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Fuzzy Real Pre-Hilbert Space and Some of Their Properties

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Abstract:

In this work, two different structures are proposed which is fuzzy real normed space (FRNS) and fuzzy real Pre-Hilbert space (FRPHS). The basic concept of fuzzy norm on a real linear space is first presented to construct $(V, \widetilde{N}_f, \Delta_c)$ space, which is a FRNS with some modification of the definition introduced by G. Rano and T. Bag. The structure of fuzzy real Pre-Hilbert space (FRPHS) is then presented which is based on the structure of FRNS. Then, some of the properties and related concepts for the suggested space FRN such as ʒ-neighborhood, closure of the set A named $GCL(A)$, the necessary condition for separable, fuzzy linear manifold (FLM) are discussed. The definition for a fuzzy seminorm on V/A is also introduced with the prove that a fuzzy seminorm on V/A is FRNS. The relationship between the $\breve{l}$-convergent sequence, $\breve{l}$-Cauchy sequence and $\breve{l}$-completeness is then investigated in this work. The structure of FRPHS with some important properties concerning on this space are introduced and proved. In addition, the property of orthogonality with some important properties for these spaces is included, for example the annihilator of the set A. The relation between FRPHS and FRNS is investigated in the present work. Finally, after introducing the structure of FRPHS, it leads naturally to the definition of the most important class of FRPHS, namely the fuzzy real Hilbert space (FRHS).

Keywords: Fuzzy Inner Product (FIP), Fuzzy Norm (FN), Fuzzy Real Hilbert Space, Fuzzy Real Normed Space, Fuzzy Orthogonality.

Introduction:

The concept of fuzzy set is a generalization of classical set theory which is first introduced by zadeh [1].Triangular conorms are an indispensable tool for the interpretation of the conjunction in fuzzy sets. They are binary operations on the closed unit interval $[0, 1]$ with neutral element 0 which is very interesting to introduce a special class of real monotone operations**.** There are many applications of fuzzy set in different areas such as multicriterion decision making [2, 3], rough sets theory [4, 5], sociology [6], etc. Numerous applications have emerged from fuzzy sets theory. This approach may potentially improve some recent results in chaos theory application, e.g., designing chaotic sensors, see [7]. Also, applications of fuzzy set theory may be considered within the actual scope of neuroscience like in [8]. First a parameter some families of continuous triangular conorm is recalled. This idea allows us to generalize the intersection and the union in a Lattice, or disjunction and conjunction in Logic. An interesting concept of fuzzy norm on a linear space has been effectively introduced by G. Rano and T. Bag [9]. A new modification of the structures fuzzy real normed space **FRNS** and fuzzy real Pre-Hilbert space **FRPHS** which can be considered as a generalization of real normed space and real Pre-Hilbert space are introduced to find fuzzy analogies of classical mathematics and help us to develop more results of functional analysis in fuzzy setting. That's why creating a suitable structure of **FRNS** and **FRPHS** which is flexible enough and study some basic properties of these spaces. Felbin [10] introduced the fuzzy norm on a linear space by using fuzzy numbers. Cheng and Mordeson [11] introduced another idea of fuzzy norm on a linear space, and in 2003 Bag and Samanta [12] defined a more suitable notion of fuzzy norm. In [13] the idea of fuzzy antinorm was introduced. On the basis of this idea, Jebril and Samanta [14] introduced the concept of fuzzy antinorm on a linear space, first applied in investigation of probabilistic metric spaces [15]. Following their concept, R. Biswas [16] and

A. M. El-Abyed & H. M. Hamouly [17] first tried to give a structure of fuzzy Pre-Hilbert space and associated fuzzy norm function. Later on many author viz. Mazumdar & Samanta [18], Hasankhani, Nazari & Saheli [19], Mukherjee & Bag [20] have introduced the structure of fuzzy Pre-Hilbert space from different points and studied a few properties and some of applications. Induced norms of these spaces may have very important applications in quantum particle physics for more details; see [21, 22]. In recent past lots (more advanced reader), numerous references to the application of these spaces have been done in this direction on fuzzy functional analysis [23-26].

Structurally, the paper comprises the following: after this introductory section, In Section 2 gives basic definitions and preliminary results which are used in the sequel. In Section 3 a structure of fuzzy norm defined on a real linear space of type $(V,\widetilde{N}_f,\Delta_c)$ as well as different properties and ideas related to this structure are presented. Section 4 devotes a structure of fuzzy inner product defined on a real linear space of type $\left(V,\tilde{P}_f,\,\Delta_c\right)$, and studies some results for this space. The implications between the given modifications structures is discussed in Section 5.

Background and Preliminaries

Some important definitions and preliminary results are presented which will be utilized in the next sections.

Definition 1 [22] A triangular conorm is a binary operation Δ_c on the interval $\mathrm{I}=[0,1]$ which is satisfying the following conditions, for all $\alpha_1,\alpha_2,\alpha_3\in[0,1]$

(H1) $\alpha_1\,\Delta_c\alpha_2=\alpha_2,\,\Delta_c\,\alpha_1$ (Commutative)

(H2) $(\alpha_1\,\Delta_c(\alpha_2\,\Delta_c\alpha_3))=((\alpha_1\,\Delta_c\,\alpha_2)\,\Delta_c\,\alpha_3)$ (Associative)

(H3) $\alpha_1\,{}_{\Delta_c}\,\alpha_2\le\alpha_1\,\Delta_c\alpha_3\;whenever\;,\alpha_2\;\le\;\alpha_3$ (Monotone)

(H4) $\alpha_1\,\Delta_c 0\;=\;\alpha_1$ (has 0 as neutral element)

If, in addition, H is continuous then H is called a continuous triangular conorm.

The following theorem introduces the characteristics of a triangular conorm

Theorem 1 [22] Let Δ_c be a triangular conorm on the set I. Then

(1) $0\;\Delta_c 0\;=\;0$

(2) $1\,\Delta_c\,0\;=\;1$

(3) $0\,\Delta c\,1\;=\;1$

(4) $1\,\Delta_c\,1\;=\;1$

(5) If $\alpha_1\le\alpha_2\,,\alpha_3\le\alpha_4$, then $\alpha_1\,\Delta_c\,\alpha_3\le\alpha_2\,\Delta_c\,\alpha_4$ for all $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in[0,1]$

(6) $\alpha_1\,\Delta_c\,\alpha_1\ge\alpha_1$ for all $\alpha_1\in[0,1]$

Example 1 [5]

(1) let $\alpha_1\,\Delta_c\;\;\alpha_2=(\alpha_1+\alpha_2-\alpha_1\alpha_2\;\;)$ for all $\alpha_1,\alpha_2\in[0,1]$ where$(\alpha_1+\alpha_2-\alpha_1\alpha_2\;\;)$ is a triangular conorm, and called the probabilistic sum

(2) let $\alpha_1\,\Delta_c\,\alpha_2=\min(\,\alpha_1+\alpha_2,1)$ for all $\alpha_1,\alpha_2\in[0,1]$ where $\min(\,\alpha_1+\alpha_2,1)$ is a triangular conorm, and named the bounded sum

Remark 1 [5] For any γ it can be find δ such that $\delta\,\Delta_c\,\delta\ge\gamma$, where $\gamma,\delta\in(0,1)$

Fuzzy Real Normed Space (FRNS) and Main Results

A real linear space is considered and a structure of fuzzy (real) normed space is introduced with a new characterization triangular conorm. In addition, some properties and subsequent results of a fuzzy real normed space are introduced and proved in this section.

Definition 2 Let Δ_c is a triangular conorm, V be a linear space over the field **R** .Let a fuzzy subset $\widetilde{N}_f:V\times[0,\infty)$ into I is a mapping called fuzzy norm on V if its following conditions are holds for all $z_1,z_2\in V$, and for all $\alpha_1\in I$:

(RN1) $\widetilde{N}_f(z_1,\alpha_1)\;>0$, for all $\alpha_1>0$

(RN2) $\widetilde{N}_f(z_1,\alpha_1)\;=1$ if and only if $z_1=o$, for all $\alpha_1>0$

(RN3)$\forall\;\boldsymbol{r}\neq\boldsymbol{0}\in\boldsymbol{R}\;,\;\widetilde{N}_f(r\,z_1,\alpha_1)=\widetilde{N}_f\left(z_1,\frac{\alpha_1}{|r|}\right)$

(RN4) $\widetilde{N}_f(z_1+z_2,\alpha_1)\le\widetilde{N}_f(z_1,\alpha_1)\,\Delta_c\,\widetilde{N}_f(z_2,\alpha_1)$

(RN5) $\widetilde{N}_f(z_1,.):[0,1]\to[0,1]$ is continuous with respect to α_1

Then $(V,\widetilde{N}_f,\Delta_c)$ is called fuzzy real normed space **(FRNS)**.

In some theorems, the following condition is considered.

Lemma 1 Suppose that $\left(V,\,\widetilde{N}_f,\,\Delta_c\right)$ is a fuzzy real normed space.Then

$\widetilde{N}_f(z_1-z_2,\alpha_1)=\;\widetilde{N}_f(z_2-z_1,\alpha_1)$ for all $z_1,z_2\in V$ and $\alpha_1>0$

The following example demonstrates the concept of a fuzzy real normed space which is necessary in this process.

Example 2 Let $(R^2,||.||)$ be a normed space, where $V=R^2$ is a linear space which is obtained if the set of ordered pairs of real numbers $z_1=(\rho_1,\rho_2)\in\;R^2$ is taken with a function $\|z_1\|=(|\rho_1|^2+|\rho_2|^2)^{\frac{1}{2}}$. Define $\widetilde{N}_f(z_1,\alpha_1)=|\|z_1\|-\alpha_1|/\alpha_1+\|z_1\|$ for $\alpha_1<\|z_1\|$ and $\widetilde{N}_f(z_1,\alpha_1)=0$ for $\alpha_1\ge\|z_1\|$. Also $a\,\Delta_c b=a+b-ab$ for all $a,b\in I$.Then $(V,\widetilde{N}_f,\,\Delta_c)$ is **FRNS**

Proof:

(RN1) Since $\|z_1\|>0,\forall z_1=(\rho_1,\rho_2)\in\;R^2$ and for all $\alpha_1>0$ so $\widetilde{N}_f(z_1,\alpha_1)>0$

(RN2) $\widetilde{N}_f(z_1,\alpha_1)=1$ if and only if $\frac{|\|z_1\|-\alpha_1|}{\alpha_1+\|z_1\|}=1$ if and only if $\|z_1\|=0$ if and only if $z_1=0$

(RN3) To verify $\widetilde{N}_f(r\,z_1,\alpha_1)=\widetilde{N}_f\left(z_1,\frac{\alpha_1}{|r|}\right)$ for all $z_1\in R^2$, $\alpha_1>0$ and $\boldsymbol{r}\neq\boldsymbol{0}\in\boldsymbol{R}$

$$\widetilde{N}_f(r\,z_1,\alpha_1)=\frac{|\|rz_1\|-\alpha_1|}{\alpha_1+\|rz_1\|}=\frac{\left||r|\|z_1\|-\alpha_1\right|}{\alpha_1+|r|\|z_1\|}$$
$$=\frac{\left|\|z_1\|-\frac{\alpha_1}{|r|}\right|}{\frac{\alpha_1}{|r|}+\|z_1\|}=\widetilde{N}_f\left(z_1,\frac{\alpha_1}{|r|}\right)$$

(RN4) let $z_1,z_2\in V$ and $\alpha_1\in I$.If $\alpha_1\geq\|z_1\|$ implies $\widetilde{N}_f(z_1,\alpha_1)=0$ also when $\alpha_1\geq\|z_2\|$ implies $\widetilde{N}_f(z_2,\alpha_1)=0$. So $\widetilde{N}_f(z_1,\alpha_1)\,\varDelta_c\,\widetilde{N}_f(z_2,\alpha_1)=0$. On the other hand, it was concluded $\alpha_1\geq\|z_1\|+\|z_2\|\geq\|z_1+z_2\|$ which implies that $\widetilde{N}_f(z_1+z_2,\alpha_1)=0$ $=\widetilde{N}_f(z_1,\alpha_1)\,\varDelta_c\,\widetilde{N}_f(z_2,\alpha_1)$

Now, If $\|z_1\|>\alpha_1$ and $\|z_2\|>\alpha_1$ which are implies $\widetilde{N}_f(z_1+z_2,\alpha_1)=\frac{\|z_1+z_2\|-\alpha_1}{\alpha_1+\|z_1+z_2\|}$

$$\leq\frac{\|z_1\|+\|z_2\|-\alpha_1}{\alpha_1+\|z_1\|+\|z_2\|}$$

$$\leq\frac{\|z_1\|-\alpha_1}{\alpha_1+\|z_1\|}+\frac{\|z_2\|-\alpha_1}{\alpha_1+\|z_2\|}-\frac{\|z_1\|-\alpha_1}{\alpha_1+\|z_1\|}\,\frac{\|z_2\|-\alpha_1}{\alpha_1+\|z_2\|}$$

$$\leq\widetilde{N}_f(z_1,\alpha_1)\,\varDelta_c\,\widetilde{N}_f(z_2,\alpha_1)$$

(RN5) It is clear that $\widetilde{N}_f(z_1,.)$ is continuous with respect to α_1

Therefore, $(V,\widetilde{N}_f,\varDelta_c)$ is **FRNS** on V.

Definition 3 Let $(V,\widetilde{N}_f,\varDelta_c)$ be a **FRNS** on V. Define a $\mathfrak{z}$-neighborhood of u by $B(z_1,\mathfrak{z},\alpha_1)$=$\{z_2\in V,\widetilde{N}_f\,(z_1-z_2,\alpha_1)>1-\mathfrak{z}\}$ with center $z_1\in V$, $r$adius $\mathfrak{z}$, $\alpha_1>0$ where $\mathfrak{z}\in(0,1)$

Definition 4 Let $(V,\widetilde{N}_f,\varDelta_c)$ be a **FRNS** on V. The set A of V is called a neighborhood of z_1 if it contains a $\mathfrak{z}$-neighborhood of z_1.

Definition 5 Let $(V,\widetilde{N}_f,\varDelta_c)$ be **FRNS** on V. A vector z_1 is called an interior vector of a set A in V if A is a neighborhood of z_1.

Definition 6 Let $(V,\widetilde{N}_f,\varDelta_c)$ be **FRNS** on V.The set A of V is called open if for any point $z_1\in V$, there exists an $\mathfrak{z}\in(0,1)$ such that the sphere with center z_1 and $r$adius $\mathfrak{z}$, $B(z_1,\mathfrak{z},\alpha_1)$ =$\{z_2\in V,\widetilde{N}_f\,(z_1-z_2,\alpha_1)=1-\mathfrak{z}\}$ is contained entirely in A. The set A of V in a **FRNS** on V is called closed if A^c is open.

Definition 7 Let $(V,\widetilde{N}_f,\varDelta_c)$ be **FRNS** on V.The set A of V is called closure of A and denoted by $GCL(A)$ if it contains the smallest closed set.

Definition 8 Let $(V,\widetilde{N}_f,\varDelta_c)$ be **FRNS** on V.The set A of V is said to be a dense in V if it $GCL(A)$ is equal to V.

Definition 9 A **FRNS** $(V,\widetilde{N}_f,\varDelta_c)$ on V is separable if it contains a dense subset that is countable.

Proposition 1 Let $(V,\widetilde{N}_f,\varDelta_c)$ be **FRNS** on V.Then $\breve{l}=\{A:A$ is the subset of V,z_1 is an interior vector of A if and only if there exists a neighborhood of z_1 contained in $A\}$.

The following definition describes the behavior of $\breve{l}$-convergent sequence in a fuzzy real normed space.

Definition 10 A sequence $\{z_n\}$ in a **FRNS** $(V,\widetilde{N}_f,\varDelta_c)$ is said to be $\breve{l}$-convergent if there exists a vector z in V such that $\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z_n}-\mathrm{z},\alpha_1)\to1$ as $\mathrm{n}\to\infty$, for all $\alpha_1>0$.

Lemma 2 let $(V,\widetilde{N}_f,\varDelta_c)$ be a **FRNS** and $\{z_n\}$ be an $\breve{l}$-convergent sequence in V.Then the limit of $\{z_n\}$ is unique

Proof:

Consider $\{z_n\}$ be a sequence in **FRNS** which is $\breve{l}$-convergent to z and $\acute{z}$. Then by (RN4)

$$\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z}+\acute{z},\alpha_1)=\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z}-\mathrm{z_n}+\mathrm{z_n}+\acute{z},\alpha_1)\text{ for all }\alpha_1>0$$

$$\leq\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z}-\mathrm{z_n},\alpha_1)\,\varDelta_c\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z_n}-(-\acute{z}),\alpha_1)$$

$$=\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z_n}-\mathrm{Z},\alpha_1)\,\varDelta_c\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z_n}-(-\acute{z}),\alpha_1)\text{ by lemma (3-2)}$$

$$\to1\,\varDelta_c1\to1\text{ by theorem ((2-2)-4)}$$

Thus, it follows that $\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z}+\acute{z},\alpha_1)=1$. From (RN2) $\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z}+\acute{z},\alpha_1)=1$ if and only if $\mathrm{z}+\acute{z}=0$, which implies that $\mathrm{z}=-\acute{z}$. Therefore the limit is unique.

Definition 11 A sequence $\{z_n\}$ in a **FRNS** $(V,\widetilde{N}_f,\varDelta_c)$ is said $\breve{l}$-Cauchy sequence if $\lim\widetilde{\mathrm{N}}_\mathrm{f}(\mathrm{z_n}-\mathrm{z_{n+s}}\,\alpha_1)=1$ as $\mathrm{n}\to\infty,\mathrm{s}>0$, for all $\alpha_1>0$.

Definition 12 A **FRNS** $(V,\widetilde{N}_f,\varDelta_c)$ is said to be $\breve{l}$-complete if every $\breve{l}$-Cauchy sequence in V is $\breve{l}$-convergent.

Definition 13 A set $\hat{A}$ in **FRNS** on V is called a fuzzy linear manifold $\hat{A}$ of $(V,\widetilde{N}_f,\varDelta_c)$ if $\hat{A}$ is a linear manifold of V considered as a vector space, with the fuzzy norm obtained by restricting the fuzzy norm on V to the set $\hat{A}$

Definition 14 A set $\hat{A}$ is called closed fuzzy linear manifold of $(V,\widetilde{N}_f,\varDelta_c)$ if $\hat{A}$ is closed in V

The following theorem describes the behavior of a fuzzy seminorm on V/A in a fuzzy real normed space**.**

Theorem 2 If $(V, \widetilde{N}_f, \Delta_c)$ is **FRNS** on V with a closed linear manifold A and let
$\widetilde{N}_f([z], \alpha_1) = \inf\{\widetilde{N}_f(z - m, \alpha_1), m \in A, \alpha_1 > 0\}$ for all $[z] \in V/A$
This defines a fuzzy seminorm on V/A, then this fuzzy seminorm is a fuzzy a norm on V/A.

Proof: Let $z \in V$ .Clearly $\widetilde{N}_f([z], \alpha_1) = \widetilde{N}_f(z - A, \alpha_1) > 0$. Suppose that $\widetilde{N}_f([z], \alpha_1) = \widetilde{N}_f(z - A, \alpha_1) = 1$, from the definition (3-12) there is a sequence $\{z_n\}$ in A such that $\widetilde{N}_f(z_n - z, \alpha_1) \to 1$ and since A is closed hence $z \in A$ and $0 + A = z + A = A$.

If $r \neq 0 \in \boldsymbol{R}, z \in V$ then

$$\widetilde{N}_f(r[z], \alpha_1) = \widetilde{N}_f(r(z - A), \alpha_1) = \widetilde{N}_f(rz - A, \alpha_1)$$
$$= \inf\{\widetilde{N}_f(rz - rz_1, \alpha_1), z_1 \in A\}$$
$$= \inf\left\{\widetilde{N}_f\left(z - z_1, \frac{\alpha_1}{|r|}\right), z_1 \in A\right\}$$
$$= \widetilde{N}_f\left([z], \frac{\alpha_1}{|r|}\right)$$

Also it is known that

$$\widetilde{N}_f([z] + [z_1], \alpha_1) = \widetilde{N}_f([z + z_1], \alpha_1) \leq \widetilde{N}_f(z + z_1 - (a + a_1), \alpha_1)$$
$$\leq \widetilde{N}_f\big((z - a) + (z_1 - a), \alpha_1\big)$$
$$\leq \widetilde{N}_f((z - a), \alpha_1)\, \Delta_c \widetilde{N}_f((z_1 - a), \alpha_1)$$

for all $a, a_1 \in A$. Taking the infimum over such a and a_1 gives the inequality

$$\widetilde{N}_f([z] + [z_1], \alpha_1) \leq \widetilde{N}_f([z], \alpha_1)\, \Delta_c \widetilde{N}_f([z_1], \alpha_1)$$

Finally, it is clear that $\widetilde{N}_f([z], .)$ from (0,1] into I continuous function

The proof is complete

Fuzzy Real Pre-Hilbert Space (FRPHS) and Main Results

The main general interesting results which hold in any fuzzy real Pre-Hilbert space will be proved. So the structure of fuzzy real Pre-Hilbert space is introduced initially.

Definition 15 *Let* Δ_c be a triangular conorm, V be a linear space over the field **R**, then $\tilde{P}_f : V \times V \times [0, \infty)$ into I is called a fuzzy real inner product on V,if for all $z_1, z_2, z_3 \in V, \alpha_1 \in I$ the following conditions are holds:

(RPH1) $\tilde{P}_f(z_1, z_2, \alpha_1)\, \Delta_c\, \tilde{P}_f(z_3, z_2, \alpha_1) \geq \tilde{P}_f(z_1 + z_3, z_2, \alpha_1)$

(RPH2) $\tilde{P}_f(z_1, z_2, \alpha_1) = \tilde{P}_f(z_2, z_1, \alpha_1)$

(RPH3) $\forall$ r $\in \boldsymbol{R}$

$$\tilde{P}_f(r\, z_1, z_2, \alpha_1) = \begin{cases} \tilde{P}_f\left(z_1, z_2, \dfrac{\alpha_1}{r}\right) & if\ r > 0 \\ 1 - \tilde{P}_f\left(z_1, z_2, \dfrac{\alpha_1}{r}\right) & if\ r < 0 \\ 1\ if\ r = 0\ and\ \alpha_1 > 0\ ,\ \ 0\ \ if\ r = 0, \alpha_1 = 0 & \end{cases}$$

(RPH4) a- for $\alpha_1 = 0$ then $\tilde{P}_f(z_1, z_1, \alpha_1) = 0$
b- for $\alpha_1 \neq 0$ then $\tilde{P}_f(z_1, z_1, \alpha_1) > 0$

(RPH5) $\tilde{P}_f(z_1, z_1, \alpha_1)\, \Delta_c \tilde{P}_f(z_2, z_2, \alpha_1) \geq \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1)$

(RPH6) $\tilde{P}_f(z_1, z_1, \alpha_1) = 1$ for all $\alpha_1 > 0$ if and only if $z_1 = 0$

(RPH7) $\tilde{P}_f(z_1, z_1, \alpha_1): [0,1] \to [0,1]$ is continuous with respect to α_1

Then $\big(V, \tilde{P}_f, \Delta_c\big)$ is called a fuzzy real Pre-Hilbert space (**FRPHS)**.

According to the previous definition, the following illustrate example is proved.

Example 3 Space $L^2[a, b]$.The vector space of all continuous real-valued functions on $[a, b]$forms pre-Hilbert space $(V, <.,.>)$ with inner product defined by $< z_1, z_2 >= \int_a^b z_1(x) z_2(x) dx$. Define $\tilde{P}_f(z_1, z_2, \alpha_1) = \frac{1}{\exp\left(\frac{<z_1, z_2>}{\alpha_1}\right)}$for all $z_1, z_2, z_3 \in V, \alpha_1 > 0$ and $\tilde{P}_f\,(z_1, z_2, \alpha_1) = 0$ for α_1=0. Then $(V, \tilde{P}_f, \Delta_c\ )$ is a **FRPHS**. Where $\alpha_1 \Delta_c\ \alpha_2 = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$ for all $\alpha_1,\ \alpha_2 \in I$.

Proof:

(RPH1) $\tilde{P}_f\,(z_1, z_2, \alpha_1) \Delta_c\ \tilde{P}_f\ \ (z_3, z_2, \alpha_1) = \tilde{P}_f(z_1, z_2, \alpha_1) + \tilde{P}_f(z_3, z_2, \alpha_1) - \tilde{P}_f(z_1, z_2, \alpha_1)\, \tilde{P}_f\,(z_3, z_2, \alpha_1)$

$$= \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha_1}\right)} + \frac{1}{\exp\left(\frac{<z_3,z_2>}{\alpha_1}\right)} - \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha_1}\right)} \cdot \frac{1}{\exp\left(\frac{<z_3,z_2>}{\alpha_1}\right)}$$

$$= \frac{\exp\left(\frac{<z_3,z_2>}{\alpha_1}\right) + \exp\left(\frac{<z_1,z_2>}{\alpha_1}\right)}{\exp\left(\frac{<z_1,z_2>}{\alpha_1} + \frac{<z_3,z_2>}{\alpha_1}\right)} - \frac{1}{\exp\left(\frac{<z_1+z_3,z_2>}{\alpha_1}\right)}$$

$$= \frac{\exp\left(\frac{<z_3,z_2>}{\alpha_1}\right) + \exp\left(\frac{<z_1,z_2>}{\alpha_1}\right) - 1}{\exp\left(\frac{<z_1+z_3,z_2>}{\alpha_1}\right)}$$

$$= \exp\left(\frac{<z_1,z_2>}{\alpha_1}\right) + \exp\left(\frac{<z_3,z_2>}{\alpha_1}\right) - \exp\frac{-<z_1+z_3,z_2>}{\alpha_1} \geq \exp\frac{-<z_1+z_3,z_2>}{\alpha_1}$$

$$= \frac{1}{\exp\left(\frac{<z_1+z_3,z_2>}{\alpha_1}\right)} = \tilde{P}_f(z_1 + z_3, z_2, \alpha_1)$$

(RPH2) $\tilde{P}_f(z_1, z_2, \alpha_1)\ \ \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha_1}\right)} = \frac{1}{\exp\left(\frac{<z_2,z_1>}{\alpha_1}\right)} = \tilde{P}_f(z_2, z_1, \alpha_1)$

(RPH3) $\forall\, r \in$ ***R*,** the following cases are considerd

a- If r > 0 , then $\tilde{P}_f(rz_1, z_2, \alpha_1) = \frac{1}{\exp\left(\frac{<rz_1,z_2>}{\alpha_1}\right)} = \frac{1}{\exp\left(\frac{r<z_1,z_2>}{\alpha_1}\right)} = \frac{1}{\exp\left(\frac{<z_1,z_2>}{\frac{\alpha_1}{r}}\right)} = \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right)$

b- If $r < 0$, then $\tilde{P}_f(rz_1, z_2, \alpha_1) = \frac{1}{\exp\left(\frac{<(-r)z_1,z_2>}{\alpha_1}\right)} = \frac{1}{\exp\left(\frac{(-r)<z_1,z_2>}{\alpha_1}\right)} = \frac{1}{\exp\left(\frac{<z_1,z_2>}{\frac{\alpha_1}{-r}}\right)} =$
$1 - \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right)$
c- If $r = 0$, then $\tilde{P}_f(rz_1, z_2, \alpha_1) = 1$ for $\alpha_1 > 0$ and $\tilde{P}_f(rz_1, z_2, \alpha_1) = 0$ for $\alpha_1 > 0$
(RPH4) a-for $\alpha_1 = 0$ it is clear that $\tilde{P}_f(z_1, z_2, \alpha_1) = 0$
b- for $\alpha_1 \neq 0$, it is clear that $\tilde{P}_f(z_1, z_2, \alpha_1) = \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha_1}\right)} > 0$
(RPH5) $\tilde{P}_f(z_1, z_1, \alpha_1)\, \Delta_c \tilde{P}_f(z_2, z_2, \alpha_1) = \tilde{P}_f(z_1, z_1, \alpha_1) + \tilde{P}_f(z_2, z_2, \alpha_1) - \tilde{P}_f(z_1, z_1, \alpha_1)\, \tilde{P}_f(z_2, z_2, \alpha_1)$
$= \frac{1}{\exp\left(\frac{<z_1,z_1>}{\alpha_1}\right)} + \frac{1}{\exp\left(\frac{<z_2,z_2>}{\alpha_1}\right)} - \frac{1}{\exp\left(\frac{<z_1,z_1>}{\alpha_1}\right)} \frac{1}{\exp\left(\frac{<z_2,z_2>}{\alpha_1}\right)}$
$= \frac{1}{\exp\left(\frac{<z_1,z_1>}{\alpha_1}\right)} + \frac{1}{\exp\left(\frac{<z_2,z_2>}{\alpha_1}\right)} - \frac{1}{\exp\left(\frac{<z_1,z_1>+<z_2,z_2>}{\alpha_1}\right)}$
$\tilde{P}_f(z_1, z_1, \alpha_1)\, \Delta_c \tilde{P}_f(z_2, z_2, \alpha_1) = \frac{\exp\left(\frac{<z_2,z_2>}{\alpha_1}\right) + \exp\left(\frac{<z_1,z_1>}{\alpha_1}\right) - 1}{\exp\left(\frac{<z_1,z_1>+<z_2,z_2>}{\alpha_1}\right)} \geq \frac{1}{\exp\left(\frac{<z_1+z_2,z_1+z_2>}{\alpha_1}\right)} = \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1)$
For all $z_1, z_2 \in V$, $\alpha_1 > 0$
(RPH6) suppose $\tilde{P}_f(z_1, z_1, \alpha_1) = 1\ \forall \alpha_1 > 0$,
$\tilde{P}_f(z_1, z_1, \alpha_1) = \frac{1}{\exp\left(\frac{<z_1,z_1>}{\alpha_1}\right)} = 1$ if and only if $\exp\left(\frac{<z_1,z_1>}{\alpha_1}\right) = 1$ if and only if $< z_1, z_1 > = 0$ if and only if $z_1 = 0$
(RPH7) Let α_n be a sequence in [0,1] such that $\alpha_n \to \alpha$.Now for every $z_1 \in V$, $\lim_{n\to\infty} \tilde{P}_f(z_1, z_2, \alpha_n) = \lim_{n\to\infty} \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha_n}\right)} =$
$\frac{1}{\exp\left(\frac{\lim_{n\to\infty} <z_1,z_2>}{\lim_{n\to\infty} \alpha_n}\right)} = \frac{1}{\exp\left(\frac{<z_1,z_2>}{\alpha}\right)} = \tilde{P}_f(z_1, z_2, \alpha)$.
So $\tilde{P}_f(z_1, z_2, \alpha_n) \to \tilde{P}_f(z_1, z_2, \alpha)$.Hence $\tilde{N}_f(z_1, .): [0,1] \to [0,1]$ is continuous with respect to α.
Therefore, $(V, \tilde{P}_f, \Delta_c)$ is **FRPHS** on V.

Lemma 3 Assume that $(V, \tilde{P}_f, \Delta_c)$ is a **FRPHS** then $\tilde{P}_f(-z_1, -z_1, \alpha_1) = \tilde{P}_f(z_1, z_1, \alpha_1)$ for all $z_1 \in V$, $\alpha_1 > 0$

Proof: To prove the equality, let
$\tilde{P}_f(-z_1, -z_1, \alpha_1) = 1 - \tilde{P}_f(z_1, -z_1, \alpha_1) = 1 - \left[1 - \tilde{P}_f(z_1, z_1, \alpha_1)\right] = \tilde{P}_f(z_1, z_1, \alpha_1)$ by (RPH3)

Remark 2 If $(V, \tilde{P}_f, \Delta_c)$ is a **FRPHS**, then for all $z_1, z_2, z_3 \in V$, the following properties are satisfied
(i) if $r > 0, s > 0,\ \alpha_1 > 0$, then
$$\tilde{P}_f(rz_1 + sz_3, z_2, \alpha_1) \leq \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right) \Delta_c \tilde{P}_f\left(z_3, z_2, \frac{\alpha_1}{s}\right)$$
(ii) if $r > 0, s = 0,\ \alpha_1 > 0$, then
$$\tilde{P}_f(rz_1 + sz_3, z_2, \alpha_1) \leq \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right) \Delta_c 1$$
(iii) if $r > 0, s < 0,\ \alpha_1 > 0$, then
$$\tilde{P}_f(rz_1 + sz_3, z_2, \alpha_1) \leq \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right) \Delta_c \left(1 - \tilde{P}_f\left(z_3, z_2, \frac{\alpha_1}{s}\right)\right)$$
(iv) if $r < 0, s < 0,\ \alpha_1 > 0$,then
$$\tilde{P}_f(rz_1 + sz_3, z_2, \alpha_1) \leq \left(1 - \tilde{P}_f\left(z_1, z_2, \frac{\alpha_1}{r}\right)\right) \Delta_c \left(1 - \tilde{P}_f\left(z_3, z_2, \frac{\alpha_1}{s}\right)\right)$$

Proposition 2 Let $(V, \tilde{P}_f, \Delta_C)$ be a **FRPHS**, Δ_C is a continuous triangular conorm. Then
$\tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1) \leq \tilde{P}_f(z_1, 2z_1, \frac{\alpha_1}{2})\Delta_C \tilde{P}_f(z_2, 2z_2, \frac{\alpha_1}{2})$ for all $z_1, z_2 \in V$ and all $\alpha_1 > 0$

Proof: Assume that $A = \tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1)$ and
$A_1 = \tilde{P}_f(z_1, 2z_1, \frac{\alpha_1}{2})\Delta_C \tilde{P}_f(z_2, 2z_2, \frac{\alpha_1}{2})$
Since $\tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1) \leq \tilde{P}_f(z_1, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(-z_2, z_1 - z_2, \alpha_1)$
$\leq \tilde{P}_f(z_1, z_1, \alpha_1)$
$\Delta_C \tilde{P}_f(z_1, -z_2, \alpha_1)\Delta_C \tilde{P}_f(-z_2, z_1, \alpha_1)\Delta_C \tilde{P}_f(-z_2, -z_2, \alpha_1)$
By the condition (PRH2) and lemma (4-3):
$\tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1) \leq \tilde{P}_f(z_1, z_1, \alpha_1)$
$\Delta_C \tilde{P}_f(-z_2, z_1, \alpha_1)\Delta_C \tilde{P}_f(-z_2, z_1, \alpha_1)\Delta_C \tilde{P}_f(z_2, z_2, \alpha_1)$
Similarly,
$\tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1) \leq \tilde{P}_f(z_1, z_1, \alpha_1)$

$\Delta_C \tilde{P}_f(z_2, z_1, \alpha_1)\Delta_C \tilde{P}_f(z_2, z_1, \alpha_1)\Delta_C \tilde{P}_f(z_2, z_2, \alpha_1)$.
Thus by Theorem ((2-2)-6), the conditions (PRH3) and (PRH5) give the following result:
$$A \leq A\, \Delta_C A = \left[\tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1)\right] \Delta_C \left[\tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1)\right]$$

$\leq \tilde{P}_f(2z_1, 2z_1, \alpha_1)\Delta_C \tilde{P}_f(2z_2, 2z_2, \alpha_1)$
Since $r = 2 > 0$, then $\tilde{P}_f(2z_1, 2z_1, \alpha_1) = \tilde{P}_f\left(z_1, 2z_1, \frac{\alpha_1}{2}\right)$
Therefore, $A = \tilde{P}_f(z_1 - z_2, z_1 - z_2, \alpha_1)\Delta_C \tilde{P}_f(z_1 + z_2, z_1 + z_2, \alpha_1) \leq \tilde{P}_f\left(z_1, 2z_1, \frac{\alpha_1}{2}\right)\Delta_C \tilde{P}_f\left(z_2, 2z_2, \frac{\alpha_1}{2}\right) = A_1$
The proof is complete.

Definition 16 A **FLM** $\hat{A}$ of **FRPHS** V is defined to be a linear manifold of V taken with the fuzzy real inner product on V restricted to $\hat{A} \times \hat{A}$.
Similarly, A **FLM** $\hat{A}$ of **FRHS** V is defined to be a linear manifold of V, regarded as the fuzzy real inner product.
Remark 3 A **FLM** $\hat{A}$ need not to be **FRHS** because $\hat{A}$ may not be $\breve{l}$-complete.
The concept of fuzzy orthogonal in a **FRPHS** is introduced in the following definition:
Definition 17 An element $z_1 \in V$ in a **FRPHS** $(\mathrm{V}, \widetilde{\mathrm{P}}_{\mathrm{f}}, \Delta_{\mathrm{C}})$ is said to be fuzzy orthogonal to an element $z_2 \in V$ if $\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, \alpha_1) = 1$, for all $\alpha_1 > 0$ and $\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, 0) = 0$. Put z_1 and z_2 are orthogonal, this means $z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} z_2$. Similarly for two sets A, B in V if $z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} z_2$ then $z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} B$ for all $z_2 \in B$ and $A \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} B$ for all z_1 in A and z_2 in B.
Lemma 4 Let $(\mathrm{V}, \widetilde{\mathrm{P}}_{\mathrm{f}}, \Delta_{\mathrm{C}})$ be a **FRPHS**. If $z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} z_2$ then
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_2, z_1 + z_2, \alpha_1) \leq$
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_1, \alpha_1) \Delta_{\mathrm{C}}\, 1\, \Delta_{\mathrm{C}} \widetilde{\mathrm{P}}_{\mathrm{f}}(z_2, z_2, \alpha_1)$ for all $\alpha_1 > 0$.

Proof: $\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_2, z_1 + z_2, \alpha_1) \leq \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_1 + z_2, \alpha_1) \Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_2, z_1 + z_2, \alpha_1)$
$= \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_2, z_1, \alpha_1) \Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_2, z_2, \alpha_1)$
$\leq \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_1, \alpha_1)$
$\Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_2, z_1, \alpha_1) \Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, \alpha_1)\, \Delta_{\mathrm{C}} \widetilde{\mathrm{P}}_{\mathrm{f}}(z_2, z_2, \alpha_1)$
Since $z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} z_2$ in a **FRPHS**, hence (RPH2) and Theorem ((2-2)-4) give
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_2, z_1 + z_2, \alpha_1) \leq$
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_1, \alpha_1) \Delta_{\mathrm{C}} 1\, \Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_2, z_2, \alpha_1)$
Definition 18 Let $\hat{A}$ be **FLM** of **FRPS**, then the orthogonal complement of $\hat{A}$ is $\hat{A}^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}} = \{z_1 \in V \mid z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} \hat{A}\}$ which is the set of all vectors fuzzy orthogonal to $\hat{A}$. An orthogonal complement is a special annihilator of a set, which defined as follows:
Definition 19 Let $(\mathrm{V}, \widetilde{\mathrm{P}}_{\mathrm{f}}, \Delta_{\mathrm{C}})$ be a **FRPHS** and let A be a non-empty set in V then the annihilator of A is defined by $A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}} = \{z_1 \in V \mid z_1 \perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}} A\}$.Thus $z_1 \in A$ if and only if $\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, \alpha_1) = 1$, for all $\alpha_1 > 0$ and $\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, 0) = 0$, for all $z_2 \in A$.
Next, the orthogonal complement set in a fuzzy real Pre-Hilbert space implies a fuzzy linear manifold is proved.
Lemma 5 $A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$ is a **FLM** of **FRPHS**
Proof: To prove that $0 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$, the following cases are considered:
Case-a Since $\widetilde{\mathrm{P}}_{\mathrm{f}}(0, z_2, \alpha_1) = 1$ for all $\alpha_1 > 0, z_2 \in$ A
Case-b Since $\widetilde{\mathrm{P}}_{\mathrm{f}}(0, z_2, 0) = 0$ for all $z_1, z_2 \in$ A then $0 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$
Now, let $z_1, z_3 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$, $\alpha_1 > 0$ and $r \in \boldsymbol{R}$
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1 + z_3, z_2, \alpha_1) \leq$
$\widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, \alpha_1) \Delta_{\mathrm{C}}\, \widetilde{\mathrm{P}}_{\mathrm{f}}(z_3, z_2, \alpha_1)$
$= 1\, \Delta_{\mathrm{C}}\, 1$
$= 1$
For every $z_2 \in A$. So $z_1 + z_3 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$
Also To prove that $rz_1 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$, the following cases are considered:
Case-a if $r > 0, \alpha_1 > 0$ and $z_1 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$
since $\widetilde{\mathrm{P}}_{\mathrm{f}}(rz_1, z_2, \alpha_1) = \widetilde{\mathrm{P}}_{\mathrm{f}}(z_1, z_2, \frac{\alpha_1}{r})$ then $\widetilde{\mathrm{P}}_{\mathrm{f}}(rz_1, z_2, \alpha_1) = \widetilde{\mathrm{P}}_{\mathrm{f}}\left(z_1, z_2, \frac{\alpha_1}{r}\right) = 1$ for all $z_2 \in A$.
Case-b if $r < 0, \alpha_1 > 0$ and $z_1 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$, then $\widetilde{\mathrm{P}}_{\mathrm{f}}(rz_1, z_2, \alpha_1) = 1 - \widetilde{\mathrm{P}}_{\mathrm{f}}\left(z_1, z_2, \frac{\alpha_1}{r}\right)$

=1-1 = 0 for all $z_2 \in A$.
Case-c if $r = 0, \alpha_1 > 0$ and $z_1 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$, then $\widetilde{\mathrm{P}}_{\mathrm{f}}(rz_1, z_2, \alpha_1) = 1$ for all $z_2 \in A$.
It is clear that $\widetilde{\mathrm{P}}_{\mathrm{f}}(rz_1, z_2, 0) = 0$ for all $z_2 \in A$.Thus $rz_1 \in A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$.
Therefore, $A^{\perp_{\widetilde{\mathrm{P}}_{\mathrm{f}}}}$ is a **FLM**

The Relation between FRPHS and FRNS

This result explains the relationship between fuzzy real Pre-Hilbert space and fuzzy real normed space in this section.
Theorem 3 Every **FRPHS** is a **FRNS**
Proof: Let $(\mathrm{V}, \tilde{P}_f, \Delta_C)$ be a **FRHPS**. Define $\tilde{N}_f(z_1, \alpha_1) = \tilde{P}_f(z_1, z_1, {\alpha_1}^2)$ for each $z_1 \in V$, $\alpha_1 > 0$ and $\tilde{N}_f(z_1, 0) = 0$.Let $z_1, z_2 \in V$, $\alpha_1 > 0$
(RN1) $\tilde{N}_f(z_1, \alpha_1) = \tilde{P}_f(z_1, z_1, {\alpha_1}^2) > 0$ since $\tilde{P}_f(z_1, z_1, \alpha_1) > 0$
(RN2) $\tilde{N}_f(z_1, \alpha_1) = 1$ if and only if $\tilde{P}_f(z_1, z_1, {\alpha_1}^2) = 1$ if and only if $\tilde{P}_f(z_1, z_1, \alpha_1) = 1$ if and only if $z_1 = 0$
(RN3) $\forall\, r \in \boldsymbol{R}$, $\tilde{N}_f(rz_1, \alpha_1) = \tilde{P}_f(rz_1, rz_1, {\alpha_1}^2) = \tilde{P}_f\left(z_1, rz_1, \frac{{\alpha_1}^2}{|r|}\right) = \tilde{P}_f\left(z_1, z_1, \frac{{\alpha_1}^2}{|r|^2}\right) = \tilde{N}_f(z_1, \frac{\alpha_1}{|r|})$
(RN4) $\tilde{N}_f(z_1 + z_2, \alpha_1) = \tilde{P}_f(z_1 + z_2, z_1 + z_2, {\alpha_1}^2)$
$\leq \tilde{P}_f(z_1, z_1 + z_2, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_1 + z_2, {\alpha_1}^2)$

$= \tilde{P}_f(z_1 + z_2, z_1, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_1 + z_2, z_2, {\alpha_1}^2)$

$\leq \tilde{P}_f(z_1, z_1, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_1, {\alpha_1}^2)$
$\Delta_{\mathrm{C}} \tilde{P}_f(z_1, z_2, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_2, {\alpha_1}^2)$

$= \tilde{P}_f(z_1, z_1, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_1, {\alpha_1}^2)$
$\Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_1, {\alpha_1}^2) \Delta_{\mathrm{C}} \tilde{P}_f(z_2, z_2, {\alpha_1}^2)$

$\geq \tilde{P}_f(z_1, z_1, {\alpha_1}^2)\Delta_C \tilde{P}_f(z_2, z_1, {\alpha_1}^2)\Delta_C \tilde{P}_f(z_2, z_2, {\alpha_1}^2)$
Theorem ((2-2)-6)
and Theorem (4-5) give $\tilde{P}_f(z_1, z_2, {\alpha_1}^2) \leq \tilde{P}_f(z_1, z_1, {\alpha_1}^2)\Delta_C \tilde{P}_f(z_2, z_2, {\alpha_1}^2)$.
Hence
$\tilde{N}_f(z_1 + z_2, \alpha_1) \leq \tilde{P}_f(z_1, z_1, {\alpha_1}^2)\Delta_C \tilde{P}_f(z_2, z_2, {\alpha_1}^2) = \tilde{N}_f(z_1, \alpha_1)\Delta_C \tilde{N}_f(z_2, \alpha_1)$

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فضاء بري هلبرت الحقيقي الضبابي وبعض خواصه

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قسم العلوم التطبيقية، الجامعة التكنولوجية، بغداد، العراق .

الخلاصة:

في هذا العمل، تم اقتراح هيكلين مختلفين هما فضاء القياس الحقيقي الضبابي وفضاء بري هلبرت الحقيقي الضبابي . يتم تقديم المفهوم الأساسي للقياس الضبابي حول الفضاء الخطي الحقيقي أولاً لبناء الفضاء $(V, \widetilde{N}_f, \Delta_c)$ وهوعبارة عن فضاء القياس الحقيقي الضبابي مع بعض التعديلات على التعريف المقدم من قبل الباحثين رانيو وباك ثم .تم عرض هيكلية فضاء بري هلبرت الحقيقي الضبابي وهو يعتمد على فضاء القياس الحقيقي الضبابي .ثم ، ناقشنا بعض الخصائص والمفاهيم ذات الصلة لفضاء القياس الحقيقي الضبابي المقترحة مثل جوار- ვ ، وإغلاق المجموعة A المسماة GCL (A) وناقشنا الشرط الضروري للفصل والمجموعة الجزئية الغامضة . ايضا تم تقديم تعريف شبه القياس الضبابي على المجال V / A مع برهان ان شبه القياس الضبابي على المجال V / A هو فضاء قياس حقيقي ضبابي .في هذا العمل تم دراسة العلاقة بين سلسلة الاقتراب– $\widetilde{l}$ ، سلسلة كوشي– $\widetilde{l}$ و.كمال– $\widetilde{l}$ ، يتم تقديم الهيكل فضاء بري هلبرت الحقيقي الضبابي مع برهان بعض الخصائص المهمة المتعلقة بـهذا الفضاء. بالأضافة الى ذلك خاصية التعامد مع بعض الخصائص الهامة، على سبيل المثال annihilator المجموعة A. يتم ادراسة العلاقة بين فضاء بري هلبرت الحقيقي الضبابي وفضاء القياس الحقيقي الضبابي. أخيرًا، بعد إدخال هيكل فضاء بري هلبرت الحقيقي الضبابي فإنه يؤدي بطبيعة الحال إلى تعريف الفئة الأكثر أهمية من فضاء بري هلبرت الحقيقي الضبابي ، والذي يسمى فضاء هلبرت الحقيقي الضبابي.

الكلمات المفتاحية: الضرب الداخلي الضبابي، القياس الضبابي، فضاء هلبرت الحقيقي الضبابي، فضاء القياس الحقيقي الضبابي، التعامد الضبابي