On (m,n)-Strongly Fully Stably Banach Algebra Modules Related to an Ideal of $A^{m \times n}$

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Abstract:

The aim of this paper is introducing the concept of (m,n) strong full stability B-Algebra-module related to an ideal. Some properties of (m,n)- strong full stability B-Algebra-module related to an ideal have been studied and another characterizations have been given. The relationship of (m,n) strong full stability B-Algebra-module related to an ideal that states, a B-module X is (m,n)- strong full stability B-Algebra-module related to an ideal $H$, if and only if for any two $m$-element sub-sets $\{N_k 1, N_k 2, \ldots, N_k 3\}$ and $\{M_j 1, M_j 2, \ldots, M_j 3\}$ of $X^n$, if $\beta_j \in \sum_{i=1}^{m} a_i A \cap X^n H$, for each $j = 1, \ldots, m$, $i = 1, \ldots, n\alpha_t \in \{N_k 1, N_k 2, \ldots, N_k 3\}$ and $\beta_j \in \{M_j 1, M_j 2, \ldots, M_j 3\}$ implies $\alpha_t \in \{M_j 1, M_j 2, \ldots, M_j 3\}$ have been proved.

Keywords: Baer-(m,n)-criterion related to an ideal, F-S-B-A-module related to an ideal, (m,n)-full-stable-B-A-module related to ideal, Multiplication-(m,n)-B-A-module relative to ideal, Pure-(m, n)- sub-module.

Introduction:

An algebra is a set $A \neq \emptyset$ and if the following conditions are satisfied, 1- the set A with addition and multiplication is satisfied through a domain $\mathcal{F}$ is a space of vectors, 2- $\alpha \ (\alpha \circ \alpha) = (\alpha \circ \alpha) \circ \alpha'$ $= \alpha \circ (\alpha \circ \alpha')$ for all $\alpha \in \mathcal{F}$, $\forall \alpha, \alpha' \in \mathcal{A}$. 3- the set A with + and $\cdot$ forms a ring by -1. $\mathcal{R}$ is called an algebra where $\mathcal{R}$ is a ring, $\mathcal{R} = \{+,-,0\}$ such that $\mathcal{R}$ and $\cdot$ are binary operations, $\cdot$ is unary and nullary element 0 satisfying, $\mathcal{R} = \{+,-,0\}$ group which is commutative, $\mathcal{R}$, which is a semi-group and $\alpha$ $(\alpha \circ \alpha) = (\alpha \circ \alpha) \circ \alpha' = (\alpha \circ \alpha) \circ \alpha'$ for all $\alpha \in \mathcal{F}$, $\forall \alpha, \alpha' \in \mathcal{A}$. Suppose that $\mathcal{A}$ is an algebra, recall that a B-algebra left module (B-A left module) is a B-space $\mathcal{E}$ insomuch as $\mathcal{E}$ is an algebra-left module, and $||\alpha|| ||x|| \geq ||\alpha \cdot x|| (\alpha \in A, x \in \mathcal{E})$ according to (1). Following (2) a map from a B-algebra left module $\mathcal{X}$ into a B-algebra left module $\mathcal{Y}$ (algebra $\mathcal{A}$ is not necessary abelian ) is called a A-multiplier (homomorphism) if it satisfies $\forall \alpha \in A, x \in \mathcal{X}$, $\forall (\alpha \cdot x) = \alpha \cdot T x$. In (1), a sub-module $\mathcal{N}$ in $\mathcal{M}$ is said to be stable, if $\mathcal{N} \supseteq f (\mathcal{N}) \forall \mathcal{R}$ $-$homomorphism f from sub-module $\mathcal{N}$ into module $\mathcal{M}$. $\mathcal{M}$ is called full stability $\mathcal{R}$-module, if each sub-module in $\mathcal{N}$ is stable. Assume that $\mathcal{X}$ is B-algebra module, $\mathcal{X}$ is called F-S-B-algebra module related to ideal $\mathcal{K}$ of algebra $\mathcal{A}$, if $\forall$ sub-module $\mathcal{N}$ in $\mathcal{X}$ and $\forall$ multiplier $\theta : \mathcal{N} \rightarrow \mathcal{X}$ holds $\mathcal{N} + \mathcal{KX} \supseteq \mathcal{O} (\mathcal{N})^{1}$. Let $\mathcal{R}^{m,n}$ be the collection of every matrices $m \times n$ over a ring $\mathcal{R}$, $\mathcal{A} \in \mathcal{R}^{m,n}$, denote $A$ is transpose of $\mathcal{A}$. In general, write $\mathcal{N}^{m,n}$ for an $\mathcal{R}$-module $\mathcal{N}$, the collection of all matrices $m \times n$ where all elements in $\mathcal{N}$. Suppose that $\mathcal{M}$ a right Banach Algebra-module and let $\mathcal{N}$ be a left $\mathcal{R}$-module. Let $\mathcal{A} \in \mathcal{M}^{x,k}, \mathcal{S} \in \mathcal{R}^{m,n}$ and $\mathcal{Y} \in \mathcal{M}^{k \times n}$, with multiplication, $\mathcal{S}$ (resp. $\mathcal{Y}$) is good defined element in $\mathcal{M}^{x,k}$ (resp. $\mathcal{N}^{m,n}$). If $\mathcal{X} \subseteq \mathcal{M}^{x,k}$, $\mathcal{S} \subseteq \mathcal{R}^{m,n}$ and $\mathcal{Y} \subseteq \mathcal{N}^{m,n}$ we define $\ell_{\mathcal{M}^{x,k}}(\mathcal{S}) = \{ \mathcal{S} \in M^{x,k} | \mathcal{S} = 0 \}$ for all $\mathcal{S} \in \mathcal{S}$ $r_{\mathcal{N}^{m,n}}(\mathcal{S}) = \{ \mathcal{S} \in N^{m,n} | \mathcal{S} = 0 \}$ for all $\mathcal{S} \in \mathcal{S}$. $\ell_{\mathcal{R}^{m,n}}(\mathcal{Y}) = \{ \mathcal{Y} \in R^{m,n} | \mathcal{S} = 0 \}$ for all $\mathcal{Y} \in \mathcal{Y}$. $r_{\mathcal{M}^{x,k}}(\mathcal{X}) = \{ \mathcal{X} \in M^{x,k} | \mathcal{S} = 0 \}$ for all $\mathcal{X} \in \mathcal{X}$. Write $\mathcal{N}^{m,n} \subseteq \mathcal{N}^{m,n}$. In our work for fixed positive integers $n,m$ the concept of (m,n)-full stability Banach Algebra modules related to an ideal have been introduced.
(m, n)-Strongly-Fully-Stable-Banach-Algebra Modules Related to ideal
A left B-algebra-module \( X \) is \( n \)-generated where \( n \in N \) if there is exist \( x_1, ..., x_n \in X \) such that for all \( x \in X \) can be represented \( x = \sum_{k=1}^{n} \alpha_k \cdot x_k \) for some \( \alpha_1, ..., \alpha_n \) in algebra. A module which is \( 1 \)-generated is called a cyclic module. A right module over \( M \) is called strongly fully (m, n)-stable relative to an ideal A of \( K^{m,n} \), if \( \forall \alpha \in K \) A \( \supseteq 0 \) for all \( n \)-generated sub-module of \( M \) and \( \theta : N \to M \) \( R \)-homomorphism.

**Definition 1:** Let \( K \) be a \( B \)-module, \( K \) is called \((m,n)\)-S-F-S-B-A-M-R to ideal \( H \) of \( A^{m,n} \), if for every \( m \)-generated sub-module of \( K \), and for each multiplier \( \theta : \theta \to K \), which satisfies \( \theta(\bar{J}) \subseteq \bar{J} \) for two fixed positive integers \( m, n \).

In (1) \( \lambda \in L \), A nonempty subset of a left \( B \)-module \( X \), the annihilator \( \text{Ann}_A(M) \) of \( B \)-module \( M \) is \( \theta \in A \); \( \lambda \in M \), \( \lambda = 0 \) for all \( \lambda \in M \) = \( \text{Ann}_A(M) \).

**Notation 1:** Suppose that \( X \) is a \( B \)-algebra-module

\[ nx_{k_1,x_{k_2},...,x_{k_n}} = \{ \theta(x_k), x \in X, \theta = 1, 2, ..., \eta \} \]

\[ M_{y_1,y_2,\cdots,y_n} = \{ \theta(m), m \in M, y \in X, i = 1, 2, \eta \} \]

**Proposition 1:** A \( B \)-module \( X = (m,n) \)-S-F-S-B-A-M-R to ideal \( H \) if and only if any \( \lambda \)-element sub-modules \( \{ nx_{k_1,x_{k_2},...,x_{k_n}} \} \) and \( \{ M_{y_1,y_2,\cdots,y_n} \} \) of \( X \), if \( \eta \notin \sum_{i=1}^{n} \alpha_i A \cap X^{m,n}H \), for each \( j = 1, \eta \), \( \alpha_i \in \{ n_{i,k_1}, n_{i,k_2},...,n_{i,k_n} \} \) and \( \beta_j \in \{ m_{j_1}, m_{j_2},\cdots,m_{j,k_n} \} \) implies

\[ r \alpha_i \in \{ n_{i,k_1}, n_{i,k_2},...,n_{i,k_n} \} \]

**Proof:** Assume that \( X = (m,n) \)-S-F-S-B-A-M-R to ideal and there exist two \( \mu \)-element sub-modules \( \{ nx_{k_1,x_{k_2},...,x_{k_n}} \} \) and \( \{ M_{y_1,y_2,\cdots,y_n} \} \) of \( M \), such that if \( M \notin \sum_{i=1}^{n} \alpha_i A \cap X^{m,n}H \), for each \( j = 1, \eta \), \( \alpha_i \in \{ n_{i,k_1}, n_{i,k_2},...,n_{i,k_n} \} \) and \( \beta_j \in \{ m_{j_1}, m_{j_2},\cdots,m_{j,k_n} \} \).

**Corollary 1:** If \( X = (m,n) \)-S-F-S-B-A-M-R to ideal \( H \) of \( A^{m,n} \), therefore any two \( \lambda \)-element sub-modules \( \{ nx_{k_1,x_{k_2},...,x_{k_n}} \} \) and \( \{ M_{y_1,y_2,\cdots,y_n} \} \) of \( X \), implies

\[ r \alpha_i \in \{ n_{i,k_1}, n_{i,k_2},...,n_{i,k_n} \} \]

**Proof:** The proof is clear.

In (2), A \( B \)-module \( X \) is called to holds Baer criterion (B-C) if all submodule of \( X \) holds Baer criterion, this mean that for every sub-module \( N \) in \( X \) and algebra- multiplier : \( N \to X \), so \( \forall \alpha \in A \) s.t

\[ \theta(\bar{N}) = \bar{N} \forall \alpha \subseteq N^n \].

**Definition 2:** A \( B \)-algebra-module \( X \) is called hold Baer-(m,n)-criterion relates (B-(m,n)-C-R) to an ideal \( H \) if each sub-module of \( X \) satisfies B-(m,n)-C-R to an ideal \( H \), this mean that, for every \( \eta \)-
generated sub-module $L$ of $X^n$ and $A$-multiplier $f:L \to X^3$ , there is a $\alpha$ such that $
exists \theta(t) = \Theta \in X^0H$ for all $\Theta \in L$.

**Proposition 2:** If $\chi$ satisfies $B-(m_1,n)-C$ to $D$ ideal and $\rho_{\lambda}(L \cap \mathbb{M}) = \rho_{\lambda}(L) + \rho_{\lambda}(\mathbb{M})$ for each $m_1,n$-generated sub-modules of $X^n$, then $\chi$ satisfies $B-(m_1,n)-C$ to $D$ ideal.

**Proof:** Let $P = A\chi + A\chi_{x_{k}+ \ldots + A\chi_{m}}$ be $m_1,n$-generated sub-module of $X^n$, $f:P \to X^3$ multiplies. Now, by induction on $m_1,n$. Clearly that $\chi$ holds $B-(m_1,n)-C$ to $D$ ideal, if $m_1,n = 1$. Suppose that $\chi$ satisfies $B-(m_1,n)-C$ to $D$ ideal for each $k$-generated submodule of $X^n$, for $n \geq k$. Write $L = A\chi_1 \cup \ldots \cup A\chi_m$, therefore for each $\Theta \in L$ and $w \in \mathbb{M}$ $f(\Theta)(w) = y_1(w), f(\Theta)(w_2) = y_2(w_2)$ for some $y_1, y_2 \in A$. It is clear $y_1, y_2 \in \rho_{\lambda}(L \cap \mathbb{M}) = \rho_{\lambda}(L) + \rho_{\lambda}(\mathbb{M})$ and $w \in \rho_{\lambda}(L), z \in \rho_{\lambda}(\mathbb{M})$ such that $y_1 = y_2 = \rho_{\lambda}(L \cap \mathbb{M})$. Then for any $w = w_1 + w_2 \in \mathbb{P}$ with $\mathbb{E}, \mathbb{W} \in \mathbb{E}$, $f(w) = f(w_1) + f(w_2) = y_1(w_1) + y_2(w_2) = y_1(w_1) + y_2(w_2) = y$. Hence $\chi$ satisfies $B-(m_1,n)-C$ to $D$ ideal.

**Corollary 2:** Let $\chi$ be a $B$-A- module. $\chi$ is $(m_1,n)$-S-F-S-B-A-M-R to $D$ ideal if and only if $\rho_{\lambda}(\Theta_{\chi_1} + \Theta_{\chi_{x_2}A} + \ldots + \Theta_{\chi_{x_{k}2-\ldots-x_{k}A}}) \subseteq \chi_{\chi_1} + \chi_{\chi_{x_2}A} + \ldots + \chi_{\chi_{x_{k}2-\ldots-x_{k}A}}\cap X^0H$, for each $\chi_{\chi_1} + \chi_{\chi_{x_2}A} + \ldots + \chi_{\chi_{x_{k}2-\ldots-x_{k}A}}\cap X^0H$, for some $t \in A$.

Following (1) "suppose that $A$ is a unital B- and assume $\alpha > 1$. Algebra-module $\chi$ is said Quasi $\alpha$-injective (Q-$\alpha$-inj), if algebra-module homomorphism $\varphi:N \to X$, s.t $\varphi \leq 1$ and there is algebra-module homomorphism $\theta:X \to Y$, s.t $\theta \circ i = \varphi$ and $|| \theta || \leq \alpha$, i is an isometry from submodule $N$ of $X$. Call $\chi$ is $-\text{inj}$, if it is Q-$\alpha$-inj for some $\alpha$.

Following (1), assume that $A$ is unital $B$- and suppose that $\alpha > 1$. Algebra-module $\chi$ is said to be Quasi-$\alpha$-injective relate to an ideal $H$ of algebra if,

$\varphi : N \to X$ is algebra-module homomorphism $s.t \ 1 \geq ||\varphi||$, and there is algebra-module homomorphism $\theta:X \to Y$, s.t $\theta \circ i = \varphi$ and $|| \theta || \leq \alpha$, i is an isometry from submodule $N$ of $X$ to $Y$.

The concepts strongly Quasi-$(m_1,n)$- $\alpha$-injective -B-A- module related to ideal for some $\alpha$ introduced.

**Definition 3:** Suppose that $A$ is a unital B-A and $1 < \alpha$. $\chi$ is said to be strongly Quasi- $(m_1,n)$- $\alpha$-injective relate to an ideal I of $A^\text{max}$ if $\beta:N \to X^\alpha$ is algebra-module homomorphisms such that $1 \geq ||\beta||$, there is a $\chi :X^\alpha \to X^\alpha$ algebra-module homomorphism, such that $(\alpha o i)(n) - \beta(n) \in X^\alpha H$ and $1 \geq ||\beta||$, $i$ is an isometry from $m_1,n$-generated submodule $N$ in $X$. $\chi$ is strongly Quasi-$(m_1,n)$-injective relate to an ideal I if $\chi$ is strongly – Quasi - $(m_1,n)$- $\alpha$- injective relate to ideal for some $\alpha$.

**Proposition 4:** If $\chi$ is $(m_1,n)$-S-F-S-B-A-M-R to $I$ ideal of an algebra, then $\chi$ is strongly Quasi $(m_1,n)$-injective B- algebra module relate to an ideal I.

**Proof:** Set $N = \alpha A + \ldots + \alpha A$, $m_1,n$-generated sub-module of $X^\alpha, \alpha \in X^\alpha$, let $\alpha$ be greater than 1 and $f$ be any algebra-modulehomomorphism from $N$ to $X^\alpha$. 

\[ f(x_1 + x_{2-\ldots-x_{k}A}) \subseteq x_{\chi_1} + \chi_{\chi_{x_2}A} + \ldots + \chi_{\chi_{x_{k}2-\ldots-x_{k}A}} \cap X^0H. \] Conversely, suppose that \[ f(x_1 + x_{2-\ldots-x_{k}A}) \subseteq x_{\chi_1} + \chi_{\chi_{x_2}A} + \ldots + \chi_{\chi_{x_{k}2-\ldots-x_{k}A}} \cap X^0H. \]
such that \( \| f \| \leq 1 \). Since \( X(\mathfrak{m}, \eta) \)-S-F-S-R to ideal, therefore \( f(aA + \cdots + aA) \subseteq a_1A + \cdots + a_nA \cap X^mI \), thus there is \( t = ( t_1, \ldots, t_n ) \in \mathbb{E}_A \), and \( w \in X^mI \). Let \( a_i = ( 0, \ldots, 1, 0, \ldots, 0 ) \) such that \( f(\sum_{i=1}^{n} a_i t_i) = ( \sum_{i=1}^{n} a_i t_i ) + w \). Define \( g : X^m \to X \) as \( g(\alpha) = t'\alpha \), clearly \( g \) is well defined algebra-module homomorphism. Now \( f(\sum_{i=1}^{n} a_i t_i) = ( \sum_{i=1}^{n} a_i t_i ) + w - t ( \sum_{i=1}^{n} a_i t_i ) = w \in X^mI \) and since for all \( y \in a_1A + \cdots + a_nA \), \( y = \sum_{i=1}^{n} a_i s_i \) for some \( s_i \in s \in A \), \( f(y) - g(y) = f(\sum_{i=1}^{n} a_i s_i) - g(\sum_{i=1}^{n} a_i s_i) = ( f(\sum_{i=1}^{n} a_i s_i) - s \in X^mI \), therefore \( X \) is strongly quasi \((\mathfrak{m}, \eta)\)-banach algebra module relative to ideal.

**Definition 4:** A sub-module \( \mathcal{N} \) of Banach \( A \)-module is called pure-(\( \mathfrak{m}, \eta \) )-sub-module if \( \mathcal{N} = \mathcal{N} \cap X^mI \) for all ideal \( \mathfrak{m} \) of \( A^{m \times \eta} \).

When the sub-module of \((\mathfrak{m}, \eta)\)-S-F-S-B-A-M-R to ideal have been partial answer in the next proposition.

**Proposition 5:** Let \( X \) be a \((\mathfrak{m}, \eta)\)-S-F-S-B-A-M-R to a non-zero ideal \( I \) of \( A^{m \times \eta} \), then every \((\mathfrak{m}, \eta)\)-pure sub-module of \( X \) is \((\mathfrak{m}, \eta)\)-S-F-S-B-A-M-R to an ideal.

**Proof:** Assume that \( \mathcal{N} \) is pure-(\( \mathfrak{m}, \eta \) )-sub-module of \( X \). For every sub-module \( L \) of \( \mathcal{N} \) and a multiplier \( f : L \to \mathcal{N} \), put \( g = id f : L \to X \) (where \( i \) is the inclusion mapping of \( \mathcal{N} \) to \( X \) ), then by assumption \( f(L) = g(L) \subseteq X^mI \), since \( f(L) \subseteq L \). Hence \( f(L) \subseteq L \cap X^mI \cap \mathcal{N} \). Because \( \mathcal{N} \) is pure-(\( \mathfrak{m}, \eta \) )-sub-module of \( X \) then \( \mathcal{N} \cap X^mI = \mathcal{N} \mathfrak{m} I \), for all ideal \( I \) of \( A^{m \times \eta} \), therefore \( f(L) \subseteq L \cap X^mI \). Therefore \( \mathcal{N} \) is \((\mathfrak{m}, \eta)\)-S-F-S-B-A-M-R to \( I \).

**Conclusion:**
In this work, the concept of \((\mathfrak{m}, \eta)\) strong full stability \( B \)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m \times \eta} \) has been introduced and it is also easy to study its properties by linking it with other concepts. The relationship of \((\mathfrak{m}, \eta)\) strong full stability \( B \)-Algebra-module related to an ideal that states, if \( X \) is \((\mathfrak{m}, \eta)\)-strong full stability \( B \)-Algebra-module related to an ideal \( I \) of algebra, then \( X \) is strongly quasi \((\mathfrak{m}, \eta)\)-injective \( B \)-algebra module relate to an ideal I have been proved, and show that every \((\mathfrak{m}, \eta)\)-pure sub-module of \( X \) strong full stability \( B \)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m \times \eta} \) is \((\mathfrak{m}, \eta)\) strong full stability \( B \)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m \times \eta} \).