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Fuzzy Convergence Sequence and Fuzzy Compact Operators on Standard Fuzzy Normed Spaces

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Abstract:

The main purpose of this work is to introduce some types of fuzzy convergence sequences of operators defined on a standard fuzzy normed space (SFN-spaces) and investigate some properties and relationships between these concepts. Firstly, the definition of weak fuzzy convergence sequence in terms of fuzzy bounded linear functional is given. Then the notions of weakly and strongly fuzzy convergence sequences of operators (Γ_n) are introduced and essential theorems related to these concepts are proved. In particular, if (Γ_n) is a strongly fuzzy convergent sequence with a limit Γ where Γ linear operator from complete standard fuzzy normed space $(U, \mathcal{N}_U, \odot)$ into a standard fuzzy normed space $(V, \mathcal{N}_V, \odot)$ then Γ belongs to the set of all fuzzy bounded linear operators $\mathbb{FB}(U, V)$. Furthermore, the concept of a fuzzy compact linear operator in a standard fuzzy normed space is introduced. Also, several fundamental theorems of fuzzy compact linear operators are studied in the same space. More accurately, every fuzzy compact linear operator $\Gamma: U \rightarrow V$ is proved to be fuzzy bounded where $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ are two standard fuzzy normed spaces.

Keywords: Fuzzy compact linear operator, Fuzzy convergence sequence of operators, Standard fuzzy normed spaces, Strongly fuzzy convergent, Weakly fuzzy convergent.

Introduction:

Fuzzy set theory was initiated by Zadeh in 1965(1), then numerous mathematicians have studied this concept and obtained different main results from various points of view(2,3). The definition of the fuzzy norm was firstly introduced by Katrasas (4) who defined the fuzzy norm in linear space. Felbin (5) presented another different type of fuzzy norm defined on a vector space. Thought of a fuzzy normed linear space and the fuzzy linear operator was introduced by Shih Chuan and John Mordeson (6). Bag and Samanta (7) studied the relation between fuzzy boundedness and fuzzy continuity and gave a notion of boundedness of a linear operator in fuzzy normed spaces. They also debated the notions of convergent sequence and Cauchy sequence in the same space. In (8) Sadeqi and Salehi introduced the concept of the fuzzy compact operator and proved the equivalence of the concepts of fuzzy boundedness and fuzzy norm continuity of linear operators. Moreover, different approaches for the space of fuzzy norm can be found in (9-22). In this paper, the first main

goal is to present the definition of weak and strong fuzzy convergence sequence of operators in a standard fuzzy normed space (U, \mathcal{N}, \odot) (briefly SFN-space) where \odot is an arbitrary continuous t-norm. Then basic properties related to these concepts are given. The second goal is to introduce the notion of fuzzy compact linear operators and discussed some important properties of them. Structurally, this paper involves the following: the basic properties of a standard fuzzy normed space are given, then, in Section 3, the definition of strongly and weakly fuzzy convergence sequence of operators in an SFN-space are presented. In Section 4, the concept of a fuzzy compact linear operator from an SFN-space into another SFN-space is given. Besides, different properties of the fuzzy compact linear operator will be proved, for example, the product of a fuzzy compact linear operator Γ and fuzzy bounded linear operator Λ on a standard fuzzy normed space (U, \mathcal{N}, \odot) is proved to be fuzzy compact and in case U is finite dimension then the linear operator Γ is proved to be

fuzzy compact. Finally, the paper finished with a conclusion section.

Preliminaries

This section presents some definitions of standard fuzzy normed spaces and basic properties related to these concepts.

Definition 1(23)

Suppose that $\odot: [0,1]^2 \rightarrow [0,1]$ a binary operation. Then \odot is called continuous t-norm if \odot satisfies the following properties for each $\rho, \delta, \beta, \sigma \in [0, 1]$:

- 1- $\rho \odot \delta = \delta \odot \rho$
- 2- $\rho \odot 1 = \rho$
- 3- $(\rho \odot \delta) \odot \beta = \rho \odot (\delta \odot \beta)$
- 4- If $\rho \leq \delta$ and $\beta \leq \sigma$ then, $\rho \odot \beta \leq \delta \odot \sigma$.

The following definition of the standard fuzzy normed space considered in (24) is given.

Definition 2 (24)

A Standard fuzzy normed space (briefly, SFN-space) is a triple (U, \mathcal{N}, \odot) where U is a linear space over field K , \odot is a continuous t-norm and \mathcal{N} is a fuzzy set on U satisfying the conditions:

- (\mathcal{N}_1) $\mathcal{N}(u) > 0, \forall u \in U$
- (\mathcal{N}_2) $\mathcal{N}(u) = 1$ if and only if $u = 0$.
- (\mathcal{N}_3) $\mathcal{N}(\rho u) = \frac{1}{|\rho|} \mathcal{N}(u), \forall u \in U$ and $\rho \neq 0$.
- (\mathcal{N}_4) $\mathcal{N}(u + w) \geq \mathcal{N}(u) \odot \mathcal{N}(w), \forall u, w \in U$.
- (\mathcal{N}_5) $\mathcal{N}(u)$ is a continuous fuzzy set.

The next proposition shows that an ordinary normed space with some conditions will be standard fuzzy normed space induced by the norm $\|\cdot\|$ as in (24).

Proposition 1(24)

Suppose that $(U, \|\cdot\|)$ be an ordinary normed space with $\|u\|$ is an integer for each $u \in U$ and $u \odot w = u \cdot w$ for each $u, w \in [0,1]$. Define

$$\mathcal{N}_{\|\cdot\|}(u) = \begin{cases} \frac{1}{\|u\|} & \text{if } u \neq 0 \\ 1 & \text{if } u = 0 \end{cases}$$

Then $(U, \mathcal{N}_{\|\cdot\|}, \odot)$ is standard fuzzy normed space deduced from the norm $\|\cdot\|$.

The following group of definitions are presenting the notions of the open ball, open and closed set convergent and Cauchy sequence, and the closure of a set respectively that given in (24).

Definition 3 (24)

Let $(U, \mathcal{N}_U, \odot)$ be an SFN-space. Put $B(u, \epsilon) = \{w \in U: \mathcal{N}(u - w) > (1 - \epsilon)\}$. Then $B(u, \epsilon)$ is

called an open ball centered at $u \in U$ with radius $0 < \epsilon < 1$.

Definition 4 (24)

Suppose that $(U, \mathcal{N}_U, \odot)$ is an SFN-space and $\mathcal{A} \subseteq U$. Then

- (1) \mathcal{A} is open set if for each $a \in \mathcal{A}$ there is an $0 < \epsilon < 1$ with $B(a, \epsilon) \subset \mathcal{A}$.
- (2) \mathcal{A} is closed set if \mathcal{A}^c is open, that is $\mathcal{A}^c = U \setminus \mathcal{A}$.

Definition 5 (24)

In an SFN-space $(U, \mathcal{N}_U, \odot)$, for any $0 < \rho < 1$ a sequence (u_n) in U is said to be:

- 1- convergent to a point $u \in U$ if there is a natural number $N \in \mathbb{N}$ with $\mathcal{N}(u_n - u) > (1 - \rho)$ for each $n \geq N$.
- 2- Cauchy if there is a natural number $N \in \mathbb{N}$ with $\mathcal{N}(u_n - u_m) > (1 - \rho)$ where $n, m \geq N$.

Definition 6 (24)

Let \mathcal{A} be a subset of an SFN-space $(U, \mathcal{N}_U, \odot)$. Then $\bar{\mathcal{A}}$ is called the closure of \mathcal{A} which is the intersection of all closed sets containing: $\bar{\mathcal{A}} = \bigcap_{\text{closed } B \supseteq \mathcal{A}} B$.

Definition 7 (24)

Suppose that $(U, \mathcal{N}_U, \odot)$ is an SFN-space and $\mathcal{A} \subseteq U$. Then \mathcal{A} is called fuzzy bounded if for each $\epsilon \in \mathcal{A}$ there is a real number say $r, 0 < r < 1$ with $\mathcal{N}(a) > (1 - r)$.

Definition 8 (24)

Suppose that $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ be two SFN-spaces and $\Gamma: D(\Gamma) \rightarrow V$ be a linear operator, where $D(\Gamma) \subset U$. The operator Γ is said to be fuzzy bounded if there is a real number $r, 0 < r < 1$ such that for all $u \in D(\Gamma), \mathcal{N}_V(\Gamma u) \geq (1 - r) \mathcal{N}_U(u)$.

The set of all fuzzy bounded operator is denoted by $\mathbb{F}\mathbb{B}(U, V)$ such that $\mathbb{F}\mathbb{B}(U, V) = \{\Gamma: U \rightarrow V | \Gamma \text{ is a fuzzy bounded operator}\}$ (25).

Theorem 1(25)

Suppose that $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ are two SFN-spaces. Then $(\mathbb{F}\mathbb{B}(U, V), \mathcal{N}, \odot)$ is SFN-space such that $\mathcal{N}(\Gamma) = \inf_{u \in D(\Gamma)} \mathcal{N}_V(\Gamma u)$ for each $\Gamma \in \mathbb{F}\mathbb{B}(U, V)$.

Definition 9 (25)

A linear functional g from an SFN-space $(U, \mathcal{N}_U, \odot)$ into the SFN-space $(K, \mathcal{N}_K, \odot)$ is called fuzzy bounded if there exists $r, 0 < r < 1$ with $\mathcal{N}_K(g(u)) \geq (1 - r) \mathcal{N}_U(u)$ for each $u \in D(g)$.

Furthermore, the standard fuzzy norm of g is $\mathcal{N}(g) = \inf_{u \in D(g)} \mathcal{N}_K(g(u))$ and $\mathcal{N}_K(g(u)) \geq \mathcal{N}(g) \circledast \mathcal{N}_U(u)$.

Theorem 2 (25)

Let $(U, \mathcal{N}_U, \circledast)$ be SFN-space. Then U is compact if and only if each sequence of elements in U has a subsequence converging to an element in U .

Fuzzy Convergence Sequence and Fuzzy Convergence Sequence of Operators

This section is devoted to introduce the notations of strongly and weakly fuzzy convergent sequence of operators and prove some properties in the standard fuzzy normed space. It is well known that if $(U, \mathcal{N}_U, \circledast)$ is an SFN-space, then the set of all fuzzy bounded functional $\mathbb{F}\mathbb{B}(U, K) = \{g: U \rightarrow K, g \text{ is fuzzy bounded linear functional}\}$ with a fuzzy norm $\mathcal{N}(g) = \inf_{u \in U} \mathcal{N}_K(g(u))$ which is called the fuzzy dual space of U (25).

First, the definition of weak fuzzy convergence sequence in terms of fuzzy bounded linear functional on a standard fuzzy normed space $(U, \mathcal{N}_U, \circledast)$ is introduce as follows.

Definition 10

Let $(U, \mathcal{N}_U, \circledast)$ be an SFN-space. A sequence (u_n) in U is called weakly fuzzy convergent if there exists an element $u \in U$ such that $\lim_{n \rightarrow \infty} g(u_n) = g(u)$ for every $g \in \mathbb{F}\mathbb{B}(U, K)$.

Theorem 3

Suppose that (u_n) be a sequence in an SFN-space $(U, \mathcal{N}_U, \circledast)$. If (u_n) fuzzy convergent sequence then (u_n) weak fuzzy convergent with the same limit.

Proof: let (u_n) be a sequence in an SFN-space $(U, \mathcal{N}_U, \circledast)$. Since (u_n) fuzzy converges to $u \in U$ then there exists an element $\rho \in (0,1)$ and a positive integer $N \in \mathbb{N}$ with $\mathcal{N}_U(u_n - u) > 1 - \rho$ for all $n \geq N$. Now for each $g \in \mathbb{F}\mathbb{B}(U, K)$

$$\mathcal{N}_K(g(u_n) - g(u)) = \mathcal{N}_K(g(u_n - u)) \geq \mathcal{N}(g) \circledast \mathcal{N}_U(u_n - u).$$

Put $\mathcal{N}(g) = 1 - \sigma$ for some $0 < \sigma < 1$, then there is $1 - \delta$, $0 < \delta < 1$ with $(1 - \sigma) \circledast (1 - \rho) > 1 - \delta$. Hence $\mathcal{N}_K(g(u_n) - g(u)) > (1 - \delta)$ and this shows that (u_n) is a weakly fuzzy convergent.

To prove that a sequence (u_n) in an SFN-space $(U, \mathcal{N}_U, \circledast)$ is weak fuzzy convergence, the following definition is introduced.

Definition 11

Suppose that $(U, \mathcal{N}_U, \circledast)$ be an SFN-space and $\mathcal{A} \subseteq U$. Then \mathcal{A} is said to be fuzzy total set if $\overline{\text{span}\mathcal{A}} = U$.

Theorem 4

Let $(U, \mathcal{N}_U, \circledast)$ be an SFN-space and (u_n) be a sequence in $(U, \mathcal{N}_U, \circledast)$. If the sequence $(\mathcal{N}(u_n))$ is a fuzzy bounded and the sequence $g(u_n)$ fuzzy converges to $g(u)$ for each element g of a fuzzy total subset $\mathcal{A} \subset \mathbb{F}\mathbb{B}(U, K)$, then the sequence (u_n) is a weakly fuzzy convergent to u .

Proof: Since $(\mathcal{N}(u_n))$ is a fuzzy bounded sequence then $\mathcal{N}(u_n) > 1 - r$ for all n and $\mathcal{N}(u) > 1 - r$ for some $0 < r < 1$. To prove that the sequence (u_n) is a weakly fuzzy convergent to u , suppose that for every $g \in \mathbb{F}\mathbb{B}(U, K)$, the sequence $g(u_n)$ should converge to $g(u)$. Since \mathcal{A} is fuzzy total in $\mathbb{F}\mathbb{B}(U, K)$, for each $g \in \mathbb{F}\mathbb{B}(U, K)$, there is a sequence (g_i) in $\text{span } \mathcal{A}$ such that g_i fuzzy converges to g . Hence for any given $0 < \epsilon < 1$ obtain $\mathcal{N}_U(g_i - g) > 1 - \epsilon$. Moreover, since $g_i \in \text{span } \mathcal{A}$, there is an N such that for all $n > N$ and $0 < \rho < 1$, $\mathcal{N}_K(g_i(u_n) - g_i(u)) > (1 - \rho)$. Hence for $n > N$, obtain

$$\begin{aligned} \mathcal{N}_K(g(u_n) - g(u)) &\geq \mathcal{N}_K(g(u_n) - g_i(u_n)) \\ &\circledast \mathcal{N}_K(g_i(u_n) - g_i(u)) \\ &\circledast \mathcal{N}_K(g_i(u) - g(u)) \end{aligned}$$

$$\begin{aligned} &\geq \mathcal{N}_U(g - g_i) \circledast \\ \mathcal{N}(u_n) \circledast (1 - \rho) \circledast \mathcal{N}_U(g_i - g) \circledast \mathcal{N}(u) & \\ &\geq (1 - \epsilon) \circledast (1 - r) \circledast \\ (1 - \rho) \circledast (1 - \epsilon) \circledast (1 - r) & \end{aligned}$$

Hence there exists δ such that $0 < 1 - \delta < 1$ and $(1 - r) \circledast (1 - c) \circledast (1 - p) \circledast (1 - r) \circledast (1 - c) > (1 - \delta)$.

This shows that the sequence (u_n) fuzzy converges weakly to u .

Definition 12

Suppose that $(U, \mathcal{N}_U, \circledast)$ is an SFN-space and $\mathcal{A} \subseteq U$. Then \mathcal{A} is called :

- (i) Rare (or nowhere dense) in U if the interior of $\overline{\mathcal{A}}$ is empty.
- (ii) Meager in U if \mathcal{A} represents the countable union of rare sets. The complement of a meager set is called residual.
- (iii) Nonmeager in U if \mathcal{A} is not meager in U .

It is well known that in an SFN-space $(U, \mathcal{N}_U, \circledast)$ every convergent sequence is Cauchy. Also, an SFN-space $(U, \mathcal{N}_U, \circledast)$ is called complete if each Cauchy sequence in U converges to an element in U . One of the essential theorems in functional analysis is Baire's theorem (26). Now, Fuzzy Baire's

theorem in the standard fuzzy normed spaces is proved.

Theorem 5

If an SFN-space $(U, \mathcal{N}_U, \odot)$ is complete, it is non-meager in itself. Hence if $U \neq \emptyset$ is complete and $U = \bigcup_{k=1}^{\infty} X_k$ (X_k closed) then there is at least one X_k contains an open subset which is not equal to \emptyset .

Proof: Assume the complete SFN-space U was meager in itself. Then

$$U = \bigcup_{k=1}^{\infty} \mathcal{A}_k \quad \dots(1)$$

with each \mathcal{A}_k nowhere in U . Cauchy sequence (a_k) will be constructed whose limit is a is in no \mathcal{A}_k this contradicts eq.(1) by assumption \mathcal{A}_1 is a nowhere dense subset in U therefore $\bar{\mathcal{A}}_1$ does not contain an open set that is not equal to \emptyset but U does. This implies $\bar{\mathcal{A}}_1 \neq U$. Thus the complement $\bar{\mathcal{A}}_1^c = U - \bar{\mathcal{A}}_1 \neq \emptyset$ and open. A point a_1 in $\bar{\mathcal{A}}_1^c$ is taken and an open ball about it, say $B_1 = B(a_1, \epsilon_1) \subset \bar{\mathcal{A}}_1^c$, for each $0 < \epsilon_1 < 1$. By assumption, \mathcal{A}_2 is nowhere dense in U therefore $\bar{\mathcal{A}}_2$ does not contain a nonempty open set. Thus $B(a_1, \frac{\epsilon_1}{2}) \not\subset \bar{\mathcal{A}}_2$, that follows $\bar{\mathcal{A}}_2^c \cap B(a_1, \frac{\epsilon_1}{2})$ is not empty and open. Now let $B_2 = B(a_2, \epsilon_2) \subset \bar{\mathcal{A}}_2^c \cap B(a_1, \frac{\epsilon_1}{2})$ where $\epsilon_2 < \frac{\epsilon_1}{2}$, then by induction, obtain a sequence of an open ball $B_k = B(a_k, \epsilon_k)$ where $\epsilon_k < 2^{-k}$, $k=1,2,\dots$ such that $B_k \cap \mathcal{A}_k = \emptyset$ and $B_{k+1} \subset B(a_k, \frac{\epsilon_k}{2}) \subset B_k$. Since $\epsilon_k < 2^{-k}$ and $1 - \epsilon_k < 1 - 2^{-k}$ the sequence (a_k) of the centers is Cauchy and converges say, $a_k \rightarrow a$ because U is complete. Now for every m and $n > m$, $B_n = B(a_n, \frac{\epsilon_n}{2})$ therefore $\mathcal{N}_U(a_m - a) \geq \mathcal{N}_U(a_m - a_n) \odot \mathcal{N}_U(a_n - a) \geq (1 - \frac{\epsilon_n}{2}) \odot \mathcal{N}_U(a_n - a) = (1 - \frac{\epsilon_n}{2}) \odot 1 = (1 - \frac{\epsilon_n}{2})$. Thus $a \in B_m$ for each m . Since $B_m \subset \bar{\mathcal{A}}_m^c$, then for each m , $a \notin \mathcal{A}_m$ therefore $a \notin \bigcup \mathcal{A}_m = U$. This contradicts $a \in U$.

According to the previous theorem (Fuzzy Baire's Theorem), the uniform fuzzy boundedness theorem is now ready to establish.

Theorem 6

Let (Γ_n) be a sequence in $\mathbb{F}\mathbb{B}(U, V)$ where $(U, \mathcal{N}_U, \odot)$ is complete SFN-space and $(V, \mathcal{N}_V, \odot)$ is an SFN-space such that $(\mathcal{N}_V(\Gamma_n u))$ is a fuzzy bounded sequence for every $u \in U$, with

$$\mathcal{N}_V(\Gamma_n u) \geq 1 - r_u \quad \dots(2)$$

where $0 < r_u < 1$. Then the sequence $(\mathcal{N}(\Gamma_n))$ is a fuzzy bounded sequence so there is a number r , $0 < r < 1$ with $\mathcal{N}(\Gamma_n) \geq 1 - r, n \in \mathbb{N}$.

Proof: For each $0 < k < 1$, let $\mathcal{A}_k \subseteq U$ be the set of all u such that $\mathcal{N}_V(\Gamma_n u) \geq 1 - k$. Now for any $u \in \bar{\mathcal{A}}_k$ there is a sequence (u_i) in \mathcal{A}_k converging to u . This means that for each fixed n , $\mathcal{N}_V(\Gamma_n u_i) \geq 1 - k$ and obtain $\mathcal{N}_V(\Gamma_n u) \geq 1 - k$ because Γ_n is fuzzy continuous and so is the fuzzy norm. Hence $u \in \mathcal{A}_k$ and \mathcal{A}_k is closed. By eq.(2), every $u \in U$ belongs to some \mathcal{A}_k . Thus $U = \bigcup_{k=1}^{\infty} \mathcal{A}_k$. Because U is complete SFN-space, Theorem 5 implies that some \mathcal{A}_k contains an open ball

$$B_1 = B(u_1, \epsilon) \subset \mathcal{A}_{k_1} \quad \dots(3)$$

Let $u \in U$ where $u \neq 0$ and put

$$s = u_1 + u \quad \dots(4)$$

with $\mathcal{N}_U(s - u_1) > (1 - \epsilon)$ that is means $s \in B(u_1, \epsilon)$. By eq.(3) and from the definition of \mathcal{A}_{k_1} , $\mathcal{N}_V(\Gamma_n s) \geq (1 - k_1)$. Also $\mathcal{N}_V(\Gamma_n u_1) \geq (1 - k_1)$ since $u_1 \in B(u_1, \epsilon)$. From eq.(4) obtain $u = s - u_1$ and:

$$\mathcal{N}_V(\Gamma_n u) = \mathcal{N}_V(\Gamma_n(s - u_1)) \geq \mathcal{N}_V(\Gamma_n s) \odot \mathcal{N}_V(\Gamma_n u_1) \geq (1 - k_1) \odot (1 - k_1).$$

Thus for all n , $\mathcal{N}(\Gamma_n) = \inf \mathcal{N}_V(\Gamma_n u) \geq (1 - r)$ for some $0 < r < 1$.

Now, some types of fuzzy convergence sequences of operators (Γ_n) in an SFN-space $\mathbb{F}\mathbb{B}(U, V)$ are established. The Strongly fuzzy convergent and Weakly fuzzy convergent sequences are introduced as follows:

Definition 13

Let $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ be an SFN-spaces. A sequence (Γ_n) of operators $\Gamma_n \in \mathbb{F}\mathbb{B}(U, V)$ is called

(i) Strongly fuzzy convergent if $(\Gamma_n u)$ fuzzy converges strongly in V for each $u \in U$. That means $\mathcal{N}_V(\Gamma_n u - \Gamma u) \rightarrow 1$ for any $n \geq \mathbb{N}$.

(ii) Weakly fuzzy convergent if $(\Gamma_n u)$ fuzzy converges weakly in V for each $u \in U$. That means that $\mathcal{N}_K(g(\Gamma_n u) - g(\Gamma u)) \rightarrow 1$ for every $g \in \mathbb{F}\mathbb{B}(V, K)$.

Lemma 1

Let (Γ_n) be a sequence of the fuzzy bounded linear operator $\Gamma_n: U \rightarrow V$ defined on a complete SFN-space $(U, \mathcal{N}_U, \odot)$ into an SFN-space $(V, \mathcal{N}_V, \odot)$. If (Γ_n) strongly fuzzy convergent sequence with

limit Γ then $\Gamma \in \text{FIB}(U, V)$.

Proof: Since $\Gamma_n u \rightarrow \Gamma u$ for every $u \in U$, the sequence $(\Gamma_n u)$ is bounded for each u . Since U is complete, $(\mathcal{N}(\Gamma_n))$ is a fuzzy bounded sequence by the uniform fuzzy boundedness theorem, say $\mathcal{N}(\Gamma_n) \geq (1-r)$ for all n and $0 < r < 1$. This follows that $\mathcal{N}_V(\Gamma_n u) \geq \mathcal{N}(\Gamma) \circledast \mathcal{N}_U(u)$. This implies $\mathcal{N}_V(\Gamma u) \geq (1-r) \circledast \mathcal{N}_U(u)$ as $n \rightarrow \infty$. For a sequence of fuzzy bounded linear functional $g_n \in \text{FIB}(U, K)$ the strongly fuzzy convergent and weakly fuzzy convergent notions are introduced as follows:

Definition 14

Let (g_n) be a sequence of fuzzy bounded linear functional on an SFN-space $(U, \mathcal{N}_U, \circledast)$. Then:

(i) Strong fuzzy convergence of (g_n) means that there exists a fuzzy bounded linear functional $g \in \text{FIB}(U, K)$ with $\mathcal{N}(g_n - g) \rightarrow 1$ which is written as $g_n \rightarrow g$.

(ii) Weak fuzzy convergence of (g_n) means that there exists a fuzzy bounded linear functional $g \in \text{FIB}(U, K)$ with $g_n(u) \rightarrow g(u)$ for each $u \in U$.

Theorem 7

Let $(U, \mathcal{N}_U, \circledast)$ and $(V, \mathcal{N}_V, \circledast)$ be a complete SFN-spaces. A sequence (Γ_n) of operators $\Gamma_n \in \text{FIB}(U, V)$ is strongly fuzzy convergence if and only if

- (a) $(\mathcal{N}(\Gamma_n))$ is a fuzzy bounded sequence.
- (b) $(\Gamma_n u)$ is Cauchy sequence in V for every u in a total subset \mathcal{A} of U .

Proof: Since $(\Gamma_n u)$ strongly fuzzy convergence sequence of operators for every $u \in U$ then condition one follows from Theorem 6 since U is complete, also condition two follows.

For the converse, assume that conditions (a) and (b) are holds, so that $\mathcal{N}(\Gamma_n) \geq (1-r)$ for all n and $0 < r < 1$. Consider any $u \in U$, the sequence $(\Gamma_n u)$ will be proved to be strongly fuzzy converges in V . Let $0 < c < 1$ be given. Since $\text{span}\mathcal{A}$ is dense in U ; therefore, there is $s \in \text{span}\mathcal{A}$ with $\mathcal{N}_U(u - s) > (1-c)$. Since $s \in \text{span}\mathcal{A}$, the sequence $(\Gamma_n s)$ is Cauchy by (b). Thus there is a natural number N with $\mathcal{N}_V(\Gamma_n s - \Gamma_m s) > (1-p)$ where $0 < p < 1$ and $n, m > N$. Now for $n, m > N$ obtain

$$\begin{aligned} \mathcal{N}_V(\Gamma_n u - \Gamma_m u) &\geq \mathcal{N}_V(\Gamma_n u - \Gamma_n s) \\ &\circledast \mathcal{N}_V(\Gamma_n s - \Gamma_m s) \\ &\circledast \mathcal{N}_V(\Gamma_m s - \Gamma_m u) \end{aligned}$$

$$\begin{aligned} &\geq \mathcal{N}(\Gamma_n) \circledast \mathcal{N}_U(u - s) \circledast \\ &\mathcal{N}_V(\Gamma_n s - \Gamma_m s) \circledast \mathcal{N}(\Gamma_m) \circledast \mathcal{N}_U(u - s) \\ &> (1-r) \circledast (1-c) \circledast (1-p) \circledast (1-r) \circledast (1-c) \end{aligned}$$

Hence there exists q such that $0 < 1-q < 1$ and $(1-r) \circledast (1-c) \circledast (1-p) \circledast (1-r) \circledast (1-c) > (1-q)$.

Hence $\mathcal{N}_V(\Gamma_n u - \Gamma_m u) > (1-q)$ for all $n, m > N$. This shows $(\Gamma_n u)$ is Cauchy in V . Since V is complete, $(\Gamma_n u)$ converges in V . Because $u \in U$ was arbitrary. This shows the sequence (Γ_n) strong fuzzy convergence.

Fuzzy Compact Linear Operators on SFN-space

In this section, the notion of fuzzy compact linear operators on a standard fuzzy normed space will be given. Moreover, the main general properties of this notion will be proved.

Definition 15

Let $(U, \mathcal{N}_U, \circledast)$ and $(V, \mathcal{N}_V, \circledast)$ be an SFN-spaces and $\Gamma: U \rightarrow V$ a linear operator between U and V . Then Γ is called fuzzy compact operator if for each bounded subset \mathcal{A} of U , $\Gamma(\mathcal{A})$ is relatively compact in V , that is, the closure $\overline{\Gamma(\mathcal{A})}$ is compact.

Theorem 8

Let $(U, \mathcal{N}_U, \circledast)$ and $(V, \mathcal{N}_V, \circledast)$ be two SFN-spaces. The linear operator $\Gamma: U \rightarrow V$ is fuzzy compact if and only if the sequence $(\Gamma(u_n))$ has a convergent subsequence in V for every fuzzy bounded sequence (u_n) in U .

Proof: Consider Γ be a linear operator from an SFN-space $(U, \mathcal{N}_U, \circledast)$ into an SFN-space $(V, \mathcal{N}_V, \circledast)$ and let Γ be fuzzy compact. Assume that (u_n) be a fuzzy bounded sequence in U . Then by Definition15, $(\Gamma(u_n))$ is relatively compact, this means that the closure of $(\Gamma(u_n))$ is compact in V . Hence by Theorem 2, the sequence $(\Gamma(u_n))$ contains a convergent subsequence.

For the converse, assume that each fuzzy bounded sequence (u_n) in U contains a subsequence, say (u_{nk}) with $(\Gamma(u_{nk}))$ converges in V . Suppose that \mathcal{A} be any fuzzy bounded subset $\mathcal{A} \subseteq U$ and consider (e_n) be any sequence in $\Gamma(\mathcal{A})$. Then $e_n = \Gamma(u_n)$ for $u_n \in \mathcal{A}$ and the sequence (u_n) is fuzzy bounded since \mathcal{A} is fuzzy bounded. The sequence $(\Gamma(u_n))$ by the assumption contains a convergent subsequence, hence the closure $\overline{\Gamma(\mathcal{A})}$ is compact. Thus Γ is fuzzy compact.

Relying on the above theorem, the next proposition discusses the relationship between the fuzzy compact linear operator and the fuzzy bounded operator.

Proposition 2

Every fuzzy compact linear operator $\Gamma: U \rightarrow V$ is fuzzy bounded where $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ are two SFN-spaces.

Proof: let Γ be a fuzzy compact linear operator the operator Γ have to proved fuzzy bounded. The set $E = B(u, \epsilon) = \{w \in U: \mathcal{N}(u - w) > (1 - \epsilon)\}$ is fuzzy bounded for each $0 < \epsilon < 1$. Now consider (e_k) an arbitrary sequence in $\Gamma(E)$ that can be written as $e_k = \Gamma(u_k)$ for some (u_k) in U . Since E is fuzzy bounded and by assumption, Γ is fuzzy compact operator, hence by Theorem 8, the sequence (e_k) contains a convergent subsequence. Now suppose that $\Gamma(E)$ is unbounded, this means that $\Gamma(E)$ contains unbounded sequence say (w_k) and this sequence has no convergent subsequence. But since each sequence (e_k) in $\Gamma(E)$ has convergent subsequence therefore $\Gamma(E)$ must be fuzzy bounded.

Proposition 3

Let $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ be an SFN-spaces and $\Gamma: U \rightarrow V$ be a linear operator. If Γ is fuzzy bounded and the dimension of $\Gamma(U)$ is finite then Γ is a fuzzy compact operator.

Proof: Let (u_n) be any fuzzy bounded sequence in U . Then by using the inequality $\mathcal{N}_V(\Gamma u_n) \geq \mathcal{N}(\Gamma) \odot \mathcal{N}_U(u_n)$ (see(24), Remark2.2) it is clear that the sequence (Γu_n) is fuzzy bounded. Thus (Γu_n) is relatively compact since the dimension of $\Gamma(U)$ is finite that follows, the sequence (Γu_n) has a convergent subsequence. Because (u_n) was an arbitrary fuzzy bounded sequence in U , hence by Theorem 2, Γ is fuzzy compact.

For a finite-dimensional SFN-space $(U, \mathcal{N}_U, \odot)$, each linear operator on U is fuzzy bounded (see(25), Theorem2.8.14). In the following, a linear operator over a finite dimension SFN-space is proved to be a fuzzy compact.

Proposition 4

Let $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ be an SFN-spaces and $\Gamma: U \rightarrow V$ be a linear operator. If the dimension of U is finite then Γ is a fuzzy compact operator.

Proof: Because the dimension of U is finite, hence Γ is fuzzy bounded operator by Theorem 2.8.14 (25) and since $\dim \Gamma(U) < \dim U$ therefore by Proposition 3, Γ is a fuzzy compact operator.

Let $(U, \mathcal{N}_U, \odot)$ and $(V, \mathcal{N}_V, \odot)$ are two SFN-spaces, then the set of all fuzzy compact linear operators is denoted by $\mathbb{FC}(U, V)$ such that $\mathbb{FC}(U, V) = \{\Gamma | \Gamma: U \rightarrow V \text{ is a fuzzy compact linear operator}\}$. Now by Theorem 8, the following result is provided.

Theorem 9

Let $(U, \mathcal{N}_U, \odot)$ be an SFN-space and $(V, \mathcal{N}_V, \odot)$ complete SFN-space. If $(\Gamma_n) \in \mathbb{FC}(U, V)$ and $(\Gamma_n) \rightarrow \Gamma$ is a fuzzy convergent sequence of operators then the limit operator Γ is fuzzy compact.

Proof: Assume that $(u_k) \in U$ be any fuzzy bounded sequence. It must prove that the image (Γu_k) has a convergent subsequence. Because Γ_1 is fuzzy compact operator, then the sequence (u_k) has a subsequence say, $(u_{1,k})$ such that $(\Gamma_1 u_{1,k})$ is Cauchy sequence. Also, the sequence $(u_{1,k})$ has a subsequence say, $(u_{2,k})$ such that $(\Gamma_2 u_{2,k})$ is Cauchy sequence. Continue in this process and then define a sequence $(w_k) = (u_{k,k})$. It is clear that (w_k) is a subsequence of (u_k) and it has the property that for each positive integer number $n \in \mathbb{N}$ the sequence $(\Gamma_n w_k)$ is Cauchy. By the assumption (u_k) is a fuzzy bounded sequence, this means that there is $0 < r < 1$ such that $\mathcal{N}_U(u_k) \geq (1 - r)$ for each k . Therefore for each k , $\mathcal{N}_U(w_k) \geq (1 - r)$, assume that (Γw_k) is a Cauchy sequence in V . Because Γ_k converges to Γ , there is $q \in \mathbb{N}$ with $\mathcal{N}(\Gamma - \Gamma_q) > (1 - \rho)$ where $0 < \rho < 1$. Since $(\Gamma_n w_k)$ is Cauchy, there is some $N \in \mathbb{N}$ with $\mathcal{N}_V(\Gamma_q w_i - \Gamma_q w_k) > (1 - \rho)$ whenever $i, k > N$. Then, for all $i, k > N$, obtained

$$\begin{aligned} \mathcal{N}_V(\Gamma w_i - \Gamma w_k) &\geq \mathcal{N}_V(\Gamma w_i - \Gamma_q w_i) \\ &\odot \mathcal{N}_V(\Gamma_q w_i - \Gamma_q w_k) \\ &\odot \mathcal{N}_V(\Gamma_q w_k - \Gamma w_k) \\ &\geq \mathcal{N}(\Gamma - \Gamma_q) \odot \mathcal{N}_U(w_i) \odot \\ \mathcal{N}_V(\Gamma_q w_i - \Gamma_q w_k) &\odot \mathcal{N}(\Gamma_q - \Gamma) \odot \mathcal{N}_U(w_k) \\ &\geq (1 - \rho) \odot (1 - r) \odot (1 - \rho) \odot \\ (1 - \rho) &\odot (1 - r). \end{aligned}$$

Hence there exists ω such that $0 < \omega < 1$ and $(1 - \rho) \odot (1 - r) \odot (1 - \rho) \odot (1 - \rho) \odot (1 - r) > (1 - \omega)$.

By Remarks (2.6) (23). Therefore, obtain $\mathcal{N}_V(\Gamma w_i - \Gamma w_k) > (1 - \omega)$ which proves that (Γw_k) is Cauchy. Since V is complete so (Γw_k) converges. Notice that (w_k) is a subsequence of

the sequence (u_k) and using Theorem 8; therefore, Γ is a fuzzy compact.

The final result explains the relationship between the product of two linear operators on standard fuzzy normed spaces.

Proposition 5

Let $\Gamma: U \rightarrow U$ be a fuzzy compact linear operator and $\Lambda: U \rightarrow U$ be a fuzzy bounded linear operator on SFN-space $(U, \mathcal{N}_U, \odot)$. Then the product $\Gamma\Lambda$ and $\Lambda\Gamma$ are fuzzy compact.

Proof: Let Γ be a fuzzy compact operator and Λ be a fuzzy bounded operator on U . To show that $\Gamma\Lambda$ is a fuzzy compact operator, take a fuzzy bounded set \mathcal{A} in U since Λ is fuzzy bounded operator so $\Lambda(\mathcal{A})$ is fuzzy bounded set and $\Gamma[\Lambda(\mathcal{A})] = \Gamma\Lambda(\mathcal{A})$ is relatively compact because Γ is a fuzzy compact operator. Hence $\Gamma\Lambda$ is a fuzzy compact operator. Now to prove that $\Lambda\Gamma$ is fuzzy compact, assume that (u_n) be any fuzzy bounded sequence in U , then the sequence (Γu_n) has a convergent subsequence say, (Γu_{n_k}) by Theorem 8 and the sequence $(\Lambda\Gamma u_{n_k})$ converges by Corollary (2.7)(i)(24). Thus $\Lambda\Gamma$ is fuzzy compact.

Conclusion:

In this paper, the convergence of the sequence of operators in a standard fuzzy normed space is studied then some of their essential properties are discussed. The concept of fuzzy compact linear operators between standard fuzzy normed spaces is given. Finally, several properties of fuzzy compact linear operators in a standard fuzzy normed space (SFN-spaces) have been studied.

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References:

1. Zadeh L. Fuzzy sets. *Inf. Control.* 1965; 8: 338-353.
2. Efe. H , Yildiz. C. Some results in fuzzy compact linear operators. *JCAA.* 2010; 12:251-262.
3. Saadati R, Vaezpour S M. Some results on fuzzy Banach spaces. *J. Appl. Math.* 2005; 17: 475-484.
4. Katsaras A. Fuzzy topological vector spaces II. *Fuzzy Sets Syst.* 1984; 12: 143-154.
5. Feblin C. Finite dimensional fuzzy normed linear spaces. *Fuzzy Sets Syst.* 1992;48:239-48.
6. Shih-Chuan C, John N. Fuzzy linear operators and fuzzy normed linear spaces. *International Conference*

- on Fuzzy Theory and Technology Proceedings. Oct 14 1992; p: 193-197.
7. Bag T, Samanta SK. Finite dimensional fuzzy normed linear spaces. *J Fuzzy Math.* 2003;11(3):687-705.
8. Sadeqi I , Salehi M . Fuzzy compact operators and topological degree theory. *Fuzzy Sets Syst.* 2009;160:1277-1285.
9. Mehmet K, Mehmet A. On Generalized Fuzzy n-Normed Spaces Including ϕ Function. *Int J Sci Eng Res.* 2013; 4(11): 2229-5518.
10. Font J, Galindo J, Macario S, Sanchis M. Mazur-Ulam type theorems for fuzzy normed spaces. *J.Nonlinear Sci. Appl.* 2017;10: 4499-4506.
11. Chatterjee S, Bag T, Samanta S. Some results on G-fuzzy normed linear space. *Int J Pure Appl Math.* 2018; 120 (5) : 1295-1320.
12. Tudor B, Flavius P, Sorin N. A Study of Boundedness in Fuzzy Normed Linear Spaces. *J Symmetry MDPI.* 2019; 11:1-13.
13. Sorin N, Tudor B , Flavius P. Some fixed point theorems for ϕ -contractive mappings in fuzzy normed linear spaces. *J. Nonlinear Sci. Appl.* 2017;10: 5668-5676.
14. Nădăban S. Fuzzy Continuous Mappings in Fuzzy Normed Linear Spaces, *IJCCC.* 2015;10(6):834-842.
15. Szabo A, Bînzar T, Nădăban S, Pater F. Some properties of fuzzy bounded sets in fuzzy normed linear space. *AIP Conference Proceedings ISSN 390009.* 2018.
16. Govindana V, Murthyb S. Solution And Hyers-Ulam Stability Of n-Dimensional Non-Quadratic Functional Equation In Fuzzy Normed Space Using Direct Method. *Fuzzy Sets Syst.* 2019; 16 : 384-391.
17. Lee K. Approximation properties in fuzzy normed spaces. *Fuzzy Sets Syst.* 2016; 282: 115-130.
18. Ju Myung K , Keun Y. Approximation Properties in Felbin Fuzzy Normed Spaces. *Math J.*2019; 22:1-14.
19. Cho Y, Rassias T, Saadati R. Fuzzy normed spaces and fuzzy metric spaces. *Fuzzy Operator Theory in Mathematical Analysis.* Springer, Cham. 2018; P: 11-43.
20. Morteza S. A comparative study of fuzzy norms of linear operators on a fuzzy normed linear Spaces. *JMM.* 2015;2(2):217-234.
21. Rana A, Buthainah A, Fadhel F. On Completeness of Fuzzy Normed Spaces. *JAM.*2015;9(8):2963-2970.
22. Mohsenialhosseini M. Error estimates for fixed point theorems in fuzzy normed spaces. *AFMI.*2016;11(1):87-95.
23. George A, Veeramani P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* 1994; 64: 395-399.
24. Kider J, Jameel R. Fuzzy Bounded and Continuous Linear Operators on Standard Fuzzy Normed Spaces. *Eng.&Tech Journal.* 2015;33(2): 178- 185.
25. Jameel R. On some results of analysis in a standard fuzzy normed space. M.Sc. Thesis, University of Technology, Iraq. 2014.
26. Kutateladze S. *Fundamentals of Functional Analysis.* Springer Science Business Media Dordrecht. Sobolev Institute of Mathematics, Novosibirsk. 1996. 276 p.

المتابعة المتقاربة الضبابية والمؤثرات المتراسة الضبابية على الفضاءات المعيارية الضبابية القياسية

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الخلاصة:

الغرض الرئيسي من هذا العمل هو تقديم بعض أنواع متابعات التقارب الضبابية للمؤثرات المعرفة على الفضاءات المعيارية الضبابية والبحث في بعض الخصائص والعلاقات بين هذه المفاهيم في البداية ، يتم تقديم تعريف متابعة التقارب (SFN-spaces) القياسية الحد نفسه تتقارب تقارباً ضبابياً ضعيفاً مع (u_n) الضبابي الضعيف بدلالة الدوال الخطية الضبابية المقيدة. بعد ذلك تم برهان أن المتابعة (fuzzy Baire's and uniform fuzzy boundedness) تم ذكر برهان نظريتين مهمتين (متقاربة ضبابياً (u_n) في حالة أن المتابعة للمؤثرات وهذه النظريات ضرورية في الاتصال مع التقارب الضبابي الضعيف. يتم تقديم مفاهيم متابعات التقارب الضبابي القوي والضعيف Γ حيث Γ متابعة متقاربة ضبابياً بقوة مع (Γ_n) . على وجه الخصوص، إذا كان واثبات النظريات الأساسية المتعلقة بهذه المفاهيم (Γ_n) تنتمي Γ فإن $(V, \mathcal{N}_V, \otimes)$ الى الفضاء المعياري الضبابي القياسي $(U, \mathcal{N}_U, \otimes)$ المؤثر الخطي من الفضاء المعياري الضبابي القياسي التام الى مجموعة كل المؤثرات الخطية الضبابية المقيدة. إضافة الى ذلك ، تم تقديم مفهوم المؤثر الخطي المتراس الضبابي في الفضاء المعياري الضبابي القياسي. أيضا ، تتم دراسة العديد من النظريات الأساسية للمؤثرات الخطية المتراسة الضبابية على نفس الفضاء. بتعبير أدق ، ثبت فضاء معياري ضبابي قياسي $(V, \mathcal{N}_V, \otimes)$ و $(U, \mathcal{N}_U, \otimes)$ أن كل مؤثر خطي متراس ضبابي يكون مقيد ضبابي بحيث أن كل من على الفضاء المعياري الضبابي القياسي Λ والمؤثر الخطي الضبابي المقيد Γ في نهاية البحث ، أثبتنا أن المؤثر الخطي الضبابي المتراس يجب أن يكون متراساً ضبابياً $(U, \mathcal{N}_U, \otimes)$.

الكلمات المفتاحية: المؤثر الخطي المتراس الضبابي، متابعة التقارب الضبابي للمؤثرات، الفضاءات المعيارية الضبابية القياسية، تقارب ضبابي قوي، تقارب ضبابي ضعيف.