Fuzzy Real Pre-Hilbert Space and Some of Their Properties

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Abstract:
In this work, two different structures are proposed which is fuzzy real normed space (FRNS) and fuzzy real Pre-Hilbert space (FRPHS). The basic concept of fuzzy norm on a real linear space is first presented to construct \((V, N_f, \Delta_c)\) space, which is a FRNS with some modification of the definition introduced by G. Rano and T. Bag. The structure of fuzzy real Pre-Hilbert space (FRPHS) is then presented which is based on the structure of FRNS. Then, some of the properties and related concepts for the suggested space FRN such as \(\delta\)-neighborhood, closure of the set \(A\) named \(GCL(A)\), the necessary condition for separable, fuzzy linear manifold (FLM) are discussed. The definition for a fuzzy seminorm on \(V/A\) is also introduced with the prove that a fuzzy seminorm on \(V/A\) is FRNS. The relationship between the \(l\)-convergent sequence, \(l\)-Cauchy sequence and \(l\)-completeness is then investigated in this work. The structure of FRPHS with some important properties concerning on this space are introduced and proved. In addition, the property of orthogonality with some important properties for these spaces is included, for example the annihilator of the set \(A\). The relation between FRPHS and FRNS is investigated in the present work. Finally, after introducing the structure of FRPHS, it leads naturally to the definition of the most important class of FRPHS, namely the fuzzy real Hilbert space (FRHS).

Keywords: Fuzzy Inner Product (FIP), Fuzzy Norm (FN), Fuzzy Real Hilbert Space, Fuzzy Real Normed Space, Fuzzy Orthogonality.

Introduction:
The concept of fuzzy set is a generalization of classical set theory which is first introduced by zadeh 1. Triangular conorms are an indispensable tool for the interpretation of the conjunction in fuzzy sets. They are binary operations on the closed unit interval \([0, 1]\) with neutral element 0 which is very interesting to introduce a special class of real monotone operations. There are many applications of fuzzy set in different areas such as multicriteria decision making 2,3, rough sets theory 4,5, sociology 6, etc. Numerous applications have emerged from fuzzy sets theory. This approach may potentially improve some recent results in chaos theory application, e.g., designing chaotic sensors, see 7. Also, applications of fuzzy set theory may be considered within the actual scope of neuroscience like in 8. First a parameter some families of continuous triangular conorm is recalled. This idea allows us to generalize the intersection and the union in a Lattice, or disjunction and conjunction in Logic. An interesting concept of fuzzy norm on a linear space has been effectively introduced by G. Rano and T. Bag 9. A new modification of the structures fuzzy real normed space FRNS and fuzzy real Pre-Hilbert space FRPHS which can be considered as a generalization of real normed space and real Pre-Hilbert space are introduced to find fuzzy analogies of classical mathematics and help us to develop more results of functional analysis in fuzzy setting. That’s why creating a suitable structure of FRNS and FRPHS which is flexible enough and study some basic properties of these spaces. Felbin 10 introduced the fuzzy norm on a linear space by using fuzzy numbers. Cheng and Mordeson 11 introduced another idea of fuzzy norm on a linear space, and in 2003 Bag and Samanta 12 defined a more suitable notion of fuzzy norm. In 13 the idea of fuzzy antinorm was introduced. On the basis of this idea, Jebril and Samanta 14 introduced the concept of fuzzy antinorm on a linear space, first applied in investigation of probabilistic metric spaces 15. Following their concept, R. Biswas 16 and
A. M. El-Abayed & H. M. Hamouly \textsuperscript{17} first tried to give a structure of fuzzy Pre-Hilbert space and associated fuzzy norm function. Later on many author viz. Mazumdar & Samanta \textsuperscript{18}, Hasankhani, Nazari & Saheli \textsuperscript{19}, Mukherjee & Bag \textsuperscript{20} have introduced the structure of fuzzy Pre-Hilbert space from different points and studied a few properties and some of applications. Induced norms of these spaces may have very important applications in quantum particle physics for more details; see \textsuperscript{21,22}. In recent past lots (more advanced reader), numerous references to the application of these spaces have been done in this direction on fuzzy functional analysis \textsuperscript{23-26}.

Structurally, the paper comprises the following: after this introductory section, In Section 2 gives basic definitions and preliminary results which are used in the sequel. In Section 3 a structure of fuzzy norm defined on a real linear space of type $(V, \tilde{N}_f, \Delta_c)$ as well as different properties and ideas related to this structure are presented. Section 4 devotes a structure of fuzzy inner product defined on a real linear space of type $(V, P_f, \Delta_c)$, and studies some results for this space. The implications between the given modifications structures is discussed in Section 5.

**Background and Preliminaries**

Some important definitions and preliminary results are presented which will be utilized in the next sections.

**Definition 1** \textsuperscript{22} A triangular conorm is a binary operation $\Delta_c$ on the interval $I = [0,1]$ which is satisfying the following conditions, for all $\alpha_1, \alpha_2, \alpha_3 \in [0,1]$

(H1) $\alpha_1 \Delta_c \alpha_2 = \alpha_2, \Delta_c \alpha_1$ (Commutative)

(H2) $(\alpha_1 \Delta_c (\alpha_2 \Delta_c \alpha_3)) = ((\alpha_1 \Delta_c \alpha_2) \Delta_c \alpha_3)$ (Associative)

(H3) $\alpha_1 \Delta_c \alpha_2 \leq \alpha_1 \Delta_c \alpha_3$ whenever, $\alpha_2 \leq \alpha_3$ (Monotone)

(H4) $\alpha_1 \Delta_c 0 = \alpha_1$ (has 0 as neutral element)

If, in addition, $H$ is continuous then $H$ is called a continuous triangular conorm.

The following theorem introduces the characteristics of a triangular conorm

**Theorem 1** \textsuperscript{22} Let $\Delta_c$ be a triangular conorm on the set $I$. Then

(1) $0 \Delta_c 0 = 0$

(2) $1 \Delta_c 0 = 1$

(3) $0 \Delta_c 1 = 1$

(4) $1 \Delta_c 1 = 1$

(5) If $\alpha_1 \leq \alpha_2, \alpha_3 \leq \alpha_4$, then $\alpha_1 \Delta_c \alpha_3 \leq \alpha_2 \Delta_c \alpha_4$ for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$

(6) $\alpha_1 \Delta_c \alpha_1 \geq \alpha_1$ for all $\alpha_1 \in [0,1]$

**Example 1** \textsuperscript{5} (1) let $\alpha_1 \Delta_c \alpha_2 = (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)$ for all $\alpha_1, \alpha_2 \in [0,1]$ where $(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)$ is a triangular conorm, and called the probabilistic sum

(2) let $\alpha_1 \Delta_c \alpha_2 = \min(\alpha_1 + \alpha_2, 1)$ for all $\alpha_1, \alpha_2 \in [0,1]$ where $\min(\alpha_1 + \alpha_2, 1)$ is a triangular conorm, and named the bounded sum

**Remark 1** \textsuperscript{5} For any $\gamma$ it can be find $\delta$ such that $\delta \Delta_c \delta \geq \gamma$, where $\gamma, \delta \in (0,1)$

**Fuzzy Real Normed Space (FRNS) and Main Results**

A real linear space is considered and a structure of fuzzy (real) normed space is introduced with a new characterization triangular conorm. In addition, some properties and subsequent results of a fuzzy real normed space are introduced and proved in this section.

**Definition 2** Let $\Delta_c$ is a triangular conorm, $V$ be a linear space over the field $R$. Let a fuzzy subset $\tilde{N}_f : V \times [0,\infty)$ into is a mapping called fuzzy norm on $V$ if its following conditions are holds for all $z_1, z_2 \in V$, and for all $\alpha_1 \in I$:

(RN1) $\tilde{N}_f(z_1, \alpha_1) > 0$, for all $\alpha_1 > 0$

(RN2) $\tilde{N}_f(z_1, \alpha_1) = 1$ if and only if $z_1 = 0$, for all $\alpha_1 > 0$

(RN3) $\forall \ r \neq 0 \in R$, $\tilde{N}_f(r z_1, \alpha_1) = \tilde{N}_f(z_1, \frac{\alpha_1}{|r|})$

(RN4) $\tilde{N}_f(z_1 + z_2, \alpha_1) \leq \tilde{N}_f(z_1, \alpha_1) \Delta_c \tilde{N}_f(z_2, \alpha_1)$

(RN5) $\tilde{N}_f(z_1, ..) : [0,1] \to [0,1]$ is continuous with respect to $\alpha_1$

Then $(V, \tilde{N}_f, \Delta_c)$ is called fuzzy real normed space (FRNS).

In some theorems, the following condition is considered.

**Lemma 1** Suppose that $(V, \tilde{N}_f, \Delta_c)$ is a fuzzy real normed space. Then

$\tilde{N}_f(z_1 - z_2, \alpha_1) = \tilde{N}_f(z_2 - z_1, \alpha_1)$ for all $z_1, z_2 \in V$ and $\alpha_1 > 0$

The following example demonstrates the concept of a fuzzy real normed space which is necessary in this process.

**Example 2** Let $(R^2, || \cdot ||)$ be a normed space, where $V = R^2$ is a linear space which is obtained if the set of ordered pairs of real numbers $z_1 = (\rho_1, \rho_2) \in R^2$ is taken with a function

$||z_1|| = (|\rho_1|^2 + |\rho_2|^2)^{\frac{1}{2}}$. Define $\tilde{N}_f(z_1, \alpha_1) = ||z_1|| - \alpha_1 ||z_1||$ for $\alpha_1 < ||z_1||$ and $\tilde{N}_f(z_1, \alpha_1) = 0$ for $\alpha_1 \geq ||z_1||$. Also $a \Delta_c b = a + b - ab$ for all $a, b \in I$. Then $(V, \tilde{N}_f, \Delta_c)$ is FRNS

**Proof:**

(RN1) Since $||z_1|| > 0, \forall z_1 = (\rho_1, \rho_2) \in R^2$ and for all $\alpha_1 > 0$ so $\tilde{N}_f(z_1, \alpha_1) > 0$
Definition 8 Let \((V, \tilde{N}_f, A_c)\) be FRNS on \(V\). The set \(A\) of \(V\) is said to be a dense in \(V\) if it \(\text{GCL}(A)\) is equal to \(V\).

Definition 9 A FRNS \((V, \tilde{N}_f, A_c)\) on \(V\) is separable if it contains a dense subset that is countable.

Proposition 1 Let \((V, \tilde{N}_f, A_c)\) be FRNS on \(V\). Then \(\ell = \{A : A\) is the subset of \(V, z_1\) is an interior of \(A\) if and only if there exists a neighborhood of \(z_1\) contained in 

The following definition describes the behavior of \(\tilde{\ell}\)-convergent sequence in a fuzzy real normed space.

Definition 10 A sequence \(\{z_n\}\) in a FRNS \((V, \tilde{N}_f, A_c)\) is said to be \(\tilde{\ell}\)-convergent if there exists a vector \(z\) in \(V\) such that \(\tilde{N}_f(z_n - z, A_c) \rightarrow 1\) as \(n \rightarrow \infty\), for all \(A_c > 0\).

Lemma 2 Let \((V, \tilde{N}_f, A_c)\) be a FRNS and \(\{z_n\}\) be an \(\tilde{\ell}\)-convergent sequence in \(V\). Then the limit of \(\{z_n\}\) is unique.

Proof: Consider \(\{z_n\}\) be a sequence in FRNS which is \(\tilde{\ell}\)-convergent to \(z\) and \(\hat{z}\). Then by \((\text{RN}4)\)

\[
\tilde{N}_f(z + \hat{z}, A_c) = \tilde{N}_f(z - z_n + z_n + \hat{z}, A_c) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ for all } A_c > 0
\]

\[
\leq \tilde{N}_f(z_n - z, A_c) \Delta_c \tilde{N}_f(z_n - (z - \hat{z}), A_c)
\]

\[
= \tilde{N}_f(z_n - z, A_c) \Delta_c \tilde{N}_f(z_n - (z - \hat{z}), A_c) \text{ by lemma (3-2)}
\]

\[
\rightarrow 1 \Delta_c 1 \rightarrow 1 \text{ by theorem ((2-2)-4)}
\]

Thus, it follows that \(\tilde{N}_f(z + \hat{z}, A_c) = 1\). From \((\text{RN}2)\)

\[
\tilde{N}_f(z + \hat{z}, A_c) = 1 \text{ if and only if } z + \hat{z} = 0, \text{ which implies that } z = -\hat{z} \text{. Therefore the limit is unique.}
\]

Definition 11 A sequence \(\{z_n\}\) in a FRNS \((V, \tilde{N}_f, A_c)\) is said \(\ell\)-Cauchy sequence if \(\lim \tilde{N}_f(z_n - z_n + s, \alpha_1) = 1\) as \(n \rightarrow \infty\), for all \(\alpha_1 > 0\).

Definition 12 A FRNS \((V, \tilde{N}_f, A_c)\) is said to be \(\ell\)-complete if every \(\ell\)-Cauchy sequence in \(V\) is \(\ell\)-convergent.

Definition 13 A set \(\hat{A}\) in FRNS on \(V\) is called a fuzzy linear manifold \(\hat{A}\) of \((V, \tilde{N}_f, A_c)\) if \(\hat{A}\) is a linear manifold of \(V\) considered as a vector space, with the fuzzy norm obtained by restricting the fuzzy norm on \(V\) to the set \(\hat{A}\).

Definition 14 A set \(\hat{A}\) is called closed fuzzy linear manifold of \((V, \tilde{N}_f, A_c)\) if \(\hat{A}\) is closed in \(V\).

The following theorem describes the behavior of a fuzzy seminorm on \(V/A\) in a fuzzy real normed space.
Theorem 2 If \((V, \tilde{N}_f, \Delta_c)\) is FRNS on \(V\) with a closed linear manifold \(A\) and let
\[
\tilde{N}_f((z, a_1)) = \inf \{\tilde{N}_f(z - m, a_1), m \in A, a_1 > 0\}
\]
for all \([z] \in V/A\)
This defines a fuzzy seminorm on \(V/A\), then this fuzzy seminorm is a norm on \(V/A\).

**Proof:** Let \(z \in V\). Cleary \(\tilde{N}_f([z], a_1) = \tilde{N}_f(z - A, a_1) > 0\). Suppose that \(\tilde{N}_f((z, a_1)) = \tilde{N}_f(z - A, a_1) = 1\), from the definition (3.12) there is a sequence \([z_n]\) in \(A\) such that \(\tilde{N}_f(z_n - z, a_1) \to 1\) and since \(A\) is closed hence \(z \in A\) and \(0 + A = z + A = A\).

If \(r \neq 0 \in R, z \in V\) then
\[
\tilde{N}_f(r[z], a_1) = \tilde{N}_f(r(z - A), a_1) = \tilde{N}_f(rz - A, a_1) = \inf \{\tilde{N}_f(rz - rz_1, a_1), z_1 \in A\}
\]
\[
= \inf \{\tilde{N}_f \left( z - z_1, \frac{a_1}{|r|} \right), z_1 \in A \}
\]
\[
= \tilde{N}_f \left( z, \frac{a_1}{|r|} \right)
\]
Also it is known that
\[
\tilde{N}_f((z) + [z_1], a_1) = \tilde{N}_f((z + z_1), a_1) \leq \tilde{N}_f((z - a) + (z_1 - a), a_1) \leq \tilde{N}_f((z - a), a_1) \Delta_c \tilde{N}_f((z_1 - a), a_1)
\]
for all \(a, a_1 \in A\). Taking the infimum over such a and \(a_1\) gives the inequality
\[
\tilde{N}_f([z] + [z_1], a_1) \leq \tilde{N}_f((z), a_1) \Delta_c \tilde{N}_f([z_1], a_1)
\]
Finally, it is clear that \(\tilde{N}_f([z], .)\) from (0,1) into \(I\) continuous function.
The proof is complete.

**Fuzzy Real Pre-Hilbert Space (FRPHS) and Main Results**
The main general interesting results which hold in any fuzzy real Pre-Hilbert space will be proved. So the structure of fuzzy real Pre-Hilbert space is introduced initially.

**Definition 15** Let \(\Delta_c\) be a triangular conorm, \(V\) be a linear space over the field \(R\), then \(\tilde{P}_f : V \times V \times [0, \infty) \to I\) is called a fuzzy real inner product on \(V\), if for all \(z_1, z_2, z_3 \in V, a_1 \in I\) the following conditions hold:

- **(RPH1)** \(\tilde{P}_f(z_1, z_2, a_1) \Delta_c \tilde{P}_f(z_3, z_2, a_1) \geq \tilde{P}_f(z_1 + z_3, z_2, a_1) \)
- **(RPH2)** \(\tilde{P}_f(z_1, z_2, a_1) = \tilde{P}_f(z_2, z_1, a_1) \)
- **(RPH3)** \(\forall r \in R\)

\[
\tilde{P}_f(r z_1, z_2, a_1) = \begin{cases} \tilde{P}_f(z_1, z_2, \frac{a_1}{r} \) & \text{if } r > 0 \\ 1 - \tilde{P}_f(z_1, z_2, a_1) & \text{if } r < 0 \end{cases}
\]
If \(r = 0\) and \(a_1 > 0\), \(0 \) if \(r = 0, a_1 = 0\) (RPH4)
- a- for \(a_1 = 0\) then \(\tilde{P}_f(z_1, z_1, a_1) = 0\)
- b- for \(a_1 \neq 0\) then \(\tilde{P}_f(z_1, z_1, a_1) > 0\)

**Example 3** Space \(L^2[a, b]\). The vector space of all continuous real-valued functions on \([a, b]\) forms a pre-Hilbert space \((V, <, >)\) with inner product defined by \(< z_1, z_2 > = \int_a^b z_1(x)z_2(x)dx\).

Define \(\tilde{P}_f(z_1, z_2, a_1) = \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right)}\) for all \(z_1, z_2, z_3 \in V, a_1 > 0\) and \(\tilde{P}_f(z_1, z_2, a_1) = 0\) if \(a_1 = 0\). Then \((V, \tilde{P}_f, \Delta_c)\) is a FRPHS. Where \(a_1 \Delta_c a_2 = a_1 + a_2 - a_1 a_2\) for all \(a_1, a_2 \in I\).

**Proof:**

- **(RPH1)** \(\tilde{P}_f(z_1, z_2, a_1) \Delta_c \tilde{P}_f(z_3, z_2, a_1) \geq \tilde{P}_f(z_1 + z_3, z_2, a_1) \)
- **(RPH2)** \(\tilde{P}_f(z_1, z_2, a_1) = \tilde{P}_f(z_2, z_1, a_1) \)
- **(RPH3)** \(\forall r \in R\)

\[
= \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
\[
= \exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_3, z_2>}{a_1} \right) - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
\[
= \exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_3, z_2>}{a_1} \right) - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
\[
= \exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_3, z_2>}{a_1} \right) - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
\[
= \exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_3, z_2>}{a_1} \right) - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
\[
= \exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_3, z_2>}{a_1} \right) - \frac{1}{\exp \left( \frac{<z_1, z_2>}{a_1} \right) + \exp \left( \frac{<z_2, z_3>}{a_1} \right)} - 1
\]
Remark

\( P(\alpha) = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} = \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} = \frac{1}{\exp\left(\frac{\xi_3 \Delta}{1 - \alpha}\right)} = 1 - \frac{\xi_1 \Delta}{1 - \alpha} \)

(2) If \( r = 0 \), then \( P_f(r \xi_1, \xi_2, \alpha_1) = 1 \) for \( \alpha_1 > 0 \) and \( P_f(r \xi_1, \xi_2, \alpha_1) = 0 \) for \( \alpha_1 > 0 \)

(RPH4) a-for \( \alpha_1 = 0 \) it is clear that \( P_f(z_1, \xi_2, \alpha_1) = 0 \)

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b- for \( \alpha_1 \neq 0 \), it is clear that \( P_f(z_1, \xi_2, \alpha_1) = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} > 0 \)

(RPH5) \( P_f(z_1, \xi_2, \alpha_1) = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( P_f(z_1 + \xi_2, z_1 + \xi_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

\( = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( P_f(z_1 + \xi_2, z_1 + \xi_2, \alpha_1) + P_f(z_2, \xi_2, \alpha_1) \)

\( = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} - \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} + \frac{1}{\exp\left(\frac{\xi_2 \Delta}{1 - \alpha}\right)} \)

\( \forall \alpha_1 > 0 \)

(RPH6) suppose \( P_f(z_1, \xi_2, \alpha_1) = 1 \) \( \forall \alpha_1 > 0 \),

\( P_f(z_1, \xi_2, \alpha_1) = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} = 1 \) if and only if

\( \exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right) = 1 \) if and only if \( < z_1, \xi_2 > = 0 \) if and only if \( z_1 = 0 \)

(RPH7) Let \( \alpha_n \) be a sequence in \( [0,1] \) such that \( \alpha_n \to \alpha \). Now for every \( z_1 \in V \),

\( \lim_{n \to \infty} \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} = \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} \)

\( \frac{1}{\exp\left(\frac{\xi_1 \Delta}{1 - \alpha}\right)} \) is a FRPHS on \( V \).

Lemma 3 Assume that \( \left(V, P_f, \alpha_1\right) \) is a FRPHS then \( P_f(-z_1, -z_2, \alpha_1) = P_f(z_1, z_1, \alpha_1) \) for all \( z_1 \in V \), \( \alpha_1 > 0 \)

Proof: To prove the equality, let

\( P_f(-z_1, -z_2, \alpha_1) = 1 - P_f(z_1, z_1, \alpha_1) = 1 - \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

by (RPH3)

Remark 2 If \( \left(V, P_f, \alpha_1\right) \) is a FRPHS, then for all \( z_1, z_2, z_3 \in V \), the following properties are satisfied

(i) if \( r > 0, s > 0, \alpha_1 > 0 \), then

\( \leq P_f(z_1, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

(ii) if \( r > 0, s = 0, \alpha_1 > 0 \), then

\( \leq P_f(z_1, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

(iii) if \( r > 0, s < 0, \alpha_1 > 0 \), then

\( \leq P_f(z_1, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

(iv) if \( r < 0, s < 0, \alpha_1 > 0 \), then

\( \leq P_f(z_1, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

Proposition 2 Let \( \left(V, P_f, \Delta \alpha_1\right) \) be a FRPHS, \( \Delta \alpha_1 \) is a continuous triangular norm. Then

\( \leq P_f(z_1, z_2, \xi_3, z_3, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

Thus by Theorem ((2-2)-6), the conditions (PRH3) and (PRH5) give the following result:

\( \leq P_f(z_1, z_2, \xi_3, z_3, z_2, \alpha_1) \Delta \frac{\xi_1 \Delta P_f(z_2, \xi_2, \alpha_1)}{1 - \alpha} \)

The proof is complete.
Definition 16 A FLM $\bar{A}$ of FRPHS $V$ is defined to be a linear manifold of $V$ taken with the fuzzy real inner product on $V$ restricted to $\bar{A} \times \bar{A}$.

Similarly, A FLM $\bar{A}$ of FRHS $V$ is defined to be a linear manifold of $V$, regarded as the fuzzy real inner product.

Remark 3 A FLM $\bar{A}$ need not to be FRHS because $\bar{A}$ may not be $\bar{l}$-complete.

The concept of fuzzy orthogonal in a FRPHS is introduced in the following definition:

Definition 17 An element $z_1 \in V$ in a FRPHS $(V, \bar{P}_f, \Delta C)$ is said to be fuzzy orthogonal to an element $z_2 \in V$ if $\bar{P}_f(z_1, z_2, \alpha_1) = 1$, for all $\alpha_1 > 0$ and $\bar{P}_f(z_1, z_2, 0) = 0$. Put $z_1$ and $z_2$ are orthogonal, this means $z_1 \perp_{\bar{P}_f} z_2$. Similarly for two sets $A, B$ in $V$ if $z_1 \perp_{\bar{P}_f} z_2$ then $z_1 \perp_{\bar{P}_f} B$ for all $z_2 \in B$ and $A \perp_{\bar{P}_f} B$ for all $z_1 \in A$ and $z_2 \in B$.

Lemma 4 Let $(V, \bar{P}_f, \Delta C)$ be a FRPHS. If $z_1 \perp_{\bar{P}_f} z_2$ then

$$\bar{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1) \leq \bar{P}_f(z_1, z_1 + z_2, \alpha_1) \Delta C \bar{P}_f(z_2, z_2, \alpha_1)$$

Proof: Let $(V, \bar{P}_f, \Delta C)$ be a FRPHS and let $A$ be a non-empty set in $V$ then the annihilator of $A$ is defined by $\lambda^{-1} P_f = \{ z_1 \in V \mid z_1 \perp_{\bar{P}_f} A \}$ which is the set of all vectors fuzzy orthogonal to $A$. An orthogonal complement is a special annihilator of a set, which defined as follows:

Definition 18 Let $\bar{A}$ be FLM of FRPS, then the orthogonal complement of $\bar{A}$ is $\bar{A}^{-1} P_f = \{ z_1 \in V \mid z_1 \perp_{\bar{P}_f} \bar{A} \}$ which is the set of all vectors fuzzy orthogonal to $\bar{A}$. An orthogonal complement is a special annihilator of a set, which defined as follows:

Definition 19 Let $(V, \bar{P}_f, \Delta C)$ be a FRPHS and let $A$ be a non-empty set in $V$ then the annihilator of $A$ is defined by $\lambda^{-1} P_f = \{ z_1 \in V \mid z_1 \perp_{\bar{P}_f} A \}$. Thus $z_1 \in A$ if and only if $\bar{P}_f(z_1, z_2, \alpha_1) = 1$, for all $\alpha_1 > 0$ and $\bar{P}_f(z_1, z_2, 0) = 0$, for all $z_2 \in A$.

Next, the orthogonal complement set in a fuzzy real Pre-Hilbert space implies a fuzzy linear manifold is proved.

Lemma 5 $A^{-1} P_f$ is a FLM of FRPHS

Proof: To prove that $0 \in A^{-1} P_f$, the following cases are considered:

Case-a Since $\bar{P}_f(0, z_2, \alpha_1) = 1$ for all $\alpha_1 > 0, z_2 \in A$

Case-b Since $\bar{P}_f(0, z_2, 0) = 0$ for all $z_1, z_2 \in A$ then $0 \in A^{-1} P_f$

Now, let $z_1, z_3 \in A^{-1} P_f$, $\alpha_1 > 0$ and $r \in R$

$$\bar{P}_f(z_1 + z_3, z_2, \alpha_1) \leq \bar{P}_f(z_1, z_2, \alpha_1) \Delta C \bar{P}_f(z_3, z_2, \alpha_1) = 1 \Delta C 1 = 1$$

For every $z_2 \in A$. So $z_1 + z_3 \in A^{-1} P_f$

Also To prove that $rz_1 \in A^{-1} P_f$, the following cases are considered:

Case-a if $r > 0$, $\alpha_1 > 0$ and $z_1 \in A^{-1} P_f$

since $\bar{P}_f(rz_1, z_2, \alpha_1) = 1$ for all $z_2 \in A$. It is clear that $\bar{P}_f(rz_1, z_2, 0) = 0$ for all $z_2 \in A$. Thus $rz_1 \in A^{-1} P_f$

Therefore, $A^{-1} P_f$ is a FLM.

The Relation between FRPHS and FRNS

This result explains the relationship between fuzzy real Pre-Hilbert space and fuzzy real normed space in this section.

Theorem 3 Every FRPHS is a FRNS

Proof: Let $(V, \bar{P}_f, \Delta C)$ be a FRPHS. Define $\bar{N}_f(z_1, \alpha_1) = \bar{P}_f(z_1, z_1, \alpha_1^2)$ for each $z_1 \in V$, $\alpha_1 > 0$ and $\bar{N}_f(z_1, 0) = 0$. Let $z_1, z_2 \in V$, $\alpha_1 > 0$

(RN1) $\bar{N}_f(z_1, \alpha_1) = \bar{P}_f(z_1, z_1, \alpha_1^2) > 0$ since $\bar{P}_f(z_1, z_1, \alpha_1) > 0$

(RN2) $\bar{N}_f(z_1, \alpha_1) = 1$ if and only if $\bar{P}_f(z_1, z_1, \alpha_1^2) = 1$ if and only if $\bar{P}_f(z_1, z_1, \alpha_1) = 1$ if and only if $z_1 = 0$

(RN3) $\forall r \in R$, $\bar{N}_f(rz_1, \alpha_1) = \bar{P}_f(rz_1, rz_1, \alpha_1^2) = \bar{P}_f(z_1, z_1, \alpha_1^2) = \bar{N}_f(z_1, \alpha_1^2)$

(RN4) $\bar{N}_f(z_1 + z_2, \alpha_1) = \bar{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1) = \bar{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1^2) \leq \bar{P}_f(z_1, z_1 + z_2, z_1 + z_2, \alpha_1^2) = \bar{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1^2) = \Delta C \bar{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1^2)$

Therefore, $A^{-1} P_f$ is a FLM.
\[ \geq \tilde{P}_f(z_1, z_1, \alpha z_1^2)\Delta C \tilde{P}_f(z_2, z_1, \alpha z_1^2) \Delta C \tilde{P}_f(z_2, z_2, \alpha z_2^2) \]

Theorem (2-2)-6)

and Theorem (4-5) give

\[ \tilde{P}_f(z_1, z_2, \alpha z_1^2) \leq \tilde{P}_f(z_1, z_2, \alpha z_2^2). \]

Hence

\[ N_f(z_1 + z_2, \alpha z_1) \leq \tilde{P}_f(z_1, z_1, \alpha z_1^2)\Delta C \tilde{P}_f(z_2, z_2, \alpha z_2^2) = \tilde{N}_f(z_1, \alpha z_1)\Delta C \tilde{N}_f(z_2, \alpha z_1) \]

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- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Technology.

References:

فضاء بري هلبرت الحقيقي الضبابي وبعض خواصه

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الخلاصة:
في هذا العمل، تم اقتراح هيكلين مختلفين هما فضاء القياس الحقيقي الضبابي وفضاء بري هلبرت الحقيقي الضبابي. يتم تقديم المفهوم الأساسي للقياس الضبابي حول الفضاء الحطي الحقيقي أولًا لبناء الفضاء 
$(V, \overline{N}, \Delta_c)$ وهو عبرية عن فضاء القياس الحقيقي الضبابي مع بعض التعديلات على التعريف المقدم من قبل الباحثين رانمو وباك ثم تم عرض هيكليا فضاء بري هلبرت الحقيقي الضبابي وهو يعتمد على فضاء القياس الحقيقي الضبابي. ثم، ناقشا بعض الخصائص والمفاهيم ذات الصلة لفضاء القياس الحقيقي الضبابي المترابطة مثل جورج ($\text{GCL}$) وناقشنا الشروط الضرورية للفصل والمجموعة الجزئية الخاصة. أيضا تم تقديم تعريف شبه الفضاء $V/A$ وتقديم هيكلية فضاء بري هلبرت الحقيقي الضبابي وتم عرض تطبيقات ودراسة العلاقة بين مجموعتين $\text{GCL}$ ($\text{A}$) وفضاء القياس الحقيقي الضبابي مع $\text{V/A}$ في هذا العمل.

الكلمات المفتاحية: الضرب الداخلي الضبابي، القياس الضبابي، فضاء هلبرت الحقيقي الضبابي، فضاء القياس الحقيقي الضبابي، التعامد الضبابي.