 there exists h in ${}_RR$ such that $x + h = as$, then $bx + bh = bas \Rightarrow bh = 0$ (by cancellative)$\Rightarrow h \in r(b)$, that is $x + h \in a\mathcal{R} \subseteq r(b) + a\mathcal{R}$ and $h \in r(b) \subseteq r(b) + a\mathcal{R}$,

but $r(b) + a\mathcal{R}$ is subtractive implies $x \in r(b) + a\mathcal{R}$. Therefore $r[\mathcal{R}b \cap l(a)] \subseteq r(b) + a\mathcal{R}$. To prove the opposite inclusion, since $r(\mathcal{R}b) \subseteq r[\mathcal{R}b \cap l(a)]$ and $r(l(a) \subseteq r[\mathcal{R}b \cap l(a)]$, then $r(\mathcal{R}b) + r(l(a)) \subseteq r[\mathcal{R}b \cap l(a)]$. But $b \in \mathcal{R}b \Rightarrow r(b) \subseteq r(\mathcal{R}b)$, $a\mathcal{R} \subseteq r(l(a))$, then $r(b) + a\mathcal{R} \subseteq r(\mathcal{R}b) + r(l(a)) \subseteq r[\mathcal{R}b \cap l(a)]$. ////

Proposition 22. Let $\mathcal{B}$ be P.Q.-injective semimodule with $Ș = End_{\mathcal{R}}(\mathcal{B})$ and let $m, n \in \mathcal{B}$.

(i) If there is an epimorphism from $\mathcal{R}m$ onto $\mathcal{R}n$, then there is a monomorphism from $nȘ$ into $mȘ$.

(ii) If there is a monomorphism from $\mathcal{R}m$ into $\mathcal{R}n$, then there is an epimorphism from $nȘ$ onto $mȘ$.

(iii) If $\mathcal{R}m \cong \mathcal{R}n$, then $nȘ \cong mȘ$.

Proof: Assume that $\beta: \mathcal{R}m \to \mathcal{R}n$ is any $\mathcal{R}$-epiomorphism, write $\beta(m) = an$ where $a \in \mathcal{R}$ and define $\delta: nȘ \to \mathcal{B}$ by $\delta[n\sigma] = a(n\sigma) = (an)\sigma = \sigma[\beta(m)]$ for all $\sigma \in Ș$. If $\beta' \in Ș$ extends β, then $\delta(n\sigma) = [\sigma[\beta(m)] = \sigma[(\beta' i(m))] = \sigma[\beta'(m)] \in mȘ$, so $\delta: nȘ \to mȘ$ is $Ș$-homomorphism.

Now to prove (i), if β is an epimorphism, then $n = \beta(bm)$ such that $b \in \mathcal{R}$. Given $\sigma(n) \in ker\delta$, thus $\sigma(n) = \sigma[\beta(bm)] = b[\sigma\beta(m)] = b\delta(n\sigma) = b0 = 0$. Hence δ is a monomorphism and $nȘ$ embeds in $mȘ$.

(ii) If β is monomorphism, then $ann_{\mathcal{R}}(\beta m) \subseteq ann_{\mathcal{R}}(m)$, in fact, let $t \in ann_{\mathcal{R}}(\beta m)$, then $t\beta(m) = \beta(tm) = 0$, so $tm \in ker(\beta)$, but β is monomorphism then $tm = 0$, hence $t \in ann_{\mathcal{R}}(m)$. So by theorem (3.16(iii)) $mȘ \subseteq \beta(m)Ș$, but

$\beta(m)$Ş ⊆ mŞ⊆ $\beta(m)$Ş. So mŞ=$\beta(m)$Ş and $\delta(n$Ş$) = m$Ş. That is $\delta: n$Ş $\rightarrow m$Ş is an epimorphism.
(iii) Follows immediately from (i) and (ii). ////

Corollary 23. Let $\mathcal{R}$ be a P-injective semiring and $a, b \in \mathcal{R}$, then
(i) If there is an epimorphism $\mathcal{R}b \longrightarrow \mathcal{R}a$, then there is a monomorphism $\mathcal{R}a \longrightarrow \mathcal{R}b$.
(ii) If there is a monomorphism $\mathcal{R}b \longrightarrow \mathcal{R}a$, then $\mathcal{R}b$ is a homomorphic image of $\mathcal{R}a$.
Proof: Since $End(_R\mathcal{R}) \cong \mathcal{R}$ and by proposition (22). ////

Definition 24 (14). A nonzero $\mathcal{R}$-subsemimodule Ų of $\mathcal{B}$ is called **essential** (large) and write Ų $\leq_e \mathcal{B}$, if Ų$\cap L \neq 0$ for every nonzero subsemimodule L of $\mathcal{B}$.

Example 25. $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule. If $K = \{\bar{0}, \bar{2}, \bar{4}\}$, then $K \not\leq_e \mathbb{Z}_6$. But if $L = \{\bar{0}, \bar{2}\} \leq \mathbb{Z}_4$, then $L \leq_e \mathbb{Z}_4$.

Definition 26 (2). Let $\mathcal{B}$ be an $\mathcal{R}$-semimodule, the sum of all simple subsemimodules of $\mathcal{B}$ is called the **socle** of $\mathcal{B}$, equal to the intersection of all essential subsemimodules of $\mathcal{B}$, it is denoted by Soc($\mathcal{B}$). If $\mathcal{B}$ has no simple subsemimodule then we put Soc($\mathcal{B}$) = 0. If Soc($\mathcal{B}$) = $\mathcal{B}$, then $\mathcal{B}$ is called semi-simple semimodule.

Remark 27 (15). An $\mathcal{R}$-semimodule is said to be semi-simple if it is a direct sum of its simple subsemimodule in $\mathcal{B}$.

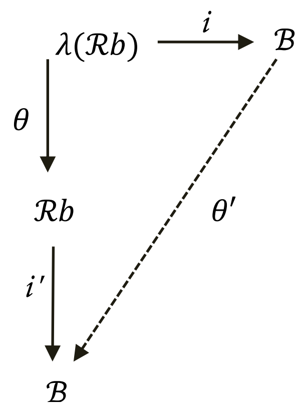
Example 28. $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is semi-simple.
Soc($\mathbb{Z}_6$)= $\{\bar{0}, \bar{2}, \bar{4}\}$+ $\{\bar{0}, \bar{3}\}$. Since $\{\bar{0}, \bar{2}, \bar{4}\}$, $\{\bar{0}, \bar{3}\}$ have no proper subsemimodules except $\{\bar{0}\}$, $\{\bar{0}, \bar{2}, \bar{4}\}$, and $\{\bar{0}\}$, $\{\bar{0}, \bar{3}\}$, respectively, then Soc($\mathbb{Z}_6$) = $\mathbb{Z}_6$. But Soc($\mathbb{Z}_4$) =$\{\bar{0}, \bar{2}\}$. Therefore $\mathbb{Z}_4$ as $\mathbb{N}$-semimodule is not semi-simple.

In (1) the relationship between the socle of $\mathcal{B}$ and P.Q.-injective modules was given. In the following, we give analogous to these properties for semimodule.

Proposition 29. Let $\mathcal{B}$ be a P.Q.-injective semimodule with Ş = $End_{\mathcal{R}}(\mathcal{B})$.
(i) If Ų is a simple subsemimodule of $\mathcal{B}$, and Ų$_1$ subsemimodule of $\mathcal{B}$ which is isomorphic to Ų, then Ų$_1$⊆ Ų Ş.
(ii) If $\mathcal{R}b$ is a simple $\mathcal{R}$-semimodule, $b \in \mathcal{B}$, then Şb is a simple Ş-semimodule.
(iii) Soc ($_R\mathcal{B}$)⊆Soc ($\mathcal{B}_{Ş}$).

Proof: (i) Let ψ: Ų⟶ Ų$_1$ be an $\mathcal{R}$-isomorphism where Ų$_1$≤ $\mathcal{B}$. If Ų = $\mathcal{R}u$, then $l(u) = l\psi(u)$, so u Ş = $\psi(u)$Ş, by Proposition(17) (iii) we have $\psi(u)$∈ u Ş ⊆ Ų Ş. If ψ' is an extension of ψ to Ş, then
Ų$_1$ = $\mathcal{R}\ \psi(u) = \mathcal{R}\ \psi'(u)$ ⊆ Ų Ş.

(ii) To prove Ş b is simple, it is enough to prove that any nonzero element of Ş has an inverse (multiplication). Consider the following diagram:



We may assume $\lambda \neq 0$. Since $\mathcal{R}b$ is simple, then $\lambda: \mathcal{R}b \rightarrow \lambda(\mathcal{R}b)$ is an isomorphism, let $\theta: \lambda(\mathcal{R}b) \rightarrow \mathcal{R}b$ be the inverse of λ, i', i are inclusion maps from $\mathcal{R}b$, $\lambda(\mathcal{R}b)$ to $\mathcal{B}$ respectively. Since $\mathcal{B}$ is P.Q.-injective semimodule, then there exists $\theta' \in$ Ş that extends θ. Now $\theta'[\lambda(b)] = \theta'[i(\lambda(b))] = i'[\theta(\lambda(b))] = \theta[\lambda(b)] = (\theta\lambda)(b) = b$. Hence $b \in$ Ş$\lambda(b) \Longrightarrow$ Şb ⊆ Ş$\lambda(b)$. That is Ş$\lambda(b)$= Şb. Hence Şb is simple. (Ş$\lambda(b)$ ⊆ Şb always holds).
(iii)This follows from (ii). ////

In (1) the notion of kasch module was introduced, where an $\mathcal{R}$-module $\mathcal{B}$ is called kasch if every simple sub-quotient of $\mathcal{B}$ can be embedded in $\mathcal{B}$, similarly, we introduce this concept for semimodule as follows, The semimodule $\mathcal{B}$ is called a kasch semimodule if every simple sub-quotient of $\mathcal{B}$ embeds in $\mathcal{B}$.i.e., there is a monomorphism from Ų/Y into $\mathcal{B}$,where Ų and Y are subsemimodules of $\mathcal{B}$ with Y is maximal subsemimodule of Ų.

Lemma 30. Let $\mathcal{B}$ be a P.Q.-injective semimodule which is kasch semimodule. if Ų is maximal subsemimodule of $_R\mathcal{R}$, then r(Ų)≠0 if and only if $l(m)$⊆ Ų for some $0 \neq m \in \mathcal{B}$. In particular, r(Ų) is a simple as right Ş-simimodule. Where r(Ų)= { $b \in \mathcal{B}$ | ub=0, ∀u∈ Ų }and $l(m)$={$t \in \mathcal{R}$| tm= 0}.
Proof: If $0 \neq m \in r$(Ų), then Ų ⊆ $l(m) \neq \mathcal{R}$, so Ų = $l(m)$ by maximality of Ų. Conversely, if $l(m)$ ⊆ Ų where m≠0, note that $\mathcal{R}m \neq$ Ų m (by maximality of Ų). Choose $\frac{X}{Ųm}$ maximal subsemimodule of $\frac{\mathcal{R}m}{Ųm}$. As $\mathcal{B}$ is kasch semimodule, let α: $\frac{\mathcal{R}m}{X} \longrightarrow \mathcal{B}$ be a monomorphism and write $\alpha(m+ X) = m'$, then Ų m' = Ų $\alpha(m + X) = \alpha$(Ųm $+X) = \alpha(X) = 0$, that is m'∈ r(Ų) and r(Ų)≠0, finally, let $0 \neq m'' \in r$(Ų), then Ų⊆ $l(m'')$, whence Ų = $l(m'')$, since $\mathcal{B}$ is a P.Q.-injective by Proposition (3. 16(ii)) then m''Ş = $r(l(m'') = r$(Ų). This shows that r(Ų) is simple as a right Ş-semimodule. ////

Proposition 31. Let $\mathcal{B}$ be a P.Q.-injective, kasch semimodule with Ș = End_R ($\mathcal{B}$), then
(i) Soc($_R\mathcal{B}$) = Soc($\mathcal{B}_{Ș}$)
(ii) Soc($\mathcal{B}_{Ș}$) $\leq_e$ $\mathcal{B}_{Ș}$

Proof: (i) We have Soc($_R\mathcal{B}$) $\subseteq$ Soc($\mathcal{B}_{Ș}$) by Proposition(29(iii))
To show that Soc($\mathcal{B}_S$) ⊆Soc($_R\mathcal{B}$), let mȘ be simple, $m\in$ $\mathcal{B}$, and let $l(m) \subseteq$ Ụ is maximal subsemimodule of $_R\mathcal{R}$. by Lemma (30), $0 \neq r$(Ụ) $\subseteq r(l(m)$ = mȘ, so mȘ = r(Ụ) by the simplicity of mȘ. Thus Ụ⊆ $l(r$(Ụ) = $l(m$Ș) = l $(m) \neq \mathcal{R}$. Since Ụ is maximal, $l(m)$ = Ụ, whence $\mathcal{R}m$ ≅ $\mathcal{R}$/ Ụ is simple. Then Soc($\mathcal{B}_{Ș}$) ⊆Soc($_R\mathcal{B}$).
(ii) let $0 \neq m\in \mathcal{B}$. If $l(m) \subseteq$ Ụ is maximal subsemimodule of $_R\mathcal{R}$, then r(Ụ) $\subseteq r(l$ (m) = mȘ, by Proposition(17(ii)). As r(Ụ) is simple Lemma(30) and r(Ụ)≠0, then Soc($\mathcal{B}_{Ș}$) $\leq_e$ $\mathcal{B}_{Ș}$. ////

Proposition 32. Let $\mathcal{B}$ be a P.Q.-injective semimodule with Ș = $End_{\mathcal{R}}(\mathcal{B})$, and let m_1, m_2, …, m_n be elements of $\mathcal{B}$.
(i) If m_1Ș$\oplus \ldots \oplus m_n$Ș is a direct sum, then any $\mathcal{R}$-homomorphism $\lambda: \mathcal{R}m_1 \oplus \ldots \oplus \mathcal{R}m_n \to \mathcal{B}$ has an extension in Ș.
(ii)If $\mathcal{R}m_1 \oplus \ldots \oplus \mathcal{R}m_n$ is a direct sum, then $(m_1+\ldots+m_n)$ Ș= m_1Ș+…+m_nȘ.

Proof: (i) Let λ_i and β denote the restrictions of λ to $\mathcal{R}m_i$ and $\mathcal{R}(m_1+\ldots+m_n)$ respectively and let $\lambda_i{}'$and β' extend $\lambda_i$and β to $\mathcal{B}$. Then $\sum_i \beta'(m_i) = \beta'(\sum_i m_i) = \lambda(\sum_i m_i) = \sum_i \lambda(m_i) = \sum_i \lambda'(m_i)$.
Since $\oplus m_i$Ș is a direct, we obtain $\beta'(m_i) = \lambda'(m_i)$, so β' extends λ.
(ii) Define $\lambda_i: \mathcal{R}(m_1 + \cdots + m_n) \to \mathcal{B}$ by $\lambda_i[r(m_1 + \cdots + m_n) = rm_i$ for all $r\in \mathcal{R}$. Then λ_i is well defined. Since $\mathcal{B}$ is P.Q.-injective semimodule, then there exists $\lambda'_i \in$ Ș that extends λ_i, hence $m_i = \lambda_i(\sum_i m_i) = \lambda'_i[i(\sum_i m_i)] = \lambda'(\sum_i m_i) \in (\sum_i m_i)$Ș and it follows that $\sum_i m_i$Ș ⊆ $(\sum_i m_i)$Ș. The reverse inclusion always holds. ////

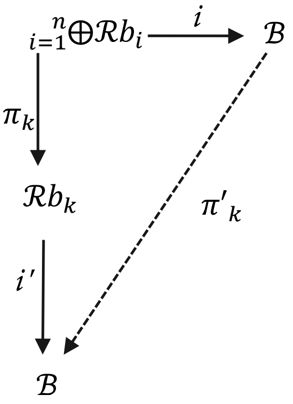
To show the next result we need the following definition.

Definition 33(8). A subsemimodule K of $\mathcal{R}$-semimodule $\mathcal{B}$ is called fully invariant if for each endomorphism f: $\mathcal{B} \to \mathcal{B}$, then $f(K) \subseteq K$.

Example 34. Every subsemimodule of $\mathbb{Z}$ as $\mathbb{Z}$-semimodule is invariant.
Let $\mathcal{B} = \mathbb{Z}$ and K= $n\mathbb{Z}$, where $n \in \mathbb{Z}$ and let f: $\mathcal{B} \to \mathcal{B}$, then $f(n\mathbb{Z}) = nf(\mathbb{Z}) \subseteq n\mathbb{Z}$.

Proposition 35. Let $\mathcal{B}$ be a P.Q.-injective semimodule with Ș = $End_{\mathcal{R}}(\mathcal{B})$, and let A, B_1, B_2, …, B_n be fully invariant subsemimodules of $\mathcal{B}$. If $B_1 \oplus \ldots . \oplus B_n$ is a direct sum of $\mathcal{B}$, then $A \cap (B_1 \oplus \ldots . \oplus B_n)$= $(A \cap B_1) \oplus \ldots \oplus (A \cap B_n)$.

Proof: It is known and easy to check that $\oplus_i (A \cap B) \subseteq A \cap (\oplus_i B_i)$.



Let $a = \sum_i b_i \in A \cap [\oplus_i B_i]$ and let $\pi_k: \oplus_{i=1}^{n} \mathcal{R}b_i \to \mathcal{R}b_k$ be the projection map and i, $'i$ are inclusion maps from $\oplus_{i=1}^{n} \mathcal{R}b_i$ and $\mathcal{R}b_k$ to $\mathcal{B}$ respectively. Since $\oplus b_i$Ș is a direct sum, because each B_i is invariant, then by Proposition (32(i)) each π_k has an extension π'_k in Ș, i.e., $\pi'_k[i(a)] = i'[\pi_k(a)]$. Since A is fully invariant, then $\pi'_k(a) = \pi'_k[i(a)] = i'[\pi_k(a)] = \pi_k(a) = b_k \in A \cap B_k$ for each k where $a \in \oplus_i (A \cap B_i)$. ////

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شبه المقاسات شبه الاغماريه رئيسية

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الخلاصة:

نقدم في هذا العمل، مفهوم شبه المقاس الرئيس شبه الاغماري، وندرس الشروط التي نحتاجها لنحصل على خصائص وصفات مشابهه كما في الموديولات.

ليكن $\mathcal{B}$ شبه مقاسا على شبه الحلقة $\mathcal{R}$ وان $\mathcal{S}$ شبه حلقة التشاكلات في شبه المقاس $\mathcal{B}$. يسمى شبه المقاس $\mathcal{B}$ رئيسا شبه اغماريا اذا كان لكل تشاكل من اي شبه مقاس جزئي دوري من $\mathcal{B}$ الى $\mathcal{B}$ يمكن توسيعه الى تشاكل في شبه حلقة التشاكلات في $\mathcal{B}$.

الكلمات المفتاحية: شبه المقاسات، شبه المقاسات شبه الاغماريه، شبه المقاسات شبه الاغماريه الرئيسية.