

New Common Fixed Points for Total Asymptotically Nonexpansive Mapping in CAT(0) Space

Ahmed Jameel Kadhim

Computer Science Department, College of Science for Women, University of Baghdad, Baghdad, Iraq
E-mail: ahmedjk_comp@cs.w.uobaghdad.edu.iq
ORCID ID: <https://orcid.org/0000-0002-1121-6515>

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Abstract:

Strong and Δ -convergence for a two-step iteration process utilizing asymptotically nonexpansive and total asymptotically nonexpansive mappings in the CAT(0) spaces have been studied. As well, several strong convergence theorems under semi-compact and condition (M) have been proved. Our results improve and extend numerous familiar results from the existing literature.

Keywords: Asymptotically nonexpansive mapping, CAT(0) space, Common fixed point, Condition (M), Δ -convergence.

Introduction:

A metric space G is a CAT(0) space, if it is geodesically connected and if each geodesic triangle in G is at least as thin as its comparison triangle in the Euclidean plane. Some typical examples of CAT(0) spaces are R-trees, Pre-Hilbert space and Euclidean buildings (1).

Fixed point theory in CAT(0) spaces was foremost initialized through Kirk (1). He proved that each nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) spaces permanently has fixed point. Therefore, the fixed point theory for single valued as well multivalued mappings in CAT(0) spaces has intensively been evolved by numerous authors. The convergence for nonexpansive mappings in CAT(0) spaces was studied by Dhompongsa-Panyanak (2). Thereafter, Khan and Abbas (3) studied the strong and Δ -convergence in CAT(0) space for an iteration process that is independent of the Ishikawa iteration process. As well, several of these results obtained for two nonexpansive mappings. It is important to remember that fixed point theorems in CAT(0) space can be stratified to graph theory, computer science and biology (1).

Let (G, d) be a metric space and $u, v \in G$ with $d(u, v) = x$. A geodesic path from u to v , this means an isometry $c: [0, x] \rightarrow c([0, 1]) \subseteq G$ such as $c(0) = u$ and $c(x) = v$. The image of every

geodesic path between u and v is called geodesic segment. Each point y in the segment is appeared by $\omega u \oplus (1 - \omega)v$, where $\omega \in [0, 1]$ that is $[u, v] = \{\omega u \oplus (1 - \omega)v : \omega \in [0, 1]\}$. The space (G, d) is called a geodesic if each two points of G are joined through a geodesic segment, and G is uniquely geodesic if there exists properly one geodesic joining u and v for every $u, v \in G$. A subset H of G is called convex if H has each geodesic segment joining any two points in H (4-6).

A geodesic triangle $\Delta(u_1, u_2, u_3)$ is a geodesic metric space (G, d) that consists of three points u_1, u_2, u_3 in G (the vertices Δ) and a geodesic segment between every pair of vertices (the edges of Δ). A comparison triangle $\bar{\Delta}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in W^2 for $\Delta(u_1, u_2, u_3)$ is a triangle in 2-dimensional Euclidean plane W^2 with $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in W^2$ such as $d(u_1, u_2) = |\bar{u}_1 - \bar{u}_2|_{W^2}$, $d(u_1, u_3) = |\bar{u}_1 - \bar{u}_3|_{W^2}$, $d(u_2, u_3) = |\bar{u}_2 - \bar{u}_3|_{W^2}$, where $|\cdot|_{W^2}$ is the Euclidean norm on W^2 (7).

CAT(0): A geodesic space is called CAT(0) space if whole geodesic triangles achieve the following comparison axiom.

Let Δ be a geodesic triangle in G and $\bar{\Delta} \subseteq W^2$ be a comparison triangle for Δ . Therefore, Δ is called to achieve the CAT(0) inequality if $\forall u, v \in \Delta$ & $\forall \bar{u}, \bar{v} \in \bar{\Delta}$, $d(u, v) \leq d_{W^2}(\bar{u}, \bar{v})$.

If u, v_1, v_2 are the points in $CAT(0)$ and if $v_0 = \frac{1}{2}(v_1 \oplus v_2)$, therefore the $CAT(0)$ inequality leads to

$$d(u, v_0)^2 \leq \frac{1}{2}d(u, v_1)^2 + \frac{1}{2}d(u, v_2)^2 - \frac{1}{4}d(v_1, v_2)^2$$

Which is the (CN) inequality of Bruhat and Tits. In verity, a geodesic space is a $CAT(0)$ space if it accomplishes (CN)(3).

Lemma (1)(8): Let (G, d) be a $CAT(0)$ space. Therefore,

$$d((1-k)a \oplus kb, c)^2 \leq (1-k)d(a, c)^2 + kd(b, c)^2 - k(1-k)d(a, b)^2$$

for all $k \in [0, 1]$ and $a, b, c \in G$.

Let $\{u_n\}$ be a bounded sequence in a $CAT(0)$ space G . For $u \in G$, setting

$$r(u, \{u_n\}) = \limsup_{n \rightarrow \infty} d(u, u_n).$$

The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given through

$$r(\{u_n\}) = \inf\{r(u, \{u_n\}) : u \in G\},$$

and the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is defined as

$$A(\{u_n\}) = \{u \in G : r(u, \{u_n\}) = r(\{u_n\})\}$$

It is familiar that in $CAT(0)$ space, $A(\{u_n\})$ has punctually one point.

Numerous iteration processes have been structured and suggested in order to approximate fixed points. The Picard iteration for a mapping $T: E \rightarrow E$ is defined by

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= T^n u_n \end{aligned} \quad (1)$$

The modified Mann iteration is considered by Schu (5), as below

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= (1 - \delta_n)u_n + \delta_n T^n u_n \end{aligned} \quad (2)$$

Where $\{\delta_n\} \in (0, 1)$.

The modified Ishikawa iteration is studied by Tan and Xu (5), as below

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= (1 - \delta_n)u_n + \delta_n T^n v_n \\ v_n &= (1 - \beta_n)u_n + \beta_n T^n u_n \end{aligned} \quad (3)$$

Where $\{\delta_n\}$ and $\{\beta_n\} \in (0, 1)$. The iteration decreases to the modified Mann iteration when $\beta_n = 0, \forall n \geq 1$.

Lately, the modified S-iteration in a Banach space is introduced by Agarwal et al. (5), as below

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= (1 - \delta_n)T^n u_n + \delta_n T^n v_n \\ v_n &= (1 - \beta_n)u_n + \beta_n T^n u_n, \quad \forall n \geq 1 \end{aligned} \quad (4)$$

where $\{\delta_n\}$ and $\{\beta_n\} \in (0, 1)$. Notice that this iteration is independent of Ishikawa and Mann iterations.

Recently, Sahin and Basarir (5) modified the above iteration in a $CAT(0)$ space, as follows

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= (1 - \delta_n)T^n u_n \oplus \delta_n T^n v_n \\ v_n &= (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \quad \forall n \geq 1 \end{aligned} \quad (5)$$

The following iteration has been studied by M. R. Yadava (9) for common fixed points of two self mappings S and T ,

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= \delta_n u_n + \gamma_n T u_n + \beta_n S v_n \\ v_n &= (1 - \gamma_n)u_n + \gamma_n T u_n, \quad n \in N \end{aligned} \quad (6)$$

Where $\{\delta_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $\delta_n + \gamma_n + \beta_n = 1$. This iteration as well decreases to Mann iteration when $T = I$ or $\gamma_n = 0$.

Inspired and motivated by the work of M. R. Yadava (6), the iteration (6) for common fixed points of two mapping asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in a $CAT(0)$ space is modified, as follows.

Deem E to be a nonempty closed convex subset of a complete $CAT(0)$ space G , $T: E \rightarrow E$ to be an asymptotically nonexpansive and $S: E \rightarrow E$ to be a total asymptotically nonexpansive mappings. Presume that $\{u_n\}$ is a sequence produced by

$$\begin{aligned} u_1 &= u \in E \\ u_{n+1} &= P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n) \\ v_n &= P((1 - \gamma_n)u_n \oplus \gamma_n T(PT)^{n-1} u_n), \quad n \in N \end{aligned} \quad (7)$$

where $\{\delta_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $\delta_n + \gamma_n + \beta_n = 1$ and P is a nonexpansive retraction of G onto E .

In this paper, a new iteration for approximating a common fixed point of asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings is constructed. Some strong convergence theorems and Δ -convergence theorem under appropriate conditions like semi-compact and condition (M) in $CAT(0)$ spaces are proved. As well, numerical example to elucidate our work is provided.

Preliminaries

Let (G, d) be a metric space & E be a nonempty subset of G . Deem $T: E \rightarrow E$ to be a mapping. A point $a \in E$ is called a fixed point of T if $Ta = a$. As well the set of common fixed points of T and S denote by F which is $F = \{a \in E : Ta = Sa = a\}$. Call that E is called retract of G if there is a continuous mapping $P: G \rightarrow E$ such as $Pa = a, \forall a \in E$. A mapping $P: G \rightarrow E$ is called a retraction if $P^2 = P$. If P is a retraction, then $Pb = b, \forall b$ in the range of P .

- A mapping $T: E \rightarrow E$ is called nonexpansive (10) if $d(Ta, Tb) \leq d(a, b), \forall a, b \in E$.
- A mapping $T: E \rightarrow E$ is called asymptotically nonexpansive (11) if \exists a sequence $\{\epsilon_n\} \subset [1, \infty)$ with $\epsilon_n \rightarrow 1$ such as $d(T^n a, T^n b) \leq \epsilon_n d(a, b), \forall n \geq 1, \forall a, b \in E$.
- A mapping $T: E \rightarrow G$ is called asymptotically nonexpansive nonself (11) if \exists a sequence $\{\epsilon_n\} \subset [1, \infty)$ with $\epsilon_n \rightarrow 1$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq \epsilon_n d(a, b), \forall n \geq 1, \forall a, b \in E$, where P is a nonexpansive retraction of G onto E .
- A mapping $T: E \rightarrow E$ is called uniformly L-lipschitzain (12) if \exists a constant $L > 0$ such as $d(T^n a, T^n b) \leq Ld(a, b), \forall n \geq 1, \forall a, b \in E$.
- A mapping $T: E \rightarrow E$ is called total asymptotically nonexpansive (11) if \exists positive sequences $\{e_n\}, \{\sigma_n\}$ with $e_n \rightarrow 0, \sigma_n \rightarrow 0$ and a strictly nondecreasing continuous function $\vartheta: [0, \infty) \rightarrow [0, \infty)$ with $\vartheta(0) = 0$ such as $d(T^n a, T^n b) \leq d(a, b) + e_n \vartheta d(a, b) + \sigma_n, \forall n \geq 1, \forall a, b \in E$.
- A mapping $T: E \rightarrow G$ is called total asymptotically nonexpansive nonself (11) if \exists positive sequences $\{e_n\}, \{\sigma_n\}$ with $e_n \rightarrow 0, \sigma_n \rightarrow 0$ and a strictly nondecreasing continuous function $\vartheta: [0, \infty) \rightarrow [0, \infty)$ with $\vartheta(0) = 0$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq d(a, b) + e_n \vartheta d(a, b) + \sigma_n, \forall n \geq 1, \forall a, b \in E$, where P is a nonexpansive retraction of G onto E .
- A mapping $T: E \rightarrow G$ is called uniformly L-lipschitzain (11) if \exists a constant $L > 0$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq Ld(a, b), \forall n \geq 1, \forall a, b \in E$.

The notion of asymptotically nonexpansive mapping was foremost introduced by Gloebel and Kirk. Therefore Alber et al. introduced the class of total asymptotically nonexpansive, which generalizes some classes of mappings that are spans of asymptotically nonexpansive. Several authors have been extensively studied these classes of mappings (6).

Definition (2)(13): A sequence $\{u_n\}$ in a CAT(0) space G is called Δ -convergence to $u \in G$ if u is the unique asymptotic center of $\{v_n\} \forall$ subsequence $\{v_n\}$ of $\{u_n\}$. Here, note down $\Delta - \lim_{n \rightarrow \infty} u_n = u$ and u is the Δ -limit of $\{u_n\}$.

Note that given $\{u_n\} \subset G, \{u_n\} \Delta$ -convergence to u and $v \in G$ with $v \neq u$ through the uniqueness of the asymptotic center that gives

$$\limsup_{n \rightarrow \infty} d(u_n, u) \leq \limsup_{n \rightarrow \infty} d(u_n, v)$$

Therefore, each CAT(0) space achieves the Opial property.

Lemma (3)(8): Let G be a CAT(0) space and $a \in G$. Presume $\{s_n\}$ is a sequence in $[z, c]$ for several $z, c \in (0, 1)$ and $\{a_n\}, \{b_n\}$ are sequences in G such as $\lim_{n \rightarrow \infty} \sup d(a_n, h^*) \leq t, \lim_{n \rightarrow \infty} \sup d(b_n, h^*) \leq t$ and $\lim_{n \rightarrow \infty} d((1 - s_n)a_n \oplus s_nb_n) = t$ for several $t \geq 0$. Thus $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$.

Lemma (4)(6): Let $\{\zeta_n\}, \{\alpha_n\}$ and $\{\lambda_n\}$ be the sequences of positive numbers such as $\zeta_{n+1} \leq (1 + \alpha_n)\zeta_n + \lambda_n, \forall n \geq 1$.

- If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, therefore $\lim_{n \rightarrow \infty} \zeta_n$ exists.
- If there is a subsequence $\{\zeta_{n_i}\} \subset \{\zeta_n\}$ such as $a_{n_i} \rightarrow 0$ thus $\lim_{n \rightarrow \infty} \zeta_n = 0$.

Lemma (5)(14): Each bounded sequence in a complete CAT(0) space G holds a Δ -convergence subsequence.

Lemma (6)(15): If E is closed convex subset of a complete CAT(0) space G and if $\{u_n\}$ is bounded sequence in E , thus the asymptotic center of $\{u_n\}$ is in E .

Theorem (7)(11): Let E be a closed convex subset of a complete CAT(0) space G . Let T be a mapping accomplishing one of the following conditions:

- $T: E \rightarrow E$ is an asymptotically nonexpansive mapping with a sequence $\{\epsilon_n\} \subset [1, \infty)$ & $\epsilon_n \rightarrow 1$.
- $T: E \rightarrow G$ is an asymptotically nonexpansive nonself mapping.
- $T: E \rightarrow E$ is a total asymptotically nonexpansive mapping.

Let $\{u_n\}$ be a bounded sequence in E such as $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$ and $\Delta - \lim_{n \rightarrow \infty} u_n = h^*$. Thus, $Th^* = h^*, h^* \in F$.

The convergence results

In this part, Δ -convergence and some strong convergence theorems by using iteration (7) for asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in CAT(0) spaces are proved.

Theorem (8): Let E be a nonempty closed convex subset of a complete CAT(0) space G . Let $T: E \rightarrow E$ be a uniformly L-lipschitzain and asymptotically nonexpansive and $S: E \rightarrow G$ be a uniformly L-lipschitzain total asymptotically nonexpansive nonself mappings with $F(T, S) \neq \emptyset$. Presume that $\{u_n\}$ is defined by (7). If $F := F(T) \cap F(S)$ and the following conditions are accomplished:

i) $\sum_{n=1}^{\infty} e_n < \infty$ & $\sum_{n=1}^{\infty} \sigma_n < \infty$.

ii) There is a constant $B^* > 0$ such as $\vartheta(\mathfrak{r}) < B^* \mathfrak{r}$, $\mathfrak{r} \geq 0$.

Thus, the sequence $\{u_n\}$ is Δ -convergence to a several points $h^* \in F$ ($F := F(T) \cap F(S)$).

Proof: Step 1: Firstly, proving that $\lim_{n \rightarrow \infty} d(u_n, h^*)$ for each $h^* \in F$ and $\lim_{n \rightarrow \infty} d(u_n, F)$ exist.

Since $h^* \in F$, $Ph^* = h^*$. Now,

$$\begin{aligned} d(v_n, h^*) &= d(P((1 - \gamma_n)u_n \\ &\quad \oplus \gamma_n T(PT)^{n-1} u_n), Ph^*) \\ &\leq d((1 - \gamma_n)u_n \oplus \gamma_n T(PT)^{n-1} u_n, Ph^*) \\ &\leq (1 - \gamma_n)d(u_n, h^*) \\ &\quad + \gamma_n d(T(PT)^{n-1} u_n, h^*) \\ &\leq (1 - \gamma_n)d(u_n, h^*) + \gamma_n \epsilon_n d(u_n, h^*) \\ &= (1 - \gamma_n + \gamma_n \epsilon_n)d(u_n, h^*) \end{aligned}$$

Thus,

$$\begin{aligned} d(u_{n+1}, h^*) &= d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq d(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n, h^*) \\ &\leq \delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) \\ &\quad + \beta_n d(S(PS)^{n-1} v_n, h^*) \\ &\leq \delta_n d(u_n, h^*) + \gamma_n \epsilon_n d(u_n, h^*) \\ &\quad + \beta_n [d(v_n, h^*) + e_n \vartheta d(v_n, h^*) \\ &\quad + \sigma_n] \\ &\leq \delta_n d(u_n, h^*) + \gamma_n \epsilon_n d(u_n, h^*) \\ &\quad + \beta_n (1 + e_n B^*) d(v_n, h^*) + \beta_n \sigma_n \\ &\leq \delta_n d(u_n, h^*) + \gamma_n \epsilon_n d(u_n, h^*) \\ &\quad + \beta_n (1 + e_n B^*) (1 - \gamma_n \\ &\quad + \gamma_n \epsilon_n) d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n \epsilon_n + \beta_n (1 + e_n B^*) (1 - \gamma_n \\ &\quad + \gamma_n \epsilon_n)] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n \epsilon_n + \beta_n + \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n \\ &\quad + \beta_n \gamma_n \epsilon_n \\ &\quad + \beta_n e_n B^* \gamma_n \epsilon_n] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n + \beta_n + \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n \\ &\quad + \beta_n \gamma_n \epsilon_n \\ &\quad + \beta_n e_n B^* \gamma_n \epsilon_n] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [1 + (\beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n \epsilon_n \\ &\quad + \beta_n e_n B^* \gamma_n \epsilon_n)] d(u_n, h^*) + \beta_n \sigma_n \\ &= (1 + \rho_n) d(u_n, h^*) + \theta_n \end{aligned}$$

where $\rho_n := \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n \epsilon_n + \beta_n e_n B^* \gamma_n \epsilon_n$ and $\theta_n := \beta_n \sigma_n$.

Whereas $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, it follows up that $\sum_{n=1}^{\infty} \rho_n < \infty$ & $\sum_{n=1}^{\infty} \theta_n < \infty$. Therefore through Lemma (4),

$\lim_{n \rightarrow \infty} d(u_n, h^*) = 0, \forall h^* \in F$ and $\lim_{n \rightarrow \infty} d(u_n, F)$ exist.

Step 2: Next, proving that

$$\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(u_n, Su_n) = 0.$$

In verity, it follows up from step (1) that for each given $h^* \in F$, $\lim_{n \rightarrow \infty} d(u_n, h^*)$ exists. Presume that

$$\lim_{n \rightarrow \infty} d(u_n, h^*) = r, \quad r \geq 0$$

$$\begin{aligned} d(v_n, h^*) &= d(P((1 - \gamma_n)u_n \\ &\quad \oplus \gamma_n T(PT)^{n-1} u_n), Ph^*) \\ &\leq (1 - \gamma_n)d(u_n, h^*) \\ &\quad + \gamma_n d(T(PT)^{n-1} u_n, h^*) \\ &\leq (1 - \gamma_n)d(u_n, h^*) + \gamma_n \epsilon_n d(u_n, h^*) \end{aligned}$$

Hence,

$$\limsup_{n \rightarrow \infty} d(v_n, h^*) \leq r \quad (8)$$

As well from $d(T(PT)^{n-1} u_n, h^*) \leq \epsilon_n d(u_n, h^*)$, $\forall n = 1, 2, \dots$, gives

$$\limsup_{n \rightarrow \infty} d(T(PT)^{n-1} u_n, h^*) \leq r$$

Now,

$$d(S(PS)^{n-1} v_n, h^*) \leq ((1 + e_n B^*)d(v_n, h^*) + \sigma_n)$$

Therefore,

$$\limsup_{n \rightarrow \infty} d(S(PS)^{n-1} v_n, h^*) \leq r$$

Moreover,

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} d(u_{n+1}, h^*) \\ &= \lim_{n \rightarrow \infty} d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq \lim_{n \rightarrow \infty} [\delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) \\ &\quad + \beta_n d(S(PS)^{n-1} v_n, h^*)] \end{aligned}$$

=

$$\begin{aligned} &(\delta_n + \gamma_n) d(T(PT)^{n-1} u_n, h^*) + \\ &\quad \beta_n d(S(PS)^{n-1} v_n, h^*) \\ &= (1 - \beta_n) d(T(PT)^{n-1} u_n, h^*) \\ &\quad + \beta_n d(S(PS)^{n-1} v_n, h^*) \end{aligned}$$

By Lemma (3), getting

$$\lim_{n \rightarrow \infty} d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) = 0$$

Now,

$$\begin{aligned} d(u_{n+1}, h^*) &= d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &= d(P(\delta_n T(PT)^{n-1} u_n \\ &\quad \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq d((1 - \beta_n) T(PT)^{n-1} u_n \\ &\quad + \beta_n S(PS)^{n-1} v_n, h^*) \\ &\leq d(T(PT)^{n-1} u_n, h^*) \\ &\quad + \beta_n d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \end{aligned}$$

Which yields that

$$r \leq \liminf_{n \rightarrow \infty} d(T(PT)^{n-1} u_n, h^*)$$

That gives

$$\lim_{n \rightarrow \infty} d(T(PT)^{n-1} u_n, h^*) = r$$

In turn,

$$\begin{aligned} &d(T(PT)^{n-1} u_n, h^*) \\ &\leq d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \\ &\quad + d(S(PS)^{n-1} v_n, h^*) \\ &\leq d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \\ &\quad + (1 + e_n B^*) d(v_n, h^*) + \sigma_n \end{aligned}$$

That implies

$$r \leq \liminf_{n \rightarrow \infty} d(v_n, h^*) \quad (9)$$

From (8) and (9), that deduces

$$\lim_{n \rightarrow \infty} d(v_n, h^*) = r$$

Again,

$$\begin{aligned} r &= \lim_{n \rightarrow \infty} d(v_n, h^*) \\ &= \lim_{n \rightarrow \infty} d(P(1 - \gamma_n)u_n \oplus \gamma_n T(PT)^{n-1} u_n, h^*) \end{aligned}$$

Through Lemma (3), getting

$$\lim_{n \rightarrow \infty} d(T(PT)^{n-1} u_n, u_n) = 0$$

Notice that

$$\lim_{n \rightarrow \infty} d(v_n, h^*) = \gamma_n d(T(PT)^{n-1} u_n, h^*)$$

Therefore,

$$\lim_{n \rightarrow \infty} d(v_n, h^*) = 0$$

Now,

$$\begin{aligned} d(u_{n+1}, u_n) &= d(P((\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \\ &\quad \oplus \beta_n S(PS)^{n-1} v_n), u_n) \\ &\leq d((\delta_n + \gamma_n)T(PT)^{n-1} u_n \\ &\quad + \beta_n S(PS)^{n-1} v_n, u_n) \\ &\leq (1 - \beta_n)d(T(PT)^{n-1} u_n, u_n) \\ &\quad + \beta_n d(S(PS)^{n-1} v_n, u_n) \end{aligned}$$

This gives

$$\lim_{n \rightarrow \infty} d(u_{n+1}, u_n) = 0$$

Thus,

$$\begin{aligned} d(u_{n+1}, v_n) &\leq d(u_{n+1}, u_n) + d(v_n, u_n) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Which gives

$$\lim_{n \rightarrow \infty} d(u_{n+1}, v_n) = 0$$

Moreover, from

$$\begin{aligned} d(u_{n+1}, S(PS)^{n-1} v_n) \\ &\leq d(u_{n+1}, u_n) + d(u_n, T(PT)^{n-1} u_n) \\ &\quad + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \end{aligned}$$

That gives

$$\lim_{n \rightarrow \infty} d(u_{n+1}, S(PS)^{n-1} v_n) = 0.$$

$$\begin{aligned} d(u_n, T(PT)^{n-1} u_n) \\ &\leq d(u_n, T(PT)^{n-1} u_n) \\ &\quad + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \\ &\quad + d(S(PS)^{n-1} v_n, T(PT)^{n-1} u_n) \end{aligned}$$

Gives that

$$\lim_{n \rightarrow \infty} d(u_n, T(PT)^{n-1} u_n) = 0$$

And

$$\begin{aligned} d(u_n, Tu_n) &\leq d(u_n, u_{n+1}) \\ &\quad + d(u_{n+1}, T(PT)^n u_{n+1}) \\ &\quad + d(T(PT)^n u_{n+1}, T(PT)^n u_n) \\ &\quad + d(T(PT)^n u_n, Tu_n) \end{aligned}$$

By uniformly L-lipschitzain, getting

$$\begin{aligned} &\leq (1 + L)d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) \\ &\quad + L d((PT)^n u_n, u_n) \\ &= (1 + L)d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) \\ &\quad + L d(PT(PT)^{n-1} u_n, u_n) \\ &\leq (1 + L)d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) \\ &\quad + L d(T(PT)^{n-1} u_n, u_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0 \quad (10)$$

and $d(u_n, Su_n) \leq d(u_n, Tu_n) + d(Tu_n, u_n) + d(u_n, Su_n)$, letting $n \rightarrow \infty$, that gives

$$d(u_n, Su_n) \leq d(u_n, Tu_n)$$

This means,

$$\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0 \quad (11)$$

Step 3: Now, proving that

$$\mathcal{Z}_\Delta(u_n) := \bigcup_{\{z_n\} \in \{u_n\}} A([z_n]) \cap F(T, S)$$

And $\mathcal{Z}_\Delta(u_n)$ has punctually one point. Since

$$\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0 \text{ \& } \lim_{n \rightarrow \infty} d(u_n, Su_n) = 0 \text{ are proved.}$$

Let $\mathcal{Z}_\Delta(u_n) := \bigcup_{\{z_n\} \in \{u_n\}} A([z_n]) \cap F(T, S)$, where the union is taken over all subsequence $\{u_n\}$ over $\{z_n\}$. To belay that Δ -convergence of $\{z_n\}$ to a common fixed point of T and S , first, elucidating that $\mathcal{Z}_\Delta(u_n) \cap F(T, S)$ & $\mathcal{Z}_\Delta(u_n)$ is a singleton set.

To show that $\mathcal{Z}_\Delta(u_n) \cap F(T, S)$, presume that $z \in \mathcal{Z}_\Delta(u_n)$. Therefore, there is a subsequence $\{z_n\}$ of $\{u_n\}$. Such that $A([z_n]) = \{z\}$. Through Lemma (5) & (6), \forall a subsequence $\{y_n\}$ of $\{z_n\}$ such as

$$\Delta - \lim_{n \rightarrow \infty} y_n = y \text{ and } y \in E.$$

Since

$\lim_{n \rightarrow \infty} d(y_n, Ty_n) = 0$ and $\lim_{n \rightarrow \infty} d(y_n, Sy_n) = 0$. It follows up from Theorem (7) that $y \in F$. By the Opial property $\lim_{n \rightarrow \infty} \sup d(y_n, y) \leq \lim_{n \rightarrow \infty} \sup d(y_n, Ty)$ and $\lim_{n \rightarrow \infty} \sup d(y_n, y) \leq \lim_{n \rightarrow \infty} \sup d(y_n, Sy)$. Thus, $Ty = y$ and $Sy = y$ i.e. $y \in F$. Now, claiming that $z = y$. If not, by step (1), $\lim_{n \rightarrow \infty} d(u_n, y)$ exists and holding to the uniqueness of the asymptotic centers,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup d(y_n, y) &< \lim_{n \rightarrow \infty} \sup d(y_n, z) \\ &\leq \lim_{n \rightarrow \infty} \sup d(z_n, z) \\ &< \lim_{n \rightarrow \infty} \sup d(z_n, y) \\ &\leq \lim_{n \rightarrow \infty} \sup d(u_n, y) \\ &= \lim_{n \rightarrow \infty} \sup d(y_n, y) \end{aligned}$$

Which is a contradiction. Thus, $z = y$. To confirm that and $\mathcal{Z}_\Delta(u_n)$ is a singleton, let $\{z_n\}$ be a subsequence of $\{u_n\}$.

By Lemma (5) & (6), there exists a subsequence $\{y_n\}$ of $\{z_n\}$ such as $\Delta - \lim_{n \rightarrow \infty} y_n = y$ and $y \in E$. Let $A([z_n]) = \{z\}$ and $A([u_n]) = \{u\}$. Previously, showing that $z = y$. Therefore, it is sufficient to show $y = u$, thus by step (1) $\lim_{n \rightarrow \infty} d(u_n, y)$ converges. By uniqueness

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup d(y_n, y) &< \lim_{n \rightarrow \infty} \sup d(y_n, u) \\ &\leq \lim_{n \rightarrow \infty} \sup d(u_n, u) \\ &< \lim_{n \rightarrow \infty} \sup d(u_n, y) \\ &= \lim_{n \rightarrow \infty} \sup d(y_n, y) \end{aligned}$$

Whish is a contradiction. Thus, $y = u$ the conclusion is belayed.

Lastly, proving $\{u_n\}$ Δ -convergence to a common fixed point of S and T . Of step (1) $d(u_n, h^*)$, $\forall h^* \in F$, and from step (2) $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$ & $\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0$, $\mathcal{Z}_\Delta(u_n)$ has

punctually one point. Hence, $\{u_n\}$ is Δ -convergence to a common fixed point of F .

Theorem (9): Under the presumption of Theorem (8). Therefore the sequence $\{u_n\}$ is defined by :

$$\begin{aligned} u_1 &= u_0 \in E \\ u_{n+1} &= \delta_n u_n \oplus \gamma_n T^n u_n \oplus \beta_n S^n v_n \\ v_n &= (1 - \gamma_n) u_n \oplus \gamma_n T^n u_n \end{aligned} \quad (12)$$

is Δ -convergence to a common fixed point of S and T .

Proof: T and S are self-mappings from E to E , take $P = I$ (the identity mapping on E).

Therefore, $(TP)^{n-1} = T^n$. The consequence of this Theorem is got of Theorem (8).

Theorem (10): Under the presumption of Theorem (8). Presume that $\{u_n\}$ is defined by (7). If $\lim_{n \rightarrow \infty} \inf d(u_n, F) = 0$ or $\lim_{n \rightarrow \infty} \sup d(u_n, F) = 0$,

where $d(u, F) = \inf_{h^* \in F} d(u, h^*)$, therefore the sequence $\{u_n\}$ converges strongly to a point in F .

Proof: Through theorem (8),

$$d(u_{n+1}, h^*) \leq (1 + \rho_n) d(u_n, F) + \theta_n \quad (13)$$

where $h^* \in F$. Since $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, through the presumption of Theorem (10).

Lemma (4) and $\lim_{n \rightarrow \infty} \inf d(u_n, F) = 0$ or $\lim_{n \rightarrow \infty} \sup d(u_n, F) = 0$, gives that

$$\lim_{n \rightarrow \infty} d(u_n, F) = 0$$

Next, $\{u_n\}$ is a cauchy sequence in E . In verity, from (13) $\forall h^* \in F$, $d(u_{n+1}, h^*) \leq (1 + \rho_n) d(u_n, h^*) + \theta_n \quad \forall n \geq 1$

Since $\forall u \geq 0, 1 + u \leq e^u$, gives that

$$\begin{aligned} d(u_{n+r}, h^*) &\leq e^{\rho_{n+r-1}} d(u_{n+r-1}, h^*) + \theta_{n+r-1} \\ &\leq e^{\rho_{n+r-1}} e^{\rho_{n+r-2}} d(u_{n+r-2}, h^*) + \\ &\quad e^{\rho_{n+r-1}} \theta_{n+r-2} + \theta_{n+r-1} \\ &\leq \dots \\ &\leq e^{\sum_{c=n}^{n+r-1} \rho_c} d(u_n, h^*) + \\ &\quad (e^{\sum_{c=n}^{n+r-1} \rho_c}) \sum_{j=n}^{n+r-1} \theta_j \\ &\leq e^{\sum_{n=1}^{\infty} \rho_n} d(u_n, h^*) + \\ &\quad (e^{\sum_{n=1}^{\infty} \rho_n}) \sum_{j=n}^{n+r-1} \theta_j \end{aligned}$$

$$\leq H d(u_n, h^*) + H \sum_{j=n}^{n+r-1} \theta_j \quad (14)$$

where

$H = e^{\sum_{n=1}^{\infty} \rho_n} < \infty$. Since $\lim_{n \rightarrow \infty} d(u_n, F) = 0$, presume a subsequence $\{u_{nw}\}$ of $\{u_n\}$ and a sequence $\{h_{nw}^*\} \in F$, $d(u_{nw}, h_{nw}^*) \rightarrow 0$ as $w \rightarrow \infty$. Thus $\forall \varepsilon > 0$, there is $w_\varepsilon > 0$

$$d(u_{nw}, h_{nw}^*) < \frac{\varepsilon}{4H} \text{ and } \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j < \frac{\varepsilon}{4H} \quad (15)$$

for all $w > w_\varepsilon$.

For any $r > 1$ and $\forall n \geq n_{w_\varepsilon}$, by (14) and (15), getting

$$d(u_{n+r}, u_n) \leq d(u_{n+r}, h_{nw}^*) + d(u_n, h_{nw}^*)$$

$$\begin{aligned} &\leq H d(u_{nw}, h_{nw}^*) + H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j + \\ &\quad H d(u_{nw}, h_{nw}^*) + H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j \\ &\leq 2H d(u_{nw}, h_{nw}^*) + 2H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j \\ &\leq 2H \cdot \frac{\varepsilon}{4H} + 2H \cdot \frac{\varepsilon}{4H} = \varepsilon \end{aligned}$$

This displays that $\{u_n\}$ is a cauchy sequence in E . Therefore, the completeness of G means that $\{u_n\}$ have to be convergent. Presume that $\lim_{n \rightarrow \infty} u_n = k^*$. Since E is closed, therefore $k^* \in E$. Next, prove that $k^* \in F$, since $\lim_{n \rightarrow \infty} d(u_n, F) = 0$, so that $d(k^*, F) = 0$. The closedness of F gives that $k^* \in F$.

Theorem (11): Under the presumption of Theorem (8). Presume that $\{u_n\}$ is defined by (7). If T and S satisfy the following condition

- (i) $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$ & $\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0$.
(ii) If the sequence $\{z_n\}$ in E satisfies $\lim_{n \rightarrow \infty} d(z_n, Tz_n) = 0$ & $\lim_{n \rightarrow \infty} d(z_n, Sz_n) = 0$ then

$\lim_{n \rightarrow \infty} \inf d(z_n, F) = 0$ or $\lim_{n \rightarrow \infty} \sup d(z_n, F) = 0$. Therefore, the sequence $\{u_n\}$ converges strongly to a point of F .

Proof: Through Theorem (8), $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$ & $\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0$ and from the second condition (ii) $\lim_{n \rightarrow \infty} \inf d(z_n, F) = 0$ or $\lim_{n \rightarrow \infty} \sup d(z_n, F) = 0$. Therefore, $\{u_n\}$ must converge strongly to a point in F through Theorem (10).

A mapping $T: E \rightarrow E$ is called semi-compact (16) if for a sequence $\{u_n\}$ in E with $\lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$, there is a subsequence $\{u_{nw}\}$ of $\{u_n\}$ such as $u_{nw} \rightarrow h^* \in E$.

Theorem (12): Under the presumption of Theorem (8). If either S or T is semi-compact, hence the sequence $\{u_n\}$ converges strongly to a point of F .

Proof: Presume that S is semi-compact. By Theorem (8), getting $\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0$. Thus, $\exists \{u_{nw}\}$ of $\{u_n\}$ such as $u_{nw} \rightarrow h^*$.

Now, by Theorem (8) encloses that $\lim_{n \rightarrow \infty} d(u_{nw}, Su_{nw}) = 0$ and thus $d(h^*, Sh^*) = 0$. In a similar way, proving that $d(h^*, Th^*) = 0$. Hence, $h^* \in F$, by (*), gives that

$d(u_{n+1}, h^*) \leq (1 + \rho_n) d(u_n, F) + \theta_n$ where $\sum_{n=1}^{\infty} \rho_n < \infty, \sum_{n=1}^{\infty} \theta_n < \infty$, by Lemma (4) $\lim_{n \rightarrow \infty} d(u_n, h^*)$ exists and $u_{nw} \rightarrow h^* \in F$ gives that $u_n \rightarrow h^*$. This proves that $\{u_n\}$ converges strongly.

Notice: The following condition is recalled:

A mapping $T: E \rightarrow E$, where E is a subset of a normed linear space G , is named to accomplish condition (N) (5) if there is a nondecreasing

function $\xi: [0, \infty) \rightarrow [0, \infty)$ with $\xi(0) = 0$ & $\xi(p) > 0$,

$\forall p \in (0, \infty)$ such as

$\|a - Ta\| \geq \xi(d(a, F(T)))$ for all $a \in E$, where

$d(a, F(T)) = \inf\{\|a - h^*\|, h^* \in F(T) \neq \emptyset\}$.

Yet, modify that definition to two mappings :

Two mappings $T: E \rightarrow E$, where E is a subset of a normed linear space G , is named to accomplish condition (M) (5) if \forall a nondecreasing function $\xi: [0, \infty) \rightarrow [0, \infty)$ with $\xi(0) = 0$ & $\xi(p) > 0$, $\forall p \in (0, \infty)$ such as $l_1\|a - Sa\| + l_2\|a - Ta\| \geq \xi(d(a, F(T)))$ for all $a \in E$, where $d(a, F) = \inf\{\|a - h^*\|, h^* \in F \neq \emptyset\}$ and l_1, l_2 are two positive real numbers such as $l_1 + l_2 = 1$. Note, the condition (M) is weaker than the compactness of the domain E . As well Condition (M) decrease to condition (N) when $S = T$.

Theorem (13): Under the presumption of Theorem (8). If S and T satisfy condition (M), hence the sequence $\{u_n\}$ converges strongly to a point of F .

Proof: Through Theorem (8), obtaining

$\lim_{n \rightarrow \infty} d(u_n, Tu_n) =$

0 and $\lim_{n \rightarrow \infty} d(u_n, Su_n) = 0$.

From condition (M),

$$\lim_{n \rightarrow \infty} \xi(d(u_n, F)) \leq l_1 \lim_{n \rightarrow \infty} d(u_n, Su_n) + l_2 \lim_{n \rightarrow \infty} d(u_n, Tu_n) = 0$$

i.e $\lim_{n \rightarrow \infty} \xi(d(u_n, F)) = 0$. Therefore

$$\lim_{n \rightarrow \infty} d(u_n, F) = 0$$

The illation just now follows up of Theorem (10).

Numerical example

Our results through the following example is elucidated (Table 1)

Example (14): Deem $G = R$ with its usual metric, so G is as well complete CAT(0) space. Let $E = [0, 1]$, which is a closed bounded convex subset of G . Define two mappings $T, S: E \rightarrow E$ by $T(u) = ku$ and $S(u) = \frac{u}{k+1}$, $0 < k < 1$. So T is asymptotically nonexpansive mapping with $\{e_n = 2\}, \forall n \in N$ and S is a total asymptotically nonexpansive nonself mapping with $e_n = \frac{1}{n^2}$ &

$\sigma_n = \frac{1}{n^3}$, $\forall n \geq 1$. Obviously, $F(T) = \{0\} = F(S)$ of the mappings T and S . Put $\delta_n = 0.2, \gamma_n = 0.5$ & $\beta_n = 0.3$ ($\delta_n + \gamma_n + \beta_n = 1$). By using Matlab, the iteration which is defined by (1) for initial points $u_1 = 0.76$ and $k = 0.59, u_1 = 0.88$ and $k = 0.75, u_1 = 0.9$ and $k = 0.9$ is calculated. Lastly, the convergence demeanors of the iteration (7) is appeared in Fig. 1. The consequence is that u_n converges to zero.

Table 1. Numerical results for 40 steps

n	$u_1 = 0.76, k = 0.59$	$u_1 = 0.88, k = 0.75$	$u_1 = 0.9, k = 0.9$
2	0.4902	0.6380	0.7200
3	0.3162	0.4626	0.5760
4	0.2039	0.3353	0.4608
5	0.1315	0.2431	0.3686
6	0.0848	0.1763	0.2949
7	0.0547	0.1278	0.2359
8	0.0353	0.0927	0.1887
9	0.0228	0.0672	0.1510
10	0.0147	0.0487	0.1208
11	0.0095	0.0353	0.0966
12	0.0061	0.0256	0.0773
13	0.0039	0.0186	0.0618
14	0.0025	0.0135	0.0495
15	0.0016	0.0098	0.0396
16	0.0011	0.0071	0.0317
17	0.0007	0.0051	0.0253
18	0.0004	0.0037	0.0203
19	0.0003	0.0027	0.0162
20	0.0002	0.0020	0.0130
21	0.0001	0.0014	0.0104
22	0.0001	0.0010	0.0083
23	0.0000	0.0007	0.0066
24	0.0000	0.0005	0.0053
25	0.0000	0.0004	0.0043
26	0.0000	0.0003	0.0034
27	0.0000	0.0002	0.0027
28	0.0000	0.0001	0.0022
29	0.0000	0.0001	0.0017
30	0.0000	0.0001	0.0014
31	0.0000	0.0001	0.0011
32	0.0000	0.0000	0.0009
33	0.0000	0.0000	0.0007
34	0.0000	0.0000	0.0006
35	0.0000	0.0000	0.0005
36	0.0000	0.0000	0.0004
37	0.0000	0.0000	0.0003
38	0.0000	0.0000	0.0002
39	0.0000	0.0000	0.0002
40	0.0000	0.0000	0.0001

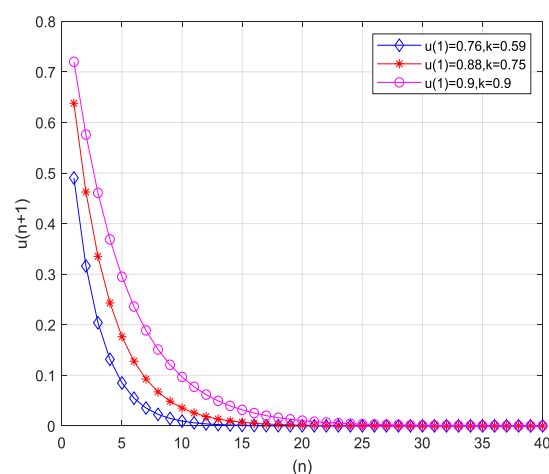


Figure 1. Convergence behaviors for different initial points for 40 steps.

Authors' declaration:

- Conflicts of Interest: None.

- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

References:

1. Abbas M, Ibrahim Y, Khan AR, Sen M. Split variational inclusion problem and fixed point problem for a class of multivalued mappings in CAT(0) spaces. *Mathematics*. 2019;7(749).
2. Dhompongsa S, Panyanak B. On Δ -convergence theorems in CAT(0) spaces, *Comput. Math. Appl.* 2008; 56: 2572-2579.
3. Khan SH, Abbas M. Strong and Δ -convergence of some iteration schemes in CAT(0) spaces. *Comput. Math. Appl.* 2011; 61: 109-116.
4. Saluja GS. On the convergence of modified S-iteration process for generalized asymptotically quasi-nonexpansive mappings CAT(0) spaces. published by faculty of Sci. Math. 2014; 6(2): 29-38.
5. Saluja GS, Postolache M. Strong and Δ -convergence theorems for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces. *Fixed Point Theory Appl.* 2015; 12.
6. Abkar A, Shekarbaigi M. A novel iteration algorithm applied tp totally asymptotically nonexpansive mappings in CAT(0) spaces. *Math.* 2017; 5(14).
7. Kimura Y. Convergence of a sequence of sets in a Hadamard space and the shrinking projection method for a real hilbert ball, *Hindawi. Abstract and Applied Analysis*, 2010.
8. Ullah K, Iqbal K, Ashed M. Some convergence results using K iteration process in CAT(0) spaces. *Fixed Point Theory Appl.* 2018; 11.
9. Yodav MR. Common fixed point for contractive operators by a faster iteration process in real Banach spaces. *J. Comp. Math. Sci.* 2014; 5(2): 251-257.
10. Abed SS, Hasan ZMM. Common fixed point of a finite-step iteration algorithm under total asymptotically quasi-nonexpansive maps. *Baghdad Sci. J.* 2019; 16(3).
11. Chang S, Wang L, Jlee HW, Chan C. Strong and Δ -convergence for mixed type total asymptotically nonexpansive mappings in CAT(0) spaces. *Fixed Point Theory Appl.* 2013; 122.
12. Abed SS, Hasan ZMM. Multi-step iteration algorithm via two finite families of total asymptotically quasi-nonexpansive maps. *ASTESJ.* 2019; 4(2).
13. Uddin I, Dala S, Imdad M. Approximating fixed points for generalized nonexpansive mapping in CAT(0) spaces. *J. Inequalities Appl.* 2014; 155.
14. Kazeem OA, Chinedu IZ, Godwin CU, Oluwatosin TM. On the proximal point algorithm and demimetric mappings in CAT(0) spaces. *Demonstr. Math.* 2018; 51: 277-294.
15. Godwin CU. Approximating a common fixed point for finite family of demimetric mappings in CAT(0) space. *Fixed point theory*, 2019; 9(1): 45-60.
16. Husain S, Singh N. Δ -convergence for proximal point algorithm and fixed point problem in CAT(0) spaces. *Fixed Point Theory Appl.* 2019; 8.

نقاط صامدة مشتركة جديدة لتطبيق اللامتددة المقاربة كلياً في فضاء CAT(0)

احمد جميل كاظم

قسم علوم الحاسوب، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة:

تم دراسة التقارب و Δ -تقارب لعملية تكرار من خطوتين باستخدام تطبيقات اللامتددة المقاربة (asymptotically nonexpansive) واللامتددة المقاربة كلياً (total asymptotically nonexpansive) في فضاء CAT(0) وكذلك تم مبرهنة بعض نظريات التقارب القوي باستخدام شبه المتراسة (semi-compact) وشرط (M). نتائجا تحسن وتطور العديد من النتائج المعروفة في الادبيات الموجودة.

الكلمات المفتاحية: تطبيق اللامتددة المقاربة، فضاء CAT(0)، نقاط صامدة تقريبية، وشرط (M)، Δ -تقارب.