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# New Common Fixed Points for Total Asymptotically Nonexpansive Mapping in CAT(0) Space

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## Abstract:

Strong and Δ-convergence for a two-step iteration process utilizing asymptotically nonexpansive and total asymptotically nonexpansive noneslf mappings in the CAT(0) spaces have been studied. As well, several strong convergence theorems under semi-compact and condition (M) have been proved. Our results improve and extend numerous familiar results from the existing literature.

**Keywords:** Asymptotically nonexpansive mapping, CAT(0) space, Common fixed point, Condition (M), Δ-convergence.

## Introduction:

A metric space G is a CAT(0) space, if it is geodesically connected and if each geodesic triangle in G is at least as thin as its comparison triangle in the Euclidean plane. Some typical examples of CAT(0) spaces are R-trees, Pre-Hilbert space and Euchlidean buldings (1).

Fixed point theory in CAT(0) spaces was foremost initialized through Kirk (1). He proved that each nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) spaces permanently has fixed point. Therefore, the fixed point theory for single valued as well multivalued mappings in CAT(0) spaces has intensively been evolved by numerous authors. The convergence for nonexpansive mappings in CAT(0) spaces was studied by Dhompongsa-Panyanak (2). Thereafter, Khan and Abbas (3) studied the strong and Δ-convergance in CAT(0) space for an iteration process that is indepent of the Ishikawa iteration process. As well, several of these results obtained for two nonexpansive mappins. It is important to remember that fixed point theorems in CAT(0) space can be stratified to graph theory, computer science and biology (1).

Let $(G,d)$ be a metric space and $u,v \in G$ with $d(u,v)=x$. A geodesic path from $u$ to $v$, this means an isometry $c:[0,x] \rightarrow c([0,1]) \subset G$ such as $c(0)=u$ and $c(x)=v$. The image of every geodesic path between $u$ and $v$ is called geodesic segment. Each point y in the segment is appeared by $\omega u \oplus (1-\omega)v$, where $\omega \in [0,1]$ that is $[u,v]=\{\omega u \oplus (1-\omega)v \colon \omega \in [0,1]\}$. The space $(G,d)$ is called a geodesic if each two points of $G$ are jouned through a geodesic segment, and $G$ is uniquely geodesic if there exists properly one geodesic jouning $u$ and $v$ for every $u,v \in G$. A subset $H$ of $G$ is called convex if $H$ has each geodesic segment joining any two points in $H$(4-6).

A geodesic triangle $\Delta(u_1,u_2,u_3)$ is a geodesic metric space $(G,d)$ that consists of three points $u_1,u_2,u_3$ in $G$ (the vertices Δ) and a geodesic segment between every pair of vertices (the edges of Δ). A comparison triangle $\bar{\Delta}(\overline{u_1},\overline{u_2},\overline{u_3})$ in $W^2$ for $\Delta(u_1,u_2,u_3)$ *is* a triangle in 2-dimensional Euclidean plane $W^2$ with $\overline{u_1},\overline{u_2},\overline{u_3} \in W^2$ such as $d(u_1,u_2)=|\overline{u_1}-\overline{u_2}|_{W^2}$, $d(u_1,u_3)=|\overline{u_1}-\overline{u_3}|_{W^2}$, $d(u_2,u_3)=|\overline{u_2}-\overline{u_3}|_{W^2}$, where $|.|_{W^2}$ is the Eulidean norm on $W^2$ (7).

**CAT(0):** A geodesic space is called CAT(0) space if whole geodesic triangles achieve the following comparison axiom.

Let Δ be a geodesic triangle in G and $\bar{\Delta} \subset W^2$ be a comparison triangle for Δ. Therefore, Δ is called to achieve the CAT(0) inequality if $\forall\, u,v \in \Delta$ & $\forall\, \bar{u},\bar{v} \in \bar{\Delta}$, $d(u,v) \le d_{W^2}(\bar{u},\bar{v})$.

If $u, v_1, v_2$ are the points in CAT(0) and if $v_0 = \frac{1}{2}(v_1 \oplus v_2)$, therefore the CAT(0) inequality leads to

$$d(u, v_0)^2 \leq \frac{1}{2} d(u, v_1)^2 + \frac{1}{2} d(u, v_2)^2 - \frac{1}{4} d(v_1, v_2)^2$$

Which is the (CN) inequality of Bruhat and Tits. In verity, a geodesic space is a CAT(0) space if it accomplishes (CN)(3).

**Lemma (1)(8):** Let $(G, d)$ be a CAT(0) space. Therefore,

$$d\big((1-k)a \oplus k\mathrm{b}, \mathrm{c}\big)^2 \leq (1-k)d(a,c)^2 + kd(b,c)^2 - k(1-k)d(a,b)^2$$

for all $k \in [0,1]$ and $a, b, c \in G$.

Let $\{u_n\}$ be a bounded sequence in a CAT(0) space $G$. For $u \in G$, setting

$$r(u, \{u_n\}) = \lim_{n\to\infty} \sup d(u, u_n).$$

The asymptotic radius $r(\{u_n\})$ of $\{u_n\}$ is given through

$$r(\{u_n\}) = \inf\{r(u, \{u_n\}) : u \in G\},$$

and the asymptotic center $A(\{u_n\})$ of $\{u_n\}$ is defined as

$$A(\{u_n\}) = \{u \in G : r(u, \{u_n\}) = r(\{u_n\})\}$$

It is familiar that in CAT(0) space, $A(\{u_n\})$ has punctually one point.

Numerous iteration processes have been structured and suggested in order to approximate fixed points. The Picard iteration for a mapping $T: E \to E$ is defined by

$$u_1 = u \in E$$
$$u_{n+1} = T^n u_n \qquad (1)$$

The modified Mann iteration is considered by Schu (5), as below

$$u_1 = u \in E$$
$$u_{n+1} = (1-\delta_n)u_n + \delta_n T^n u_n \qquad (2)$$

Where $\{\delta_n\} \in (0, 1)$.

The modified Ishikawa iteration is studied by Tan and Xu (5), as below

$$u_1 = u \in E$$
$$u_{n+1} = (1-\delta_n)u_n + \delta_n T^n v_n$$
$$v_n = (1-\beta_n)u_n + \beta_n T^n u_n \qquad (3)$$

Where $\{\delta_n\}$ and $\{\beta_n\} \in (0, 1)$. The iteration decreases to the modified Mann iteration when $\beta_n = 0, \forall n \geq 1$.

Lately, the modified S-iteration in a Banach space is introduced by Agarwal et al. (5), as below

$$u_1 = u \in E$$
$$u_{n+1} = (1-\delta_n)T^n u_n + \delta_n T^n v_n$$
$$v_n = (1-\beta_n)u_n + \beta_n T^n u_n, \ \forall n \geq 1 \qquad (4)$$

where $\{\delta_n\}$ and $\{\beta_n\} \in (0, 1)$. Notice that this iteration is independent of Ishikawa and Mann iterations.

Recently, Sahin and Basarir (5) modified the above iteration in a CAT(0) space, as follows

$$u_1 = u \in E$$
$$u_{n+1} = (1-\delta_n)T^n u_n \oplus \delta_n T^n v_n$$
$$v_n = (1-\beta_n)u_n \oplus \beta_n T^n u_n, \ \forall n \geq 1 \qquad (5)$$

The following iteration has been studied by M. R. Yadava (9) for common fixed points of two self mappins $S$ and $T$,

$$u_1 = u \in E$$
$$u_{n+1} = \delta_n u_n + \gamma_n T u_n + \beta_n S v_n$$
$$v_n = (1-\gamma_n)u_n + \gamma_n T u_n, \ n \in N \qquad (6)$$

Where $\{\delta_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $\delta_n + \gamma_n + \beta_n = 1$. This iteration as well decreases to Mann iteration when $T = I$ or $\gamma_n = 0$.

Inspired and motivated by the work of M. R. Yadava (6), the iteration (6) for common fixed points of two mapping asymptotically naonexpansive and total asymptotically nonexpansive nonself mappings in a CAT(0) space is modified, as follows.

Deem E to be a nonempty closed convex subset of a complete CAT(0) space G, $T: E \to E$ to be an asymptotically nonexpansive and $S: E \to E$ to be a total asymptotically nonexpansive mappings. Presume that $\{u_n\}$ is a sequence produced by

$$u_1 = u \in E$$
$$u_{n+1} = P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n)$$
$$v_n = P((1-\gamma_n)u_n \oplus \gamma_n T(PT)^{n-1} u_n), \ n \in N \ (7)$$

where $\{\delta_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ with $\delta_n + \gamma_n + \beta_n = 1$ and $P$ is a nonexpansive retraction of $G$ onto $E$.

In this paper, a new iteration for approximating a common fixed point of asymptotically nonexpansive and total asymptotically nonexpensive nonself mappings is constructed. Some strong convergence theorems and Δ-convergence theorem under appropriate conditions like semi-compact and condition (M) in CAT(0) spaces are proved. As well, numerical example to elucidate our work is provided.

## Preliminaries

Let $(G, d)$ be a metric space & $E$ be a nonempty subset of $G$. Deem $T: E \to E$ to be a mapping. A point $a \in E$ is called a fixed point of $T$ if $Ta = a$. As well the set of common fixed points of $T$ and $S$ denote by $F$ which is

$F = \{a \in E : Ta = Sa = a\}$. Call that $E$ is called retract of $G$ if there is a continuous mapping $P: G \to E$ such as $Pa = a, \forall a \in E$. A mapping $P: G \to E$ is called a retraction if $P^2 = P$. If $P$ is a retraction, then $Pb = b$, $\forall\, b$ in the range of $P$.

- A mapping $T: E \to E$ is called nonexpansive (10) if $d(Ta, Tb) \leq d(a, b), \forall a, b \in E$.
- A mapping $T: E \to E$ is called asymptotically nonexpansive (11) if ∃ a sequence $\{\epsilon_n\} \subset [1, \infty)$ with $\epsilon_n \to 1$ such as $d(T^n a, T^n b) \leq \epsilon_n d(a, b)$, $\forall n \geq 1, \forall a, b \in E$.
- A mapping $T: E \to G$ is called asymptotically nonexpansive nonself (11) if ∃ a sequence $\{\epsilon_n\} \subset [1, \infty)$ with $\epsilon_n \to 1$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq \epsilon_n d(a, b), \forall n \geq 1, \forall a, b \in E$, where $P$ is a nonexpansive retractiopn of $G$ onto $E$.
- A mapping $T: E \to E$ is called uniformly L-lipschitzain (12) if ∃ a constant $L > 0$ such as $d(T^n a, T^n b) \leq Ld(a, b), \forall n \geq 1$, $\forall a, b \in E$.
- A mapping $T: E \to E$ is called total asymptotically nonexpansive (11) if ∃ positive sequences $\{e_n\}, \{\sigma_n\}$ with $e_n \to 0, \sigma_n \to 0$ and a strictly nondecreasing continuous function $\vartheta: [0, \infty) \to [0, \infty)$ with $\vartheta(0) = 0$ such as $d(T^n a, T^n b) \leq d(a, b) + e_n \vartheta d(a, b)$ $+\sigma_n, \forall n \geq 1, \forall a, b \in E$.
- A mapping $T: E \to G$ is called total asymptotically nonexpansive nonself (11) if ∃ positive sequences $\{e_n\}, \{\sigma_n\}$ with $e_n \to 0, \sigma_n \to 0$ and a strictly nondecreasing continuous function $\vartheta: [0, \infty) \to [0, \infty)$ with $\vartheta(0) = 0$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq d(a, b) + e_n \vartheta d(a, b) + \sigma_n, \forall n \geq 1, \forall a, b \in E$, where $P$ is a nonexpansive retraction of $G$ onto $E$.
- A mapping $T: E \to G$ is called uniformly L-lipschitzain (11) if ∃ a constant $L > 0$ such as $d(T(PT)^{n-1}a, T(PT)^{n-1}b) \leq Ld(a, b), \forall n \geq 1, \forall a, b \in E$.

The notion of asymptotcally nonexpansive mapping was foremost introduced by Gloebel and Kirk. Therefore Alber et al. introduced the class of total asymptotically nonexpensive, which generalizes some classes of mappings that are spans of asymptotically nonexpensive. Several authours have been extensively studied these classes of mappings (6).

**Definition (2)(13):** A sequence $\{u_n\}$ in a CAT(0) space $G$ is called Δ-convergence to $u \in G$ if $u$ is the unique asymptotic center of $\{v_n\}$ ∀ subsequence $\{v_n\}$ of $\{u_n\}$. Here, note down $\Delta - \lim_{n\to\infty} u_n = u$ and $u$ is the Δ-limit of $\{u_n\}$.

Note that given $\{u_n\} \subset G, \{u_n\}\ \Delta -$ convergence to $u$ and $v \in G$ with $v \neq u$ through the uniqueness of the asymptotic center that gives

$$\lim_{n\to\infty} \sup d(u_n, u) \leq \lim_{n\to\infty} \sup d(u_n, v)$$

Therefore, each CAT(0) space achieves the Opial property.

**Lemma (3)(8):** Let $G$ be a CAT(0) space and $a \in G$. Presume $\{s_n\}$ is a sequence in $[z, c]$ for several $z, c \in (0,1)$ and $\{a_n\}, \{b_n\}$ are sequences in $G$ such as $\lim_{n\to\infty} sup\ d(a_n, h^*) \leq t$, $\lim_{n\to\infty} sup\ d(b_n, h^*) \leq t$ and $\lim_{n\to\infty} d((1 - s_n)\, a_n \oplus s_n b_n) = t$ for several $t \geq 0$. Thus $\lim_{n\to\infty} d(a_n, b_n) = 0$.

**Lemma (4)(6):** Let $\{\varsigma_n\}, \{\alpha_n\}$ and $\{\lambda_n\}$ be the sequences of positive numbers such as $\varsigma_{n+1} \leq (1 + \alpha_n)\varsigma_n + \lambda_n, \forall n \geq 1$.

- If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, therefore $\lim_{n\to\infty} \varsigma_n$ exists.
- If there is a subsequence $\{\varsigma_{ni}\} \subset \{\varsigma_n\}$ such as $a_{ni} \to 0$ thus $\lim_{n\to\infty} \varsigma_n = 0$.

**Lemma (5)(14):** Each bounded sequence in a complete CAT(0) space $G$ holds a Δ-convergence subsequence.

**Lemma (6)(15):** If $E$ is closed convex subset of a complete CAT(0) space $G$ and if $\{u_n\}$ is bounded sequence in $E$, thus the asympitotic center of $\{u_n\}$ is in $E$.

**Theorem (7)(11):** Let $E$ be a closed convex subset of a complete CAT(0) space $G$. Let T be a mapping accomplishing one of the following conditions:

- $T: E \to E$ is an asymptotically nonexpansive mapping with a sequence $\{\epsilon_n\} \subset [1, \infty)$ & $\epsilon_n \to 1$.
- $T: E \to G$ is an asymptotically nonexpansive nonself mapping.
- $T: E \to E$ is a total asymptotically nonexpansive mapping.

Let $\{u_n\}$ be a bounded sequence in $E$ such as $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\Delta - \lim_{n\to\infty} u_n = h^*$. Thus, $Th^* = h^*, h^* \in F$.

## The convergence results

In this part, Δ-convergance and some strong convergnce theorems by using iteration (7) for asymptotically nonexpansive and total asymptotically nonexpansive nonself mappings in CAT(0) spaces are proved.

**Theorem (8):** Let $E$ be a nonempty closed convex subset of a complete CAT(0) space $G$. Let $T: E \to E$ be a uniformly L-lipschitzain and asymptotically nonexpansive and $S: E \to G$ be a uniformly L-lipschitzain total asymptotically nonexpansive nonself mappings with $F(T, S) \neq \emptyset$. Presume that $\{u_n\}$ is defined by (7). If $F := F(T) \cap F(S)$ and the following conditions are accomplished:

i) $\sum_{n=1}^{\infty} e_n < \infty$ & $\sum_{n=1}^{\infty} \sigma_n < \infty$.
ii) There is a constant $B^* > 0$ such as $\vartheta(ꞵ) < B^* ꞵ$, $ꞵ \geq 0$.
Thus, the sequence $\{u_n\}$ is Δ-convergence to a several points $h^* \in F$ $\big(F \coloneqq F(T) \cap F(S)\big)$.

**Proof: Step 1:** Firstly, proving that $\lim_{n\to\infty} d(u_n, h^*)$ for each $h^* F$ and $\lim_{n\to\infty} d(u_n, F)$ exist.
Since $h^* \in F,\ Ph^* = h^*$. Now,

$$\begin{aligned} d(v_n, h^*) &= d\big(P\big((1-\gamma_n)u_n \oplus \gamma_n \mathrm{T}(PT)^{n-1} u_n\big), Ph^*\big) \\ &\leq d\big((1-\gamma_n)u_n \oplus \gamma_n \mathrm{T}(PT)^{n-1} u_n, Ph^*\big) \\ &\leq (1-\gamma_n) d(u_n, h^*) + \gamma_n d(\mathrm{T}(PT)^{n-1} u_n, h^*) \\ &\leq (1-\gamma_n) d(u_n, h^*) + \gamma_n б_n d(u_n, h^*) \\ &= (1-\gamma_n + \gamma_n б_n) d(u_n, h^*) \end{aligned}$$

Thus,

$$\begin{aligned} d(u_{n+1}, h^*) &= d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq d(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n, h^*) \\ &\leq \delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) + \beta_n d(S(PS)^{n-1} v_n, h^*) \\ &\leq \delta_n d(u_n, h^*) + \gamma_n б_n d(u_n, h^*) + \beta_n [d(v_n, h^*) + e_n \vartheta d(v_n, h^*) + \sigma_n)] \\ &\leq \delta_n d(u_n, h^*) + \gamma_n б_n d(u_n, h^*) + \beta_n (1 + e_n B^*) d(v_n, h^*) + \beta_n \sigma_n \\ &\leq \delta_n d(u_n, h^*) + \gamma_n б_n d(u_n, h^*) + \beta_n (1 + e_n B^*)(1-\gamma_n + \gamma_n б_n) d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n б_n + \beta_n (1 + e_n B^*)(1-\gamma_n + \gamma_n б_n)] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n б_n + \beta_n + \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n б_n + \beta_n e_n B^* \gamma_n б_n] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [\delta_n + \gamma_n + \beta_n + \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n б_n + \beta_n e_n B^* \gamma_n б_n] d(u_n, h^*) + \beta_n \sigma_n \\ &\leq [1 + (\beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n б_n + \beta_n e_n B^* \gamma_n б_n)] d(u_n, h^*) + \beta_n \sigma_n \\ &= (1 + \rho_n) d(u_n, h^*) + \theta_n \end{aligned}$$

where $\rho_n \coloneqq \beta_n e_n B^* - \beta_n \gamma_n - \beta_n e_n B^* \gamma_n + \beta_n \gamma_n б_n + \beta_n e_n B^* \gamma_n б_n$ and $\theta_n \coloneqq \beta_n \sigma_n$.
Whereas $\sum_{n=1}^{\infty} e_n < \infty$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, it follows up that $\sum_{n=1}^{\infty} \rho_n < \infty$ & $\sum_{n=1}^{\infty} \theta_n < \infty$. Therefore through Lemma (4), $\lim_{n\to\infty} d(u_n, h^*), \forall\, h^* \in F$ and $\lim_{n\to\infty} d(u_n, F)$ exist.

**Step 2:** Next, proving that $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ and $\lim_{n\to\infty} d(u_n, Su_n) = 0$.
In verity, it follows up from step (1) that for each given $h^* \in F$, $\lim_{n\to\infty} d(u_n, h^*)$ exists. Presume that

$$\lim_{n\to\infty} d(u_n, h^*) = r, \quad r \geq 0$$

$$\begin{aligned} d(v_n, h^*) &= d\big(P\big((1-\gamma_n)u_n \oplus \gamma_n \mathrm{T}(PT)^{n-1} u_n\big), Ph^*\big) \\ &\leq (1-\gamma_n) d(u_n, h^*) + \gamma_n d(\mathrm{T}(PT)^{n-1} u_n, h^*) \\ &\leq (1-\gamma_n) d(u_n, h^*) + \gamma_n б_n d(u_n, h^*) \end{aligned}$$

Hence,

$$\lim_{n\to\infty} sup\ d(v_n, h^*) \leq r \qquad (8)$$

As well from $d(\mathrm{T}(PT)^{n-1} u_n, h^*) \leq б_n d(u_n, h^*)$, $\forall\, n = 1, 2, \ldots$, gives

$$\lim_{n\to\infty} sup\, d(\mathrm{T}(PT)^{n-1} u_n, h^*) \leq r$$

Now,

$$d(S(PS)^{n-1} v_n, h^*) \leq ((1 + e_n B^*) d(v_n, h^*) + \sigma_n)$$

Therefore,

$$\lim_{n\to\infty} sup\, d(S(PS)^{n-1} v_n, h^*) \leq r$$

Moreover,

$$\begin{aligned} r &= \lim_{n\to\infty} d(u_{n+1}, h^*) \\ &= \lim_{n\to\infty} d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq \lim_{n\to\infty} [\delta_n d(u_n, h^*) + \gamma_n d(T(PT)^{n-1} u_n, h^*) + \beta_n d(S(PS)^{n-1} v_n, h^*) \\ &= (\delta_n + \gamma_n) d(T(PT)^{n-1} u_n, h^*) + \beta_n d(S(PS)^{n-1} v_n, h^*) \\ &= (1-\beta_n) d(T(PT)^{n-1} u_n, h^*) + \beta_n d(S(PS)^{n-1} v_n, h^*) \end{aligned}$$

By Lemma (3), getting

$$\lim_{n\to\infty} d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) = 0$$

Now,

$$\begin{aligned} d(u_{n+1}, h^*) &= d(P(\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &= d(P(\delta_n T(PT)^{n-1} u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n), Ph^*) \\ &\leq d\big((1-\beta_n) T(PT)^{n-1} u_n + \beta_n S(PS)^{n-1} v_n, h^*\big) \\ &\leq d(\mathrm{T}(PT)^{n-1} u_n, h^*) + \beta_n d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) \end{aligned}$$

Which yields that

$$r \leq \lim_{n\to\infty} inf\, d(\mathrm{T}(PT)^{n-1} u_n, h^*)$$

That gives

$$\lim_{n\to\infty} d(\mathrm{T}(PT)^{n-1} u_n, h^*) = r$$

In turn,

$$\begin{aligned} d(\mathrm{T}(PT)^{n-1} u_n, h^*) &\leq d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) + d(S(PS)^{n-1} v_n, h^*) \\ &\leq d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) + (1 + e_n B^*) d(v_n, h^*) + \sigma_n \end{aligned}$$

That implies

$$r \leq \lim_{n\to\infty} \inf d(v_n, h^*) \qquad (9)$$

From (8) and (9), that deduces

$$\lim_{n\to\infty} d(v_n, h^*) = r$$

Again,

$$r = \lim_{n\to\infty} d(v_n, h^*)$$
$$= \lim_{n\to\infty} d\left(P(1-\gamma_n)u_n \oplus \gamma_n \mathrm{T}(PT)^{n-1} u_n), h^*\right)$$

Through Lemma (3), getting

$$\lim_{n\to\infty} d(\mathrm{T}(PT)^{n-1} u_n, u_n) = 0$$

Notice that

$\lim_{n\to\infty} d(v_n, h^*) = \gamma_n \mathrm{d}(\mathrm{T}(PT)^{n-1} u_n, h^*)$

Therefore,

$$\lim_{n\to\infty} d(v_n, h^*) = 0$$

Now,

$$d(u_{n+1}, u_n) = d(P((\delta_n u_n \oplus \gamma_n T(PT)^{n-1} u_n \oplus \beta_n S(PS)^{n-1} v_n), u_n)$$
$$\leq d((\delta_n + \gamma_n) T(PT)^{n-1} u_n + \beta_n S(PS)^{n-1} v_n, u_n)$$
$$\leq (1-\beta_n) d(T(PT)^{n-1} u_n, u_n) + \beta_n d(S(PS)^{n-1} v_n, u_n)$$

This gives

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = 0$$

Thus,

$$d(u_{n+1}, v_n) \leq d(u_{n+1}, u_n) + d(v_n, u_n) \to 0 \ as \ n \to \infty$$

Which gives

$$\lim_{n\to\infty} d(u_{n+1}, v_n) = 0$$

Moreover, from

$$d(u_{n+1}, S(PS)^{n-1} v_n)$$
$$\leq d(u_{n+1}, u_n) + d(u_n, T(PT)^{n-1} u_n) + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n)$$

That gives

$$\lim_{n\to\infty} d(u_{n+1}, S(PS)^{n-1} v_n) = 0.$$

$$d(u_n, T(PT)^{n-1} u_n)$$
$$\leq d(u_n, T(PT)^{n-1} u_n) + d(T(PT)^{n-1} u_n, S(PS)^{n-1} v_n) + d(S(PS)^{n-1} v_n), T(PT)^{n-1} u_n)$$

Gives that

$$\lim_{n\to\infty} d(u_n, T(PT)^{n-1} u_n) = 0$$

And

$$d(u_n, Tu_n) \leq d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) + d(T(PT)^n u_{n+1}, T(PT)^n u_n) + d(T(PT)^n u_n, Tu_n)$$

By uniformly L-lipschitzain, getting

$$\leq (1+L) d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) + L\, d((PT)^n u_n, u_n)$$
$$= (1+L) d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) + L\, d(PT(PT)^{n-1} u_n, u_n)$$
$$\leq (1+L) d(u_n, u_{n+1}) + d(u_{n+1}, T(PT)^n u_{n+1}) + L\, d(T(PT)^{n-1} u_n, u_n) \to 0 \ as \ n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0 \qquad (10)$$

and $d(u_n, Su_n) \leq d(u_n, Tu_n) + d(Tu_n, u_n) + d(u_n, Su_n)$, letting $n \to \infty$, that gives

$$d(u_n, Su_n) \leq d(u_n, Su_n)$$

This means,

$$\lim_{n\to\infty} d(u_n, Su_n) = 0 \qquad (11)$$

**Step 3:** Now, proving that

$$\text{Ƨ}_\Delta(u_n) \coloneqq \bigcup_{\{z_n\} \square \{u_n\}} A([z_n]) \square F(T,S)$$

And $\text{Ƨ}_\Delta(u_n)$ has punctually one point. Since $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ & $\lim_{n\to\infty} d(u_n, Su_n) = 0$ are proved.

Let $\text{Ƨ}_\Delta(u_n) \coloneqq \cup_{\{z_n\} \square \{u_n\}} A([z_n]) \square F(T,S)$, where the union is taked over all subsequence $\{u_n\}$ over $\{z_n\}$. To belay that $\Delta$-convergance of $\{z_n\}$ to a common fixed point of $T$ and $S$, first, elucidating that $\text{Ƨ}_\Delta(u_n) \square F(T,S)$ & $\text{Ƨ}_\Delta(u_n)$ is a singleton set.

To show that $\text{Ƨ}_\Delta(u_n) \square F(T,S)$, presume that $z \in \text{Ƨ}_\Delta(u_n)$. Therefore, there is a subsequence $\{z_n\}$ of $\{u_n\}$. Such that $A([z_n]) = \{z\}$. Through Lemma (5) & (6), $\forall$ a subsequence $\{y_n\}$ of $\{z_n\}$ such as $\Delta - \lim_{n\to\infty} y_n = y$ and $y \in E$.

Since $\lim_{n\to\infty} d(y_n, Ty_n) = 0$ and $\lim_{n\to\infty} d(y_n, Sy_n) = 0$. It follows up from Theorem (7) that $y \in F$. By the Opial property $\lim_{n\to\infty} \sup d(y_n, y) \leq \lim_{n\to\infty} \sup d(y_n, Ty)$ and $\lim_{n\to\infty} \sup d(y_n, y) \leq \lim_{n\to\infty} \sup d(y_n, Sy)$. Thus, $Ty = y$ and $Sy = y$ $i.e\ y \in F$. Now, claiming that $z = y$. If not, by step (1), $\lim_{n\to\infty} d(u_n, y)$ exists and holding to the uniqness of the asymptotic centers,

$$\lim_{n\to\infty} \sup d(y_n, y) < \lim_{n\to\infty} \sup d(y_n, z)$$
$$\leq \lim_{n\to\infty} \sup d(z_n, z)$$
$$< \lim_{n\to\infty} \sup d(z_n, y)$$
$$\leq \lim_{n\to\infty} \sup d(u_n, y)$$
$$= \lim_{n\to\infty} \sup d(y_n, y)$$

Which is a contradication. Thus, $z = y$. To confirm that and $\text{Ƨ}_\Delta(u_n)$ is a singleton, let $\{z_n\}$ be a subsequence of $\{u_n\}$.

By Lemma (5) & (6), there exists a subsequence $\{y_n\}$ of $\{z_n\}$ such as $\Delta - \lim_{n\to\infty} y_n = y$ and $y \in E$. Let $A([z_n]) = \{z\}$ and $A([u_n]) = \{u\}$. Previoulsy, showing that $z = y$. Threrefore, it is sufficient to show $y = u$, thus by step (1) $\lim_{n\to\infty} d(u_n, y)$ converges. By uniqueness

$$\lim_{n\to\infty} \sup d(y_n, y) < \lim_{n\to\infty} \sup d(y_n, u)$$
$$\leq \lim_{n\to\infty} \sup d(u_n, u)$$
$$< \lim_{n\to\infty} \sup d(u_n, y)$$
$$= \lim_{n\to\infty} \sup d(y_n, y)$$

Whish is a contradication. Thus, $y = u$ the conclusion is belayed.

Lastly, proving $\{u_n\}$ $\Delta$-convergence to a common fixed point of $S$ and $T$. Of step (1) $d(u_n, h^*), \forall\, h^* \in F$, and from step (2) $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ & $\lim_{n\to\infty} d(u_n, Su_n) = 0, \text{Ƨ}_\Delta(u_n)$ has punctually one point. Hence, $\{u_n\}$ is Δ-convergence to a common fixed point of $F$.

**Theorem (9):** Under the presumption of Theorem (8). Therefore the sequence $\{u_n\}$ is defined by :

$$u_1 = u_0 \in E$$
$$u_{n+1} = \delta_n u_n \oplus \gamma_n T^n u_n \oplus \beta_n S^n v_n$$
$$v_n = (1-\gamma_n)u_n \oplus \gamma_n T^n u_n \qquad (12)$$

is Δ-convergence to a common fixed point of $S$ and $T$.

**Proof:** $T$ and $S$ are self-mappings from $E$ to $E$, take $P = I$ ( the identity mapping on $E$).

Therefore, $(TP)^{n-1} = T^n$. The consequence of this Theorem is got of Theorem (8).

**Theorem (10):** Under the presumption of Theorem (8). Presume that $\{u_n\}$ is defined by (7). If $\lim_{n\to\infty} \inf d(u_n, F) = 0$ or $\lim_{n\to\infty} \sup d(u_n, F) = 0$, where $d(u,F) = inf_{h^*\in F} d(u,h^*)$, therefore the sequence $\{u_n\}$ converges strongly to a point in $F$.

**Proof:** Through theorem (8),

$$d(u_{n+1}, h^*) \le (1+\rho_n) d(u_n, F) + \theta_n \qquad (13)$$

where $h^* \in F$. Since $\sum_{n=1}^{\infty} \rho_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$, through the presumption of Theorem (10). Lemma (4) and $\lim_{n\to\infty} \inf d(u_n, F) = 0$ or $\lim_{n\to\infty} \sup d(u_n, F) = 0$, gives that

$$\lim_{n\to\infty} d(u_n, F) = 0$$

Next, $\{u_n\}$ is a cauchy sequence in $E$. In verity, from (13) $\forall\, h^* \in F$, $d(u_{n+1}, h^*) \le (1+\rho_n) d(u_n, h^*) + \theta_n \quad \forall n \ge 1$

Since $\forall\, u \ge 0, 1+u \le e^u$, gives that

$$d(u_{n+r}, h^*) \le e^{\rho_{n+r-1}} d(u_{n+r-1}, h^*) + \theta_{n+r-1}$$
$$\le e^{\rho_{n+r-1}} e^{\rho_{n+r-2}} d(u_{n+r-2}, h^*) + e^{\rho_{n+r-1}} \theta_{n+r-2} + \theta_{n+r-1}$$
$$\le \cdots$$
$$\le e^{\sum_{c=n}^{n+r-1} \rho_c} d(u_n, h^*) + (e^{\sum_{c=n}^{n+r-1} \rho_c}) \sum_{j=n}^{n+r-1} \theta_j$$
$$\le e^{\sum_{n=1}^{\infty} \rho_n} d(u_n, h^*) + (e^{\sum_{n=1}^{\infty} \rho_n}) \sum_{j=n}^{n+r-1} \theta_j$$

$$\le H d(u_n, h^*) + H \sum_{j=n}^{n+r-1} \theta_j \qquad (14)$$

where

$H = e^{\sum_{n=1}^{\infty} \rho_n} < \infty$. Since $\lim_{n\to\infty} d(u_n, F) = 0$, presume a subsequence $\{u_{nw}\}$ of $\{u_n\}$ and a sequence $\{h^*_{nw}\} \in F$, $d(u_{nw}, h^*_{nw}) \to 0$ as $w \to \infty$. Thus $\forall\, \varepsilon > 0$, there is $w_\varepsilon > 0$

$$d(u_{nw}, h^*_{nw}) < \frac{\varepsilon}{4H} \text{ and } \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j < \frac{\varepsilon}{4H} \qquad (15)$$

for all $w > w_\varepsilon$.

For any $r > 1$ and $\forall\, n \ge n_{w_\varepsilon}$, by (14) and (15), getting

$$d(u_{n+r}, u_n) \le d(u_{n+r}, h^*_{nw}) + d(u_n, h^*_{nw})$$
$$\le H d(u_{nw}, h^*_{nw}) + H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j + H d(u_{nw}, h^*_{nw}) + H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j$$
$$\le 2H\, d(u_{nw}, h^*_{nw}) + 2H \sum_{j=n_{w_\varepsilon}}^{\infty} \theta_j$$
$$\le 2H.\frac{\varepsilon}{4H} + 2H.\frac{\varepsilon}{4H} = \varepsilon$$

This displays that $\{u_n\}$ is a cauchy sequence in $E$. Therefore, the completeness of $G$ means that $\{u_n\}$ have to be convergent. Presume that $\lim_{n\to\infty} u_n = k^*$. Since E is closed, therefore $k^* \in E$. Next, prove that $k^* \in F$, since $\lim_{n\to\infty} d(u_n, F) = 0$, so that $d(k^*, F) = 0$. The closedness of F gives that $k^* \in F$.

**Theorem (11):** Under the presumption of Theorem (8). Presume that $\{u_n\}$ is defined by (7). If $T$ and $S$ satisfy the following condition

(i) $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ & $\lim_{n\to\infty} d(u_n, Su_n) = 0$.

(ii) If the sequence $\{z_n\}$ in $E$ satisfies $\lim_{n\to\infty} d(z_n, Tz_n) = 0$ & $\lim_{n\to\infty} d(z_n, Sz_n) = 0$, then $\lim_{n\to\infty} \inf d(z_n, F) = 0$ or $\lim_{n\to\infty} \sup d(z_n, F) = 0$. Therefore, the sequence $\{u_n\}$ converges strongly to a point of $F$.

**Proof:** Through Theorem (8), $\lim_{n\to\infty} d(u_n, Tu_n) = 0$ & $\lim_{n\to\infty} d(u_n, Su_n) = 0$ and from the second condition (ii) $\lim_{n\to\infty} \inf d(z_n, F) = 0$ or $\lim_{n\to\infty} \sup d(z_n, F) = 0$.Therefore, $\{u_n\}$ must converge strongly to a point in F through Theorem (10).

A mapping $T: E \to E$ is called semi-compact (16) if for a sequence $\{u_n\}$ in $E$ with $\lim_{n\to\infty} d(u_n, Tu_n) = 0$, there is a subsequence $\{u_{nw}\}$ of $\{u_n\}$ such as $u_{nw} \to h^* \in E$.

**Theorem (12):** Under the presumption of Theorem (8). If either $S$ or $T$ is semi-compact, hence the sequence $\{u_n\}$ converges strongly to a point of $F$.

**Proof:** Presume that S is semi-compact. By Theorem (8), getting $\lim_{n\to\infty} d(u_n, Su_n) = 0$. Thus, $\exists\, \{u_{nw}\}$ of $\{u_n\}$ such as $u_{nw} \to h^*$.

Now, by Theorem (8) encloses that $\lim_{n\to\infty} d(u_{nw}, Su_{nw}) = 0$ and thus $d(h^*, Sh^*) = 0$. In a similar way, proving that $d(h^*, Th^*) = 0$. Hence, $h^* \in F$, by (*), gives that

$$d(u_{n+1}, h^*) \le (1+\rho_n) d(u_n, F) + \theta_n$$

where $\sum_{n=1}^{\infty} \rho_n < \infty, \sum_{n=1}^{\infty} \theta_n < \infty$, by Lemma (4) $\lim_{n\to\infty} d(u_n, h^*)$ exists and $u_{nw} \to h^* \in F$ gives that

$u_n \to h^*$. This proves that $\{u_n\}$ converges strongly.

**Notice:** The following condition is recalled:

A mapping $T: E \to E$, where $E$ is a subset of a normed linear space $G$, is named to accomplish condition (N) (5) if there is a nondecreasing

function $\xi:[0,\infty)\to[0,\infty)$ with $\xi(0)=0$ & $\xi(Þ)>0$,
$\forall\, Þ\in(0,\infty)$ such as
$\|a-Ta\|\geq\xi(d(a,F(T)))$ for all $a\in E$, where $d(a,F(T))=\inf\{\|a-h^*\|, h^*\in F(T)\neq\emptyset\}$.
Yet, modify that definition to two mappings :
Two mappings $T:E\to E$, where $E$ is a subset of a normed linear space $G$, is named to accomplish condition (M) (5) if $\forall$ a nondecreasing function $\xi:[0,\infty)\to[0,\infty)$ with $\xi(0)=0$ &$\xi(Þ)>0$, $\forall\, Þ\in(0,\infty)$ such as $l_1\|a-Sa\|+l_2\|a-Ta\|\geq\xi(d(a,F(T))$ for all $a\in E$, where $d(a,F)=\inf\{\|a-h^*\|, h^*\in F\neq\emptyset\}$ and $l_1,l_2$ are two positive real numbers such as $l_1+l_2=1$. Note, the condition (M) is weaker than the compactness of the domain $E$. As well Condition (M) decrease to condition (N) when $S=T$.

**Theorem (13):** Under the presumption of Theorem (8). If S and T satisfy condition (M), hence the sequence $\{u_n\}$ converges strongly to a point of $F$.

**Proof:** Through Theorem (8), obtaining $\lim_{n\to\infty} d(u_n,Tu_n)=0$ and $\lim_{n\to\infty} d(u_n,Su_n)=0$.
From condition (M),

$$\lim_{n\to\infty}\xi(d(u_n,F))\leq l_1\lim_{n\to\infty} d(u_n,Su_n)+l_2\lim_{n\to\infty} d(u_n,Tu_n)=0$$

i.e $\lim_{n\to\infty}\xi(d(u_n,F))=0$. Therefore

$$\lim_{n\to\infty} d\,(u_n,F)=0$$

The illation just now follows up of Theorem (10).

## Numerical example

Our results through the following example is elucidated (Table 1)

**Example (14):** Deem $G=R$ with its usual metric, so $G$ is as well complete CAT(0) space. Let $E=[0,1]$, which is a closed bounded convex subset of $G$. Define two mappings $T,S:E\to E$ by $T(u)=ku$ and $S(u)=\frac{u}{k+1}$, $0<k<1$. So $T$ is asymptotically nonexpansive mapping with $\{6_n=2\}, \forall\, n\in N$ and $S$ is a total asymptotically nonexpansive nonself mapping with $e_n=\frac{1}{n^2}$ & $\sigma_n=\frac{1}{n^3}$, $\forall\, n\geq 1$. Obviously, $F(T)=\{0\}=F(S)$ of the mappings $T$ and $S$. Put $\delta_n=0.2, \gamma_n=0.5$ & $\beta_n=0.3$ $(\delta_n+\gamma_n+\beta_n=1)$. By using Matlab,the iteration which is defined by (1) for initial points $u_1=0.76$ and $k=0.59, u_1=0.88$ and $k=0.75, u_1=0.9$ and $k=0.9$ is calculated. Lastly, the convergence demeanors of the iteration (7) is appeared in Fig. 1. The consequence is that $u_n$ converges to zero.

**Table 1. Numerical results for 40 steps**

| n | $u_1=0.76, k=0.59$ | $u_1=0.88, k=0.75$ | $u_1=0.9, k=0.9$ |
|---|---|---|---|
| 2 | 0.4902 | 0.6380 | 0.7200 |
| 3 | 0.3162 | 0.4626 | 0.5760 |
| 4 | 0.2039 | 0.3353 | 0.4608 |
| 5 | 0.1315 | 0.2431 | 0.3686 |
| 6 | 0.0848 | 0.1763 | 0.2949 |
| 7 | 0.0547 | 0.1278 | 0.2359 |
| 8 | 0.0353 | 0.0927 | 0.1887 |
| 9 | 0.0228 | 0.0672 | 0.1510 |
| 10 | 0.0147 | 0.0487 | 0.1208 |
| 11 | 0.0095 | 0.0353 | 0.0966 |
| 12 | 0.0061 | 0.0256 | 0.0773 |
| 13 | 0.0039 | 0.0186 | 0.0618 |
| 14 | 0.0025 | 0.0135 | 0.0495 |
| 15 | 0.0016 | 0.0098 | 0.0396 |
| 16 | 0.0011 | 0.0071 | 0.0317 |
| 17 | 0.0007 | 0.0051 | 0.0253 |
| 18 | 0.0004 | 0.0037 | 0.0203 |
| 19 | 0.0003 | 0.0027 | 0.0162 |
| 20 | 0.0002 | 0.0020 | 0.0130 |
| 21 | 0.0001 | 0.0014 | 0.0104 |
| 22 | 0.0001 | 0.0010 | 0.0083 |
| 23 | 0.0000 | 0.0007 | 0.0066 |
| 24 | 0.0000 | 0.0005 | 0.0053 |
| 25 | 0.0000 | 0.0004 | 0.0043 |
| 26 | 0.0000 | 0.0003 | 0.0034 |
| 27 | 0.0000 | 0.0002 | 0.0027 |
| 28 | 0.0000 | 0.0001 | 0.0022 |
| 29 | 0.0000 | 0.0001 | 0.0017 |
| 30 | 0.0000 | 0.0001 | 0.0014 |
| 31 | 0.0000 | 0.0001 | 0.0011 |
| 32 | 0.0000 | 0.0000 | 0.0009 |
| 33 | 0.0000 | 0.0000 | 0.0007 |
| 34 | 0.0000 | 0.0000 | 0.0006 |
| 35 | 0.0000 | 0.0000 | 0.0005 |
| 36 | 0.0000 | 0.0000 | 0.0004 |
| 37 | 0.0000 | 0.0000 | 0.0003 |
| 38 | 0.0000 | 0.0000 | 0.0002 |
| 39 | 0.0000 | 0.0000 | 0.0002 |
| 40 | 0.0000 | 0.0000 | 0.0001 |

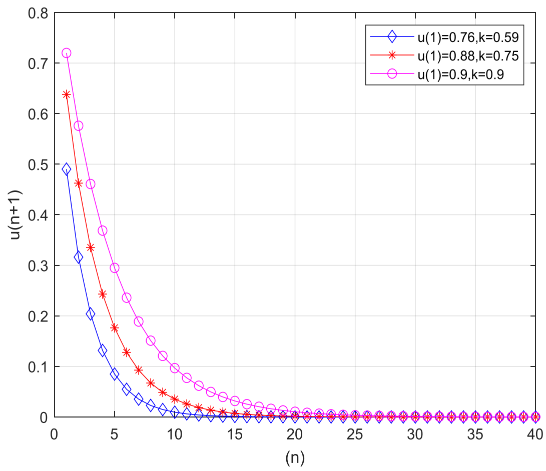


**Figure 1. Convergence behaviors for different initial points for 40 steps.**

## Authors' declaration:

- Conflicts of Interest: None.

- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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# نقاط صامدة مشتركة جديدة لتطبيق اللامتمددة المقاربة كلياً في فضاء CAT(0)

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**الخلاصة:**

تم دراسة التقارب و Δ- تقارب لعملية تكرار من خطوتين بأستخدام تطبيقات اللامتمددة المقاربة ( asymptotically nonexpansive) واللامتمددة المقاربة كلياً (total asymptotically nonexpansive) في فضاء CAT(0) وكذلك تم مبرهنة بعض نظريات التقارب القوي باستخدام شبه المتراصة (semi-compact) وشرط (M). نتائجنا تحسن وتطور العديد من النتائج المعروفة في الادبيات الموجودة.

**الكلمات المفتاحية:** تطبيق اللامتمددة المقاربة, فضاءCAT(0), نقاط صامدة تقريبية, وشرط (M) , Δ-تقارب.