Open Access
Published Online First: February 2021

Baghdad Science Journal
2021,18(3):543-546

P-ISSN: 2078-8665
E-ISSN: 2411-7986

DOI: http://dx.doi.org/10.21123/bsj.2021.18.3.0543

Some Common Fixed Points Theorems of Four Weakly Compatible Mappings in Metric Spaces

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Received 22/11/2019, Accepted 15/9/2020, Published Online First 21/2/2021, Published 1/9/2021

Abstract:

In this paper, we proved coincidence points theorems for two pairs mappings which are defined on nonempty subset A in metric spaces X by using condition (1.1). As application, we established a unique common fixed points theorems for these mappings by using the concept weakly compatible (R-weakly commuting) between these mappings.

Key words: Coincidence points, Common fixed points, Weakly compatible mappings.

Introduction:

The common fixed point theory is so powerful in Mathematics. It can be used in various fields, for example, variation al inequalities, optimization, and approximation theory. The common fixed point theorems for mappings achieving certain contractive conditions in metric spaces have been continually studied for decades (see (1, 2, 3, 4, 5, 6)) and references contained therein.

In 2015 (7) Chahan, Imdad, Kadelbrg and Vetro proved a fixed and coincidence point theorem for a hybrid pair of compatible mappings via an implicit relation.

In 2017 (8) Xu, Chen and Aleksic proved common random fixed point theorems for the generalized quasi-contraction with the spectral radius $r\ (\lambda)$ of the quasi-contractive constant vector λ satisfying $r\ (\lambda) \in [0,\frac{1}{\delta})$ in the setting of cone b-metric spaces over Banach algebras, where the coefficient δ satisfies $\delta \geq 1$.

In 2018 (9) Tomar and Sharma proved common fixed point theorems for a pair of self-maps in metric space.

In 2019 (10), Mitrovic, Aydi, Hussain and Mukheimer proved certain common fixed point theorems in $\mathcal{F}$-metric spaces.

" Let A be a nonempty subset of a metric space X and let f and g be self-mappings of A. A point $a \in A$ is a common fixed [coincidence] point of f and g if $f(a) = g(a) = a$ $[f(a) = g(a)]$ [9]". The set of coincidence points of f and g is denoted by (f,g) , the set of fixed points of f is denoted by $F(f)$ and the closure of the set A is denoted by $\bar{A}$. The pair (f,g) said to be

(a) " Commuting maps if $fg(a) = gf(a)$, for all $a \in A$."(11);

(b) " R-weakly commuting maps if for all $a \in A$ there exists $R > 0$ such that $d(fg(a), gf(a)) < Rd(f(a), g(a))$, if R=1, then the maps are called weakly commuting." (12);

(c) " weakly compatible if they commute at their coincidence points, that is $fg(a) = gf(a)$ whenever $f(a) = g(a)$." (13).

In 2016 (14) Choudhury and Som proved unique fixed point results for quasi contraction mappings on a metric space satisfying some generalized inequality conditions and unique common fixed point result for asymptotically regular mappings of certain kind and satisfying a generalized contraction condition .

Main Results:

Theorem (1.1).

Let $f_1, f_2, g_1, g_2: A \to A$ such that for all $a_1, a_2 \in A$ and $0 \leq M < \frac{1}{2}$, the pair (f_1, f_2) satisfy the following condition

$$d\big(f_1(a_1), f_2(a_2)\big) \leq M \max\{d\big(g_1(a_1), g_2(a_2)\big), d\big(g_1(a_1), f_1(a_1)\big), d\big(g_2(a_2), f_2(a_2)\big), d\big(g_1(a_1), f_2(a_2)\big),$$

$d(g_2(a_2), f_1(a_1))\} \dots (1.1)$

If $\overline{f_1(A)} \subseteq g_2(A), \overline{f_2(A)} \subseteq g_1(A)$ and one of the Subsets $\overline{f_1(A)}, \overline{f_2(A)}, \overline{g_1(A)}$ or $\overline{g_2(A)}$ is complete, then $c(f_1, g_1) \neq \emptyset$ and $c(f_2, g_2) \neq \emptyset$.

Proof:

Let $a_\circ$ be arbitrary, Using $\overline{f_1(A)} \subseteq g_2(A)$and $\overline{f_2(A)} \subseteq g_1(A)$, there exists two sequences $\{b_{2k}\}$ and $\{b_{2k+1}\}$, $k \in \mathbb{N}$, such that

$b_{2k} = f_1(a_{2k}) = g_2(a_{2k+1})$.......(1.2)

$b_{2k+1} = f_2(a_{2k+1}) = g_1(a_{2k+2})$.......(1.3)

From (1.2), (1.3) and (1.1), we have

$$d(b_{2k+2}, b_{2k+1}) = d(f_1(a_{2k+2}), f_2(a_{2k+1}))$$
$$\leq M \max\{d(g_1(a_{2k+2}), g_2(a_{2k+1})), d(g_1(a_{2k+2}), f_1(a_{2k+2})), d(g_2(a_{2k+1}), f_2(a_{2k+1})), d(g_1(a_{2k+2}), f_2(a_{2k+1})), d(g_2(a_{2k+1}), f_1(a_{2k+2}))\}$$
$$= M \max\{d(b_{2k+1}, b_{2k}), d(b_{2k+1}, b_{2k+2}), d(b_{2k}, b_{2k+1}), d(b_{2k+1}, b_{2k+1}), d(b_{2k}, b_{2k+2})\}$$
$$= M \max\{d(b_{2k+1}, b_{2k}), d(b_{2k+1}, b_{2k+2}), d(b_{2k}, b_{2k+2})\}$$

Using the following condition of metric space $(x, z) \leq d(x, y) + d(y, z)$, we have

$$\leq M \max\{d(b_{2k+1}, b_{2k}), d(b_{2k+1}, b_{2k+2}), d(b_{2k}, b_{2k+1}) + d(b_{2k+1}, b_{2k+2})\}$$
$$= Md(b_{2k}, b_{2k+1}) + Md(b_{2k+1}, b_{2k+2})$$
$$d(b_{2k+1}, b_{2k+2}) - Md(b_{2k+1}, b_{2k+2}) \leq Md(b_{2k}, b_{2k+1})$$
$$(1 - M)d(b_{2k+1}, b_{2k+2}) \leq Md(b_{2k}, b_{2k+1})$$

Hence, $d(b_{2k+1}, b_{2k+2}) \leq \gamma d(b_{2k}, b_{2k+1})$, where $\gamma = \frac{M}{1-M} < 1$, by similar way we have $d(b_{2k-1}, b_{2k}) \leq \gamma d(b_{2k-2}, b_{2k-1})$, therefore

$$d(b_{2k}, b_{2k+1}) \leq \gamma d(b_{2k-1}, b_{2k}) \leq \gamma^2 d(b_{2k-2}, b_{2k-1}) \leq \cdots \leq \gamma^k d(b_0, b_1)$$

To prove $\langle b_k \rangle$ is a Cauchy sequence, for $k, z \in \mathbb{N}, k > z$

$$d(b_k, b_z) \leq d(b_k, b_{k-1}) + d(b_{k-1}, b_{k-2}) + \cdots + d(b_{z+1}, b_z)$$
$$\leq (\gamma^{k-1} + \gamma^{k-2} + \cdots + \gamma^z) d(b_0, b_1)$$
$$\leq \frac{\gamma^z}{1-\gamma} d(b_0, b_1)$$

Let $\varepsilon > 0$ be given, choose a natural number v large enough such that $\frac{\gamma^z}{1-\gamma} < \varepsilon$ for every $k > z \geq v$. So, $\langle b_k \rangle$is a Cauchy sequence. Suppose that $\overline{g_2(A)}$ is a complete subspace of X, this implies the sequence $\langle b_k \rangle$ has a limit b. Since $\overline{f_1(A)} \subseteq g_2(A)$, then $g_2(a) = b$ for some $a \in A$. Thus we have

$$b = \lim_{k\to\infty} b_{2k} = \lim_{k\to\infty} f_1(a_{2k}) = \lim_{k\to\infty} g_2(a_{2k+1}) = \lim_{k\to\infty} g_1(a_{2k+1}) = \lim_{k\to\infty} f_2(a_{2k})$$

Using (1.2) and (1.1), we have

$$d(b_{2k}, f_2(a)) = d(f_1(a_{2k}), f_2(a))$$
$$\leq M \max\{d(g_1(a_{2k}), g_2(a)), d(g_1(a_{2k}), f_1(a_{2k})), d(g_2(a), f_2(a)), d(g_1(a_{2k}), f_2(a)), d(g_2(a), f_1(a_{2k}))\}$$

Taking $k \to \infty$, we get

$$d(b, f_2(a)) \leq M \max\{d(b, g_2(a)), d(b, b), d(g_2(a), f_2(a)), d(b, f_2(a)), d(g_2(a), b)\}$$
$$= M \max\{d(b, g_2(a)), d(g_2(a), f_2(a)), d(b, f_2(a))\}$$

From $b = g_2(a)$, we get

$d(b, f_2(a)) \leq$M $d(b, f_2(a)) \Rightarrow$

$(1 - M)d(b, f_2(a)) \leq 0$

Hence, $b = f_2(a) = g_2(a)$......(1.4)

Therefore $c(f_2, g_2) \neq \emptyset$.

Since $\overline{f_2(A)} \subseteq g_1(A)$, then $b \in g_1(A)$. We are obtained $w \in A$ such that $g_1(w) = b$......(1.5).

Using (1.4), (1.1) and (1.5), we have

$$d(f_1(w), b) = d(f_1(w), f_2(a))$$
$$\leq M \max\{d(g_1(w), g_2(a)), d(g_1(w), f_1(w)), d(g_2(a), f_2(a)), d(g_1(w), f_2(a)), d(g_2(a), f_1(w))\}$$
$$= M \max\{d(b, b), d(b, f_1(w)), d(b, b), d(b, b), d(b, f_1(w))\}$$

This implies $d(b, f_1(w)) = 0$ therefore $b = f_1(w) = g_1(w)$. Hence w a coincidence point of g_1 and f_1 . we conclude that

$f_1(w) = g_1(w) = b = f_2(a) = g_2(a)$......(1.6).

Theorem (1.2).

Let $f_1, f_2, g_1, g_2, \overline{f_1(A)}, \overline{f_2(A)}, \overline{g_1(A)}$ and $\overline{g_2(A)}$ as in theorem (1.1). If the pairs $(f_1, g_1)$and (f_2, g_2) are weakly compatible (R-weakly commuting), then $F(f_1) \cap F(f_2) \cap F(g_1) \cap F(g_2)$ is a single element.

Proof:

By theorem(1.1), there exists coincidence point a of f_2 and g_2 such that $f_2(a) = g_2(a)$ and coincidence point w of f_1 and $g_1$such that $f_1(w) = g_1(w)$. If the pairs $(f_1, g_1)$and (f_2, g_2) are weakly compatible, then

$f_1(g_1(w)) = g_1(f_1(w))$ and

$f_2(g_2(a)) = g_2(f_2(a))$

By (1.6), we get

$f_1(b) = g_1(b)$ and $f_2(b) = g_2(b)$.....(1.7)

Using (1.6), (1.1) and (1.7), we get

$$d(b, f_2(b)) = d(f_1(w), f_2(b))$$
$$\leq M \max\{d(g_1(w), g_2(b)), d(g_1(w), f_1(w)), d(g_2(b), f_2(b)), d(g_1(w), f_2(b)), d(g_2(b), f_1(w))\}$$
$$= M \max\{d(b, f_2(b)), d(b, b), d(g_2(b), g_2(b)), d(b, f_2(b)), d(b, f_2(b))\}$$
$$= M \{d(b, f_2(b))\}$$

Then we get, $(1 - M)d(b, f_2(b)) \leq 0$, this implies $b = f_2(b)$, using (1.7) we have

$b = g_2(b) = f_2(b)$......(1.8)

Again, by (1.8), (1.1) and (1.7), we have

$d(f_1(b),b) = d(f_1(b),f_2(b))$
$\leq M \max\{d(g_1(b),g_2(b)), d(g_1(b),f_1(b)),$
$d(g_2(b),f_2(b)), d(g_1(b),f_2(b)), d(g_2(b),f_1(b))\}$
$= M \max\{d(f_1(b),b), d(f_1(b),f_1(b)), d(b,b),$
$d(f_1(b),b), d(b,f_1(b))\}$
$= M\, d(f_1(b),b)$
Getting $(1-M)d(f_1(b),b) \leq 0$, this implies $b = f_1(b)$, using (1.7) we have
$b = g_1(b) = f_1(b)$……(1.9)
Using (1.8). we have
$b = g_1(b) = f_1(b) = g_2(b) = f_2(b)$…..(1.10)
Thus b is a common fixed point of f_1, f_2, g_1 and g_2.
Singleton: let c be another common fixed point of f_1, f_2, g_1 and g_2, then from (1.1) we get
$d(b,c) = d(f_1(b),f_2(c))$
$\leq M \max\{d(g_1(b),g_2(c)), d(g_1(b),f_1(b)),$
$d(g_2(c),f_2(c)), d(g_1(b),f_2(c)),, d(g_2(c),f_1(b))\}$
$= M \max\{d(b,c), d(b,b), d(c,c), d(b,c), d(c,b)$
$= M\, d(b,c)$
We get $(1-M)\, d(b,c) \leq 0$, this implies $b = c$.
Suppose that (f_1, g_1) is R-weakly commuting and w is a coincidence point of f_1 and g_1, then
$\mathrm{d}(f_1 g_1(w), (g_1 f_1(w)) \leq R\mathrm{d}(f_1(w), (g_1(w)) = 0$, hence $f_1 g_1(w) = g_1 f_1(w)$, therefore the pair (f_1, g_1) is weakly compatible.
Similarly, we have (f_2, g_2) is weakly compatible, then the same steps above we can show that b is a unique common fixed point of f_1, f_2, g_1 and g_2.
Remark. Put $f_1 = f_2$, $g_{1=}g_2$ and $0 \leq M < 1$ in the above theorem, we have the theorem (2.1) in (13).

Conclusion:

In this work, common fixed (coincidence) points of two pairs mappings which are defined on nonempty subset A of metric space X are found using the condition (1.1) and some relations between the pair of mappings.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Al-Mustansiriyah University.

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بعض مرهنات النقاط الصامدة المشتركة لأربع تطبيقات متوافقة بضعف في الفضاءات المترية

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الخلاصة:
في هذا البحث، سنثبت مبرهنات نقاط التطابق لزوجين من التطبيقات معرفة على مجموعة جزئية غير خالية A من الفضاء المتري X بأستخدام الشرط (1.1). كتطبيق سنثبت مبرهنات وحدانية النقاط الصامدة المشتركة لهذه التطبيقات باستخدام مفهوم التطابق بضعف أوR - ابدالية بضعف بين هذه التطبيقات.

الكلمات المفتاحية: نقاط التطابق، النقاط الصامدة المشتركة، تطبيقات متوافقة بضعف .