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The Gumbel- Pareto Distribution: Theory and Applications

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Abstract:

In this paper, for the first time we introduce a new four-parameter model called the Gumbel- Pareto distribution by using the *T-X* method. We obtain some of its mathematical properties. Some structural properties of the new distribution are studied. The method of maximum likelihood is used for estimating the model parameters. Numerical illustration and an application to a real data set are given to show the flexibility and potentiality of the new model.

Key words: Gumbel distribution, Hazard function, Maximum likelihood estimation, Moments، Pareto distribution, Quantile function, *T-X* method.

Introduction:

Eugene et al. (1) for the first time introduced the beta-generated family of distributions. "They noted that the probability density function pdf of the beta random variable and the cumulative distribution function CDF of any distribution are between 0 and 1". They used the beta distribution as a generator to develop the so called family of beta-generated distributions. The beta-generated random variable X is defined with the following CDF and pdf

$$\text{“}\ G(x) = \int_0^{F(x)} b(t)dt\ \text{”} \quad \ldots \quad (1)$$

and

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)F(x)^{\alpha-1}[1-F(x)]^{\beta-1}$$

where $b(t)$ is the pdf of the beta random variable with parameters α and β, $F(x)$ and $f(x)$ are the CDF and the pdf of any random variable.

Many authors derived and studied many beta-generated distributions in the literature, for example beta-Gumbel (Nadarajah and Kotz, (2)), beta-Weibull (Famoye et al. (3)), beta-exponential (Nadarajah and Kotz, (4)), beta-gamma (Kong et al., (5)), beta-Pareto (Akinsete et al., (6)), beta-generalized exponential (Barreto-Souza et al., (7)), beta-generalized Pareto (Mahmoudi, (8)), and beta-Cauchy (Alshawarbeh, et al., (9)).

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Recently, Alzaatreh et al. (10) presented a new general method for generating new distributions, called *T-X* family of distributions. This method depends on replacing the beta pdf in (1) with a pdf of any continuous random variable and applying a function $W(F(x))$ that satisfies the following conditions:

1- $W(F(x)) \in [a, b]$.
2- $W(F(x))$ is differentiable and monotonically non-decreasing.
3- $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$.

"Let X be a random variable with pdf $f(x)$ and CDF $F(x)$, and let T be a continuous random variable with pdf $r(t)$ and CDF $R(t)$ defined on $[a, b]$ for $-\infty < a < b < \infty$. Alzaatreh et al. (10) defined the CDF and pdf of a new family of distributions as"

$$G(x) = \int_a^{W(F(x))} r(t)dt = R\{W(F(x))\}$$

and

$$g(x) = \left[\frac{d}{dx} W\big(F(x)\big)\right] r\big[W\big(F(x)\big)\big]$$

A different $W(F(x))$ will give a new family of distributions. The form of $W(F(x))$ depends on the support of the random variable T . Alzaatreh et al. (10) gave the following examples of $W(.)$:

1- When the support of T is bounded: Without loss of generality, assume the support of T *is* [0, 1]. Distributions for such T include uniform (0, 1), beta, Kumaraswamy and other types of generalized beta distributions. In such cases, $W(F(x))$ can be defined as $F(x)$, which gives the beta-generated family of distributions that have been well studied during the recent decade.

2- When the support of T is$[a,\infty)$, $a \geq 0$: Without loss of generality, assuming a = 0. $W(F(x))$ can be defined as $-\log(1-F(x))$, $F(x)$ / $(1-F(x))$, $-\log(1-F^{\alpha}(x))$, and $F^{\alpha}(x)$ / $(1-F^{\alpha}(x))$, where $\alpha > 0$.

3- When the support of T is $(-\infty,\infty)$: $W(F(x))$ can be defined as $\log[-\log(1-F(x))]$, $\log[F(x)$ / $(1-F(x))]$, $\log[-\log(1-F^{\alpha}(x))]$, and $\log[F^{\alpha}(x)$ / $(1-F^{\alpha}(x))]$.

$$G(x) = exp\left(-e^{\frac{\mu}{\sigma}}\left(\frac{f(x)}{(1-F(x))}\right)^{-\frac{1}{\sigma}}\right), -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \quad .... \quad (2)$$

$$g(x) = \frac{\lambda f(x)}{\sigma F(x)(1-F(x))}\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}} exp\left(-\lambda\left(\frac{F(x)}{1-F(x)}\right)^{-\frac{1}{\sigma}}\right)$$

$$-\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \sigma > 0, \quad \lambda = e^{\frac{\mu}{\sigma}} > 0 \quad .... \quad (3)$$

Al-Aqtash et al. (11) defined and studied a family of T-X distributions arising from the logit function of the CDF of a continuous random variable when the random variable T is defined on $(-\infty,\infty)$. “They used $W(F(x))= \log\{F(x)/(1-F(x))\}$ which is one of the transformation functions suggested by Alzaatreh et al”. (10). Al-Aqtash et al. (11) obtained the following CDF and pdf of the gumbel-X family as Al-Aqtash et al. (11,12) proposed the gumbel- Weibull distribution by taking T as the gumbel random variable and X as the weibull distribution. Tahir et al. (13) defined the gumbel-Lomax distribution by taking T as the gumbel random variable and X as the Lomax distribution.

The Gumbel- Pareto Distribution(*GPD*)

If the random variable X be the Pareto distribution with pdf and CDF given, respectively, by

“$f(x) = \frac{k\theta^k}{x^{k+1}}, x > 0,\ k,\theta > 0$”

and

“$F(x) = 1 - \left(\frac{\theta}{x}\right)^k, x > 0,\ k,\theta > 0$”

then using (2) and (3) the gumbel- pareto distribution (*GPD*) is given with the CDF and pdf

$$G(x) = exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right), \ x > 0, \ \lambda,\sigma,k,\theta > 0 \quad (4)$$

and

$$g(x) = \frac{k\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1}}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\frac{1}{\sigma}+1}} exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right), \ x > 0, \ \lambda,\sigma,k,\theta > 0 \ ...(5)$$

respectively.

“Figs. 1 and 2 illustrate some of the possible shapes of the pdf and CDF of *GPD* for selected values of the parameters k,λ,θ and , respectively” ( )

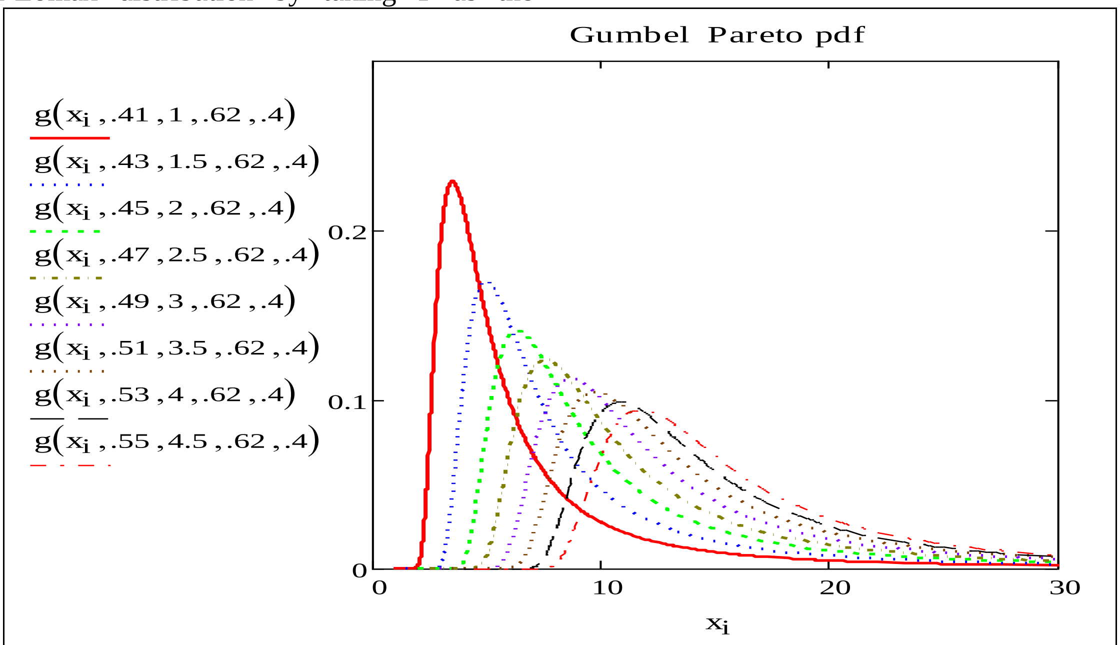

Figure 1. density function $g(x;k,\lambda,\theta,\sigma)$ **of the *GPD***

Figure 1. Indicate that the *GPD* is unimodal.

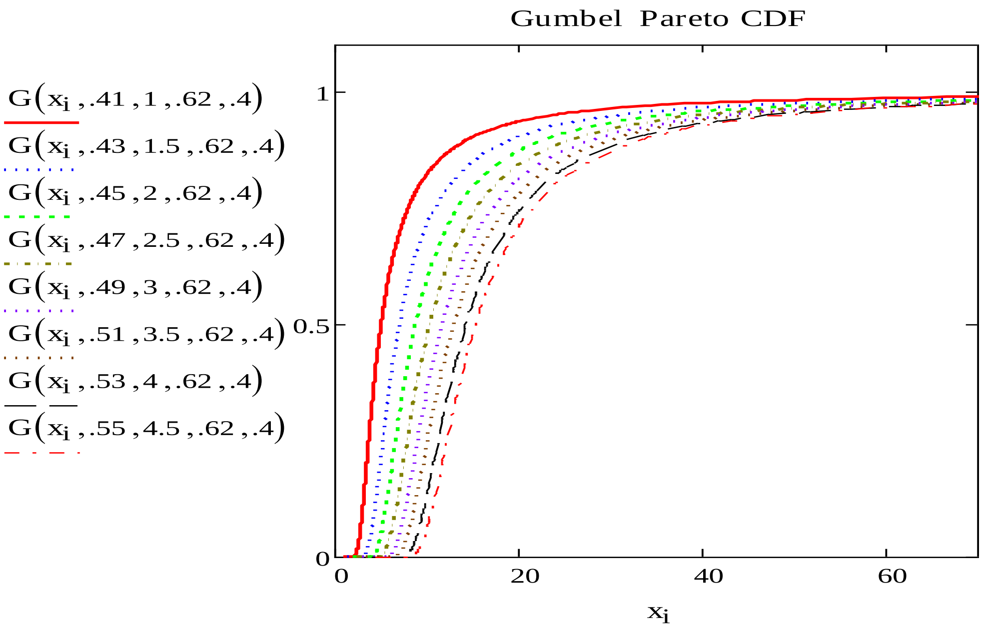

Figure 2. cumulative function $G(x;k,\lambda,\theta,\sigma)$ **of the** ***GPD***

Transformation

If a random variable Y follows the gumbel distribution with parameters μ and , then the random variable $X=\theta(e^{Y}+1)^{\frac{1}{k}}$ follows the *GPD.*

Survival and hazard functions

The survival function of the *GPD* is

$$S(x)=1-exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)$$

“and the hazard function and the cumulative hazard function of the *GPD* is given by”

$$h(x)=\frac{k\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1}exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{\frac{1}{\sigma}+1}\left(1-exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)\right)}$$

”

and

$$H(x)=-\log\left(1-exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)\right)$$

Figures 3 and 4 illustrate some of the possible shapes of the survival and hazard rate function of *GPD* for selected values of the parameters k,λ,θ and , respectively

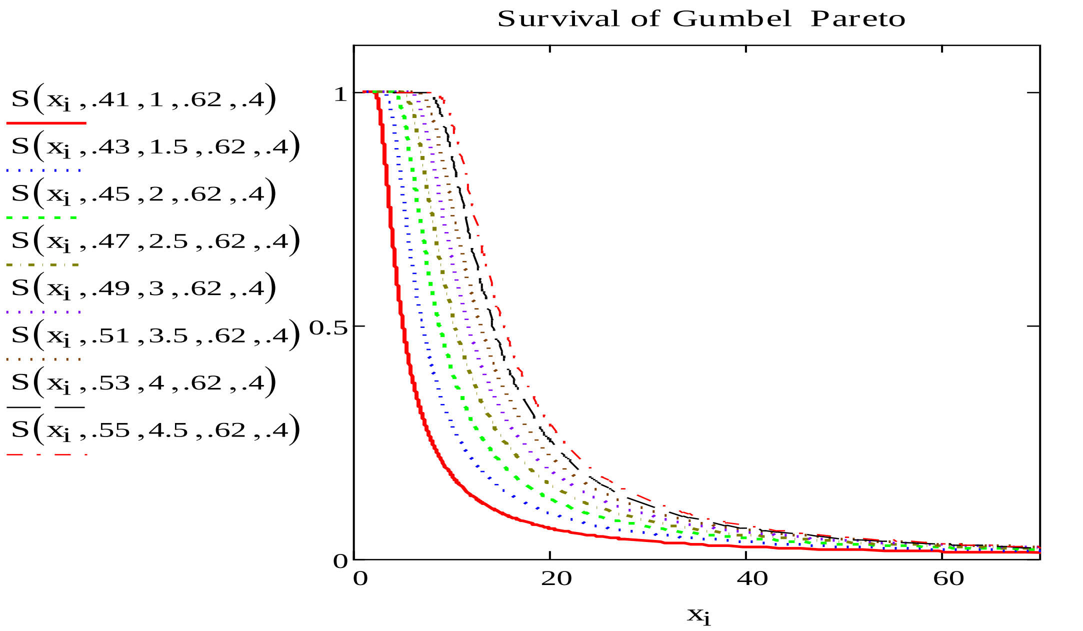

Figure 3. survival function $S(x; k, \lambda, \theta, \sigma)$ **of the** ***GPD***

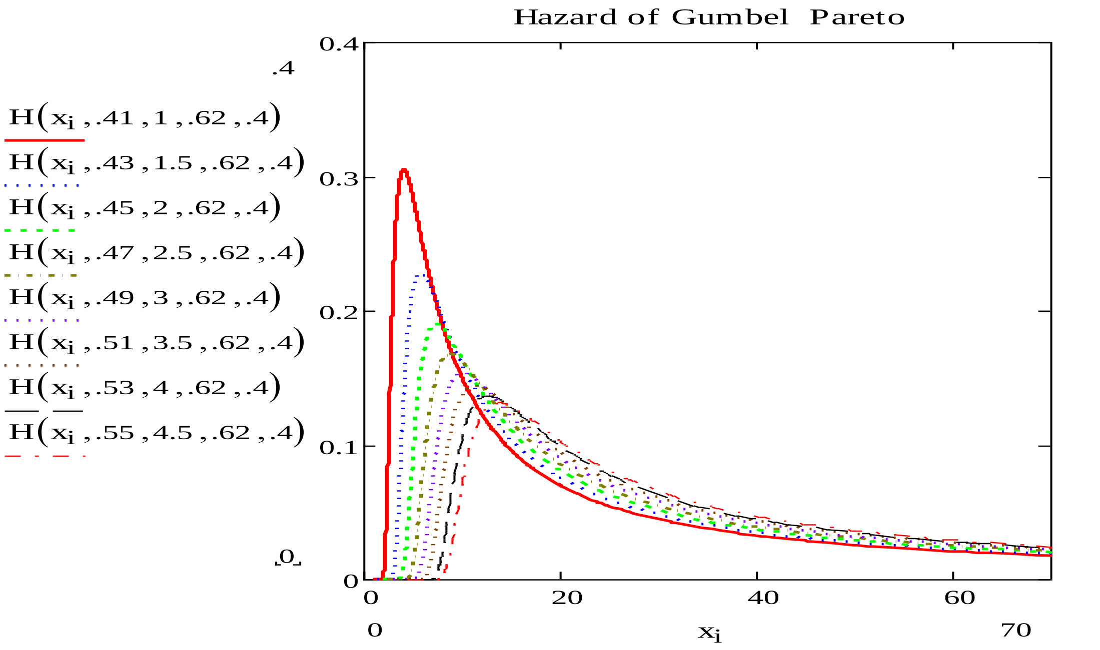

Figure 4. hazard function $H(x; k, \lambda, \theta, \sigma)$ **of the** ***GPD***

Quantile function, Median, and simulation

The quantile function of *GPD* can be written as

$$Q_{X_n}(P) = \theta\left\{\left(\frac{1}{\lambda}log\left(\frac{1}{P}\right)\right)^{-\sigma} + 1\right\}^{\frac{1}{k}}, \qquad 0 < P < 1$$

Consequently, the median of *GPD* is

$$Q_{X_n}(0.5) = \theta\left\{\left(\frac{0.30103}{\lambda}\right)^{-\sigma} + 1\right\}^{\frac{1}{k}}$$

Let a uniform be variant on the unit interval (0,1). Thus by means of the inverse transformation method, we consider the random variable X given by:

$$X = \theta\left\{\left(\frac{1}{\lambda}log\left(\frac{1}{U}\right)\right)^{-\sigma} + 1\right\}^{\frac{1}{k}} \quad (6)$$

This follows the *GPD*.

Mixture representation

From (5):

$$g(x)=\frac{k\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1} exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{\frac{1}{\sigma}+1}}$$,,

let

$$A=exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{1}{\sigma}}\right)$$

By expanding the quantity A in power series,

$$A=\sum_{i=1}^{\infty}\frac{(-1)^{i}\lambda^{i}}{i\,!}\left(\frac{\theta}{x}\right)^{\frac{ki}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{-\frac{i}{\sigma}}$$

we can write

$$g(x)=\frac{k}{\sigma\theta}\sum_{i=1}^{\infty}\frac{(-1)^{i}\lambda^{(i+1)}}{i\,!}\frac{\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}(i+1)+1}}{\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{\frac{(i+1)}{\sigma}+1}}$$

let $$B=\left(1-\left(\frac{\theta}{x}\right)^{k}\right)^{\frac{(i+1)}{\sigma}+1}$$

By expanding the quantity B in power series such that $\left(\frac{\theta}{x}\right)^{k}<1$, we obtain

$$B=\sum_{0}^{\infty}(-1)^{j}\binom{\frac{-(i+1)}{\sigma}+1}{j}\left(\frac{\theta}{x}\right)^{kj}$$

Combining the last two results, we have

$$g(x)=\frac{k}{\sigma\theta}\sum_{i=1}^{\infty}\frac{(-1)^{i+j}\lambda^{(i+1)}}{i\,!\,(i+1+\sigma j)}\binom{\frac{-(i+1)}{\sigma}+1}{j}\frac{k(i+1+\sigma j)}{\sigma\theta}\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}(i+1+\sigma j)+1}$$

In a more simplified form, the pdf of X can be expressed as

$g(x)=\sum_{i,j=0}^{\infty}V_{i,j}\ \ g_{\frac{k}{\sigma}(i+1+\sigma j),\theta}(x)$ (7)

where

$$V_{i,j}=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\lambda^{(i+1)}}{i\,!\,(i+1+\sigma j)}\binom{\frac{-(i+1)}{\sigma}+1}{j}$$

and

$$g_{\frac{k}{\sigma}(i+1+\sigma j),\theta}(x)=\frac{k(i+1+\sigma j)}{\sigma\theta}\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}(i+1+\sigma j)+1}$$

Equation (7) reveals that the *GPD* density function has a mixture representation of pareto densities with parameters $\frac{k}{\sigma}(i+1+\sigma j)$ and θ. Thus, several of its structural properties can be derived from those of the pareto distribution. The coefficients $V_{i,j}$ depend only on the generator parameters.

Moments

The r^{th} ordinary moment of the *GPD* is

$$E(X)=\frac{\theta^{r}}{\sigma}\int_{-\infty}^{\infty}(e^{Y}+1)^{\frac{Y}{k}}\ \lambda e^{-\frac{Y}{\sigma}}exp\left(-\lambda e^{-\frac{Y}{\sigma}}\right)dY$$

$$=\frac{\theta^{r}}{\sigma}\sum_{i=0}^{\infty}\binom{\frac{r}{k}}{i}\int_{-\infty}^{\infty}e^{Yi}\ \lambda e^{-\frac{Y}{\sigma}}exp\left(-\lambda e^{-\frac{Y}{\sigma}}\right)dY$$

$$=\frac{\theta^{r}}{\sigma}\sum_{i=0}^{\infty}\binom{\frac{r}{k}}{i}\int_{-\infty}^{\infty}\lambda e^{-\frac{Y}{\sigma}(1-\sigma i)}\ exp\left(-\lambda e^{-\frac{Y}{\sigma}}\right)dY$$

$$=\frac{\theta^{r}}{\sigma}\sum_{i=0}^{\infty}\lambda^{\sigma i}\binom{\frac{r}{k}}{i}\int_{-\infty}^{\infty}\left(\lambda e^{-\frac{Y}{\sigma}}\right)^{(1-\sigma i)}\ \left(e^{-\lambda\theta}\right)^{-\frac{Y}{\sigma}}dY$$

$$=\frac{\theta^{r}}{\sigma}\sum_{i=0}^{\infty}\lambda^{\sigma i}\binom{\frac{r}{k}}{i}\ \Gamma(1-\sigma i)$$

$=\frac{\theta^{r}}{\sigma}\sum_{i=0}^{\infty}W_{i,r}\,\Gamma(1-\sigma i)$, $\sigma i>0$ (8)

where

$$W_{i,r}=\lambda^{\sigma i}\binom{\frac{r}{k}}{i}$$

Setting $r=1$ in (8), the mean of X reduces to

$$\acute{\mu}_{1}=\frac{\theta}{\sigma}\sum_{i=0}^{\infty}W_{i,1}\,\Gamma(1-\sigma i)$$

Setting r = 2 in (8), the $\acute{\mu_2}$ reduces to

$$\acute{\mu_2} = \frac{\theta^2}{\sigma}\sum_{i=0}^{\infty} W_{i,2}\,\Gamma(1-\sigma i)$$

and

$$Var(X) = \frac{\theta^2}{\sigma}\sum_{i=0}^{\infty} W_{i,2}\,\Gamma(1-\sigma i) - \left(\frac{\theta}{\sigma}\sum_{i=0}^{\infty} W_{i,1}\,\Gamma(1-\sigma i)\right)^2$$

Moment generating function

"By definition the moment generating function (mgf) of *GPD* is"

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx}\,g(x)dx\text{"}$$

Using Taylor series

$$M_X(t) = \int_0^{\infty}\left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} + \cdots\right)g(x)dx$$

$$= \sum_{P=0}^{\infty}\frac{t^P E(X^P)}{P!}$$

$$= \sum_{i,P=0}^{\infty}\frac{t^P\theta^P}{P!\,\sigma}\sum_{i=0}^{\infty} W_{i,P}\,\Gamma(1-\sigma i)\,, \quad \sigma i > 0 \ ,$$

Distribution of the order statistics

In this section, we derive closed form expressions for the pdfs of the r^{th}order statistic of the *GPD*. Let $X_1, X_2, \dots, X_n$ be a simple random sample from *GPD* distribution with CDF and pdf given by (4) and (5), respectively. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of the r^{th}order statistic of the *GPD* is

$$g_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!}\frac{k\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1}}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\frac{1}{\sigma}+1}}\left(exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right)\right)^r\left(1 - exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right)\right)^{n-r}$$

"The pdf of the largest order statistic $X_{(n)}$ is therefore"

$$g_{X_{(n)}}(x) = \frac{nk\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1}}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\frac{1}{\sigma}+1}}\left(exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right)\right)^n$$

"and the pdf of the smallest order statistic $X_{(n)}$ is given by"

$$g_{X_{(1)}}(x) = \frac{nk\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}+1}}{\sigma\theta\left(1-\left(\frac{\theta}{x}\right)^k\right)^{\frac{1}{\sigma}+1}}\left(exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right)\right)^1\left(1 - exp\left(-\lambda\left(\frac{\theta}{x}\right)^{\frac{k}{\sigma}}\left(1-\left(\frac{\theta}{x}\right)^k\right)^{-\frac{1}{\sigma}}\right)\right)^{n-1}$$

Parameter estimation of *GPD*:

"Let $X_1, X_2, \dots, X_n$ be a random sample from a *GPD* with parameters k, λ, θ and σ, then the log-likelihood function from (5) is given by"

$$logL(k,\lambda,\theta,\sigma) = \ell = \sum_{i=1}^{n} log(g(x_i)) = n[log\lambda + logk - log\sigma - log\theta] + (\frac{k}{\sigma} +$$

$1)\sum_{i=1}^{n} log\left(\frac{\theta}{x_i}\right) - (\frac{1}{\sigma}+1)\sum_{i=1}^{n} log\left(1-\left(\frac{\theta}{x_i}\right)^{k}\right) - \lambda\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}}$ (9)

The first partial derivatives of (9) are

$$\frac{\partial\ell}{\partial k} = \frac{n}{k} + \frac{1}{\sigma}\sum_{i=1}^{n} log\left(\frac{\theta}{x_i}\right) + (\frac{1}{\sigma}+1)\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} log\left(\frac{\theta}{x_i}\right) - \frac{\lambda}{\sigma}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{nk}{\sigma\theta} + \frac{k}{\theta}(\frac{1}{\sigma}+1)\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} - \frac{\lambda k}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k}$$

$$\frac{\partial\ell}{\partial\sigma} = -\frac{n}{\sigma} + \frac{k}{\sigma^2}\sum_{i=1}^{n} log\left(\frac{\theta}{x_i}\right) + \frac{1}{\sigma^2}\sum_{i=1}^{n} log\left(1-\left(\frac{\theta}{x_i}\right)^{k}\right) - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}} log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)$$

"On setting these derivatives equal to zero and solving the system of equations iteratively using MathCAD, we obtain the maximum likelihood estimates of the *GPD*".

The second derivatives with respect to k, λ, θ and σ will be:

$$\frac{\partial^2\ell}{\partial k^2} = -\frac{n}{k^2} + (\frac{1}{\sigma}+1)\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-2}\left(\frac{\theta}{x_i}\right)^{-k}\left(log\left(\frac{\theta}{x_i}\right)\right)^{2} - \frac{\lambda(1+\sigma)}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+2)}\left(\frac{\theta}{x_i}\right)^{-2k}\left(log\left(\frac{\theta}{x_i}\right)\right)^{2} + \frac{\lambda}{\sigma}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k}\left(log\left(\frac{\theta}{x_i}\right)\right)^{2}$$

$$\frac{\partial^2\ell}{\partial k\partial\lambda} = -\frac{1}{\sigma}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right)$$

$$\frac{\partial^2\ell}{\partial k\partial\theta} = \frac{n}{\sigma\theta} + \frac{k}{\theta}\left(\frac{1}{\sigma}+1\right)\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-2}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right) + \frac{1+\sigma}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} - \frac{\lambda k(1+\sigma)}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\left(\frac{1}{\sigma}+2\right)}\left(\frac{\theta}{x_i}\right)^{-2k} log\left(\frac{\theta}{x_i}\right) - \frac{\lambda k}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\left(\frac{1}{\sigma}+1\right)}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right) - \frac{\lambda}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\left(\frac{1}{\sigma}+1\right)}\left(\frac{\theta}{x_i}\right)^{-k}$$

$$\frac{\partial^2\ell}{\partial k\partial\theta} = -\frac{1}{\sigma^2}\sum_{i=1}^{n} log\left(\frac{\theta}{x_i}\right) - \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} log\left(\frac{\theta}{x_i}\right) + \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right) - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k} log\left(\frac{\theta}{x_i}\right) log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)$$

$$\frac{\partial^2\ell}{\partial\lambda^2} = -\frac{n}{\lambda^2}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\theta} = \frac{k}{\sigma\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\sigma} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}} log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)$$

$$\frac{\partial^2 \ell}{\partial\theta^2} = -\frac{nk}{\sigma\theta^2} - \frac{k(1+\sigma)}{\sigma\theta^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} + \frac{k^2}{\theta^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-2}\left(\frac{\theta}{x_i}\right)^{-k} + \frac{\lambda k}{\sigma\theta^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k} + \frac{\lambda k^2}{\sigma\theta^2}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-(\frac{1}{\sigma}+1)}\left(\frac{\theta}{x_i}\right)^{-k}$$

$$\frac{\partial^2 \ell}{\partial\theta\partial\sigma} = \frac{nk}{\sigma^2\theta} - \frac{k}{\sigma^2\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-1} + \frac{\lambda k}{\sigma^2\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\left(\frac{1}{\sigma}+1\right)}\left(\frac{\theta}{x_i}\right)^{-k} - \frac{\lambda k}{\sigma^3\theta}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\left(\frac{1}{\sigma}+1\right)}\left(\frac{\theta}{x_i}\right)^{-k}\log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)$$

$$\frac{\partial^2 \ell}{\partial\sigma^2} = \frac{n}{\sigma^2} + \frac{2k}{\sigma^3} - \frac{2}{\sigma^3}\sum_{i=1}^{n}\log\left(1-\left(\frac{\theta}{x_i}\right)^{k}\right) + \frac{2\lambda}{\sigma^3}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}}\log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right) - \frac{\lambda}{\sigma^4}\sum_{i=1}^{n}\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)^{-\frac{1}{\sigma}}\left(\log\left(\left(\frac{\theta}{x_i}\right)^{-k}-1\right)\right)^2$$

Now we can derive the elements of the Fisher information matrix as follows:

$$I_{1,1} = -E\left(\frac{\partial^2 \ell}{\partial k^2}\right),$$

$$I_{2,2} = -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right),$$

$$I_{3,3} = -E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right),$$

$$I_{4,4} = -E\left(\frac{\partial^2 \ell}{\partial \sigma^2}\right),$$

$$I_{1,2} = -E\left(\frac{\partial^2 \ell}{\partial k\partial\lambda}\right),$$

$$I_{1,3} = -E\left(\frac{\partial^2 \ell}{\partial k\partial\theta}\right),$$

$$I_{1,4} = -E\left(\frac{\partial^2 \ell}{\partial k\partial\sigma}\right),$$

$$I_{2,3} = -E\left(\frac{\partial^2 \ell}{\partial \lambda\partial\theta}\right),$$

$$I_{2,4} = -E\left(\frac{\partial^2 \ell}{\partial \lambda\partial\sigma}\right),$$

and

$$I_{3,4} = -E\left(\frac{\partial^2 \ell}{\partial \theta\partial\sigma}\right)$$

then the Fisher information matrix is

"
$$I = \begin{pmatrix} I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} \\ I_{2,1} & I_{2,2} & I_{2,3} & I_{2,4} \\ I_{3,1} & I_{3,2} & I_{3,3} & I_{3,4} \\ I_{4,1} & I_{4,2} & I_{4,3} & I_{4,4} \end{pmatrix}$$
"

The variance-covariance matrix of $(\hat{k},\hat{\lambda},\hat{\theta},\hat{\sigma})$ is obtained by inverting the Fisher information matrix as follows:

"$$I^{-1}(\hat{k},\hat{\lambda},\hat{\theta},\hat{\sigma}) = \begin{pmatrix} Var(\hat{k}) & Cov(\hat{k},) & Cov(\hat{k},) & Cov(\hat{k},\hat{\sigma}) \\ Cov(\hat{k},\hat{\lambda}) & Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\theta}) & Cov(\hat{\lambda},\hat{\sigma}) \\ Cov(\hat{k},\hat{\theta}) & Cov(\hat{\lambda},\hat{\theta}) & Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\sigma}) \\ Cov(\hat{k},\hat{\sigma}) & Cov(\hat{\lambda},\hat{\sigma}) & Cov(\hat{\theta},\hat{\sigma}) & Var(\hat{\sigma}) \end{pmatrix}$$"

Numerical Illustration

In this section, random numbers are generated using the CDF of the *GPD* distribution, and then the maximum likelihood estimates are obtained.

Generate 1000 samples of each of sizes 10, 15,…, 30 from the *GPD* distribution for different values of the parameters k, λ, θ and σ, using the CDF of *GPD*, and then the maximum likelihood estimates for each sample will be obtained, along with the mean, root of the mean square error, bias and standard error of those estimates. The steps of this procedure will be as the following:

1. Set initial values for the parameters k, λ, θ and σ.
2. Generate 1000 samples of each of sizes 10, 15,…, 30, using (6).
3. "Obtain the maximum likelihood estimates for k, λ, θ and σ for the different sample sizes".
4. "Obtain the mean, biases, root of the mean square error and standard errors for the MLE estimates for the different sample sizes".
5. Repeat steps 1:4 for different values of k, λ, θ and σ.

Results are listed in Table 1.

Table 1. Means, Biases, Root of the Mean Square Errors and Standard Errors for the MLEs of *GPD* distribution for different values of parameters.

n		$k=0.5$	$\lambda=0.15$	$\theta=0.06$	$\sigma=0.25$	$k=0.75$	$\lambda=0.25$	$\theta=0.16$	$\sigma=0.5$
10	**Mean**	1.817	0.084	0.311	0.887	2.233	0.168	0.459	1.427
	Biase	1.317	-0.066	0.251	0.637	1.483	-0.082	0.299	0.927
	R.MSE	3.221	0.086	0.416	1.449	3.53	0.13	0.482	1.966
	S.E	0.294	0.005497	0.033	0.13	0.32	0.01	0.038	0.173
15	**Mean**	1.465	0.077	0.31	0.653	2.105	0.165	0.507	1.205
	Biase	0.965	-0.073	0.25	0.403	1.355	-0.085	0.347	0.705
	R.MSE	2.965	0.089	0.455	1.117	3.386	0.125	0.57	1.658
	S.E	0.187	0.003303	0.025	0.07	0.207	0.00612	0.03	0.1
20	**Mean**	1.17	0.074	0.273	0.531	1.792	0.16	0.502	0.995
	Biase	0.67	-0.076	0.213	0.281	1.042	-0.09	0.342	0.495
	R.MSE	2.498	0.088	0.414	0.882	2.995	0.123	0.598	1.362
	S.E	0.12	0.002284	0.018	0.042	0.14	0.00418	0.025	0.063
25	**Mean**	0.979	0.073	0.256	0.455	1.578	0.159	0.491	0.881
	Biase	0.479	-0.077	0.196	0.205	0.828	-0.091	0.331	0.381
	R.MSE	1.978	0.088	0.388	0.677	2.729	0.12	0.598	1.155
	S.E	0.077	0.001705	0.013	0.026	0.104	0.0031	0.02	0.044
30	**Mean**	0.862	0.072	0.232	0.405	1.211	0.158	0.432	0.735
	Biase	0.362	-0.078	0.172	0.155	0.461	-0.092	0.272	0.235
	R.MSE	1.916	0.088	0.348	0.596	2.013	0.117	0.516	0.87
	S.E	0.063	0.001323	0.01	0.019	0.065	0.0024	0.015	0.028

In general the more sample size increases the more Means, Biases, R.MSEs and S.Es decreases.

Application

This section presents application of *GPD* using real data set. In this application, we obtain the maximum likelihood estimates of the parameters of the fitted distributions. *GPD* is compared with other distributions (Gumbel- Weibull Distribution (*GWD*), Gumbel- Lomax Distribution (*GLD*) and Gumbel Distribution (*GD*)) based on the maximized log-likelihood, the Kolmogorov-Smirnov (K-S) test along with the corresponding p-value, Akaike Information Criterion (AIC)**,** Bayesian Information Criteria (BIC) and Cramer von Mises statistic (CM) where

$$AD=\left\{-\sum_{i=1}^{n}\frac{2i-1}{n}\{\ln F_0(x_i)+\ln[1-F_0(x_{n-i+1})]\}\right\}-n,$$

$$CM=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-F_0(x_i)\right]^2.$$

$$AIC=-2log(likelihood)+2k,$$

$$BIC=k\ln(n)-2\log\text{ (likelihood)},$$

where n is the sample size of the training set, k is the number of estimated parameters, $F_0(x_i)$is the cumulative distribution function evaluated at x_i.

In addition, the histogram of the data is presented for graphical illustration of the goodness of fit".

The data set in Table 2 from Choulakian and Stephens (14), is the exceedances of flood peaks of Wheaton River, Yukon Territory, Canada. The data consists of 72 exceedances for the years 1958-1984, rounded to one decimal place. It is a right-skewed data (skewness = 1.5 and kurtosis = 3.19) with a long right tail.

Table 2. Exceedances of the Wheaton River data

9.3	12.0	13.0	1.9	0.7	5.3	20.6	0.4	1.1	14.4	2.2	1.7
37.6	1.7	4.4	2.5	1.1	22.1	14.1	11.6	25.5	8.5	18.7	1.4
0.6	1.1	0.1	1.7	22.9	7.3	11.0	15.0	0.3	39.0	2.2	0.6
3.6	30.0	10.7	10.4	9.9	14.1	2.8	0.4	20.1	7.0	1.7	9.0
64.0	2.7	36.4	27.6	21.5	11.9	3.4	25.5	4.2	13.3	30.8	5.6
27.0	2.5	27.5	9.7	5.3	16.8	20.2	27.1	1.0	27.4	2.5	1.5

Table 3. MLEs for the distributions Parameters from the Exceedances of the Wheaton River data

Model		MLEs	-2 log L	AIC	BIC	KS	P value	CM
GPD	$\hat{k}$	0.036	726.568	734.569	743.675	0.186	0.014	0.5
	$\hat{\theta}$	1.252×10^{-5}						
	$\hat{\lambda}$	0.029						
	$\hat{\sigma}$	0.166						
GWD	$\hat{\alpha}$	2.344	2056	2059	2061	0.213	0.002991	0.715
	$\hat{\beta}$	315.954						
	$\hat{\lambda}$	0.026						
	$\hat{\sigma}$	3.092						
GLD	$\hat{\delta}$	0.009009	1589.894	1598	1607	0.2	0.006479	0.655
	$\hat{\gamma}$	7.75						
	$\hat{\lambda}$	0.01						
	$\hat{\sigma}$	1.314						
GD	$\hat{\mu}$	6.968	15660	15660	15670	0.564	0	5.54
	$\hat{\sigma}$	8.189						

"The values in Table (3), indicate that the *GPD* is a strong competitor to other distributions used here for fitting data set". The variance covariance matrix of the MLEs under the *GPD* for data set is computed as

$$I^{-1} = \begin{pmatrix} 5.946\times10^{-4} & 1.542\times10^{-3} & 2.856\times10^{-6} & 3.25\times10^{-6} \\ 1.542\times10^{-3} & 5.374\times10^{-3} & 5.823\times10^{-6} & 6.992\times10^{-3} \\ 2.856\times10^{-6} & 5.823\times10^{-6} & 1.569\times10^{-8} & 1.757\times10^{-5} \\ 3.25\times10^{-6} & 6.992\times10^{-3} & 1.757\times10^{-5} & 0.02 \end{pmatrix}$$

The symmetry of the variance covariance matrix of the MLEs indicates numerical accuracy.

Conclusions:

This article for the first time defines a new four-parameter model using the *T-X* method, called the new gumbel- pareto distribution (*GPD*). Various properties of the distribution are studied. The moments, survival function, hazard function and the maximum likelihood estimates of the parameters, have been investigated. "The application of the new distribution has also been demonstrated with real life data". The results, compared with other known distributions, reveal that the *GPD* provides a better fit for modeling real life data.

Conflicts of Interest: None.

References:

1. Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Comm. Statist. Theory Methods 2002;31:497-512.
2. Nadarajah S, Kotz S. The beta Gumbel distribution. Math. Probl. Eng. 2004;4:323-332.
3. Famoye F, Lee C, Olumolade O. The beta-Weibull distribution. J. Stat. Theory Appl. 2005;4(2):121-136.
4. Nadarajah S, Kotz S. The beta exponential distribution. Reliab. Eng. Syst. Saf. 2006;91:689-697.
5. Kong L, Lee C, Sepanski J H. On the properties of beta-gamma distribution, J. Mod. Appl. Statist. Methods 2007;6(1):187-211.
6. Akinsete A, Famoye F, Lee C. The beta-Pareto distribution. Statistics. 2008;42(6): 547-563.
7. Barreto-Souza W, Santos A H S, Cordeiro G M. The beta generalized exponential distribution. J. Stat. Comput. Simul. 2010;80(2):159-172.
8. Mahmoudi E. The beta generalized Pareto distribution with application to lifetime data. Math. Comput. Simul. 2011;81:2414-2430.
9. Alshawarbeh E, Lee C, Famoye F. The beta-Cauchy distribution. J. Probab. Statist. Sci. 2012;10:41-58.
10. Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. Metron. 2013;71(1):63-79.
11. Al-Aqtash R, Famoye F, Lee C. On generating a new family of distributions using the logit function. J. Probab. Statist. Sci. 2015;13(1):135-152.
12. Al-Aqtash R, Lee C, Famoye F. Gumbel-Weibull distribution: Properties and applications. J. Mod. Appl. Statist. Methods. 2014;13:201–225.
13. Tahir M.H, Hussain M A, Cordeiro G M, Hamedani G G, Mansoor M, Zubair M. The Gumbel-Lomax Distribution: Properties and Applications. J. Stat. Theory Appl. 2016;15(1):61-79.
14. Choulakian V, Stephens M A. Goodness-of-fit for the generalized Pareto distribution. Technometrics. 2001;43(4):478-484.

توزيع جمبل باريتو- الخصائص والتطبيقات

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الخلاصة:

فى هذه المقالة قدمنا للمرة الاولى توزيعا جديدا به أربع معالم باستخدام طريقة *T-X* هذا التوزيع يسمى جمبل باريتو و قمنا بدراسة خصائصه وكذلك تقدير معالمه كما قمنا بتطبيقه على مجموعة من البيانات الحقيقية كمثال لتطبيقات هذا التوزيع و قارناه بتوزيعات أخرى وأثبت تفوقه فى تمثيل هذه البيانات.

الكلمات المفتاحية: توزيع جمبل، دالة هازرد، تقدير الامكان الأعظم، توزيع باريتو، دالة كوانتيل، طريقة *T-X*.