Open Access **Baghdad Science Journal** **P-ISSN: 2078-8665**
Published Online First: April 2021 **E-ISSN: 2411-7986**

DOI: http://dx.doi.org/10.21123/bsj.2021.18.4.1234

On (ɱ,ɳ)-Strongly Fully Stably Banach Algebra Modules Related to an Ideal of $A^{ɱ\times ɳ}$

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Received 3/2/2020, Accepted 27/9/2020, Published Online First 30/4/2021

Abstract:

The aim of this paper is introducing the concept of (ɱ,ɳ) strong full stability B-Algebra-module related to an ideal. Some properties of (ɱ,ɳ)- strong full stability B-Algebra-module related to an ideal have been studied and another characterizations have been given. The relationship of (ɱ,ɳ) strong full stability B-Algebra-module related to an ideal that states, a B- Ạ-module Ҳ is (ɱ,ɳ)- strong full stability B-Algebra-module related to an ideal Ḥ , if and only if for any two ɱ-element sub-sets $\{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2},\cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$and $\{\acute{M}_{\grave{y}_1},\acute{M}_{\grave{y}_1,\grave{y}_2},\cdots,\acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$of $Ҳ^{ɳ}$, if $\beta_j \notin \sum_{i=1}^{n}\alpha_i A \cap Ҳ^{ɱ}Ḥ$, for each j = 1, …, ɱ, i = 1,…, ɳ$\alpha_i \in \{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2},\cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$ and $\beta_j \in \{\acute{M}_{\grave{y}_1},\acute{M}_{\grave{y}_1,\grave{y}_2},\cdots,\acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$ implies$r_{Ạɳ}(\{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2},\cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \nsubseteq r_{Ạɳ}(\{\acute{M}_{\grave{y}_1},\acute{M}_{\grave{y}_1,\grave{y}_2},\cdots,\acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\})$ have been proved..

Keywords: Baer-(ɱ,ɳ)-criterion related to an ideal, F-S-B-Ạ-module related to an ideal, (ɱ,ɳ)-full-stable-B-Ạ-module related to ideal, Multiplication-(ɱ,ɳ)-B-Ạ-module relative to ideal, Pure-(ɱ, ɳ)- sub-module.

Introduction:

An algebra is a set $A \neq \emptyset$ and if the following conditions are satisfied**,** 1- the set Ạ with addition and multiplication are satisfied through a domain $\mathcal{F}$ is a space of vectors**,** 2- α (å∘ď) =(αå) ∘ď = å∘ (αď) for all$\alpha \in \mathcal{F}$, ∀ å, ď ∈ Ạ, 3- the set Ạ with + and ∘ forms a ring by -1-. $\mathfrak{R}$ is called an algebra where $\mathfrak{R}$ is a ring, $[\mathfrak{R},+,\cdot,-,0]$ such that+ and· are binary operations ,− is unary and nullaryelement is 0 satisfying, $[\mathfrak{R},+,-,0]$ group which is commutative, $[\mathfrak{R},.]$ which is a semi-group and å.(ể + ȍ) = (å.ể) + (å.ȍ) and (å + ể).ȍ = (å.ȍ) + (ể + ȍ)(1). Suppose that Ạ is an algebra, recall that a B- algebra- left module (B-Ạ-left module) is a B-space Ė insomuch as Ė is an algebra-left module, and ‖å‖‖ẋ‖ ≥ ‖å.ẋ‖(å ∈ Ạ, ẋ ∈ Ė) according to (1). Following (2) a map from a B-algebra- left module Ҳ into a B−algebra − left module Ў (algebra Ạ is not necessary abelian) is called a Ạ-multiplier (homomorphism) if it satisfies∀å ∈ Ạ, ẋ ∈ Ҳ, Ṯ(å.ẋ) = å.Ṯẋ. In (1), a sub-moduleṄ in Ṁ is said to be stabile, if Ṅ ⊇ f(Ṅ) ∀ $\mathfrak{R}$ −homomorphism f from sub-module Ṅ into module Ṁ. M is called full stability $\mathfrak{R}$-module ,if each sub-module in Ṁ is stable . Assume that Ҳ is B−algebra − module, Ҳ is called F-S-B−algebra −module related to an ideal Ḳ of algebra Ạ, if ∀ sub-module Ṅ in Ҳ and,∀ multiplier θ: Ṅ ⟶ Ҳ holds Ṅ + ḲҲ ⊇ θ(Ṅ)" (1). Let $\mathfrak{R}^{ɱ\times ɳ}$ be the collection of every matrices $ɱ\times ɳ$ over a ring $\mathfrak{R}$. $\ddot{A} \in \mathfrak{R}^{m\times n}$, denote $\ddot{A}^T$ is transpose of Ä. In general, write $Ṅ^{ɱ\times ɳ}$ for an $\mathfrak{R}$-module $\dot{N}$, the collection of all matrices $ɱ\times ɳ$ where all elements in $\dot{N}$. Suppose that $\dot{M}$ a right Banach Algebra-module and let $\dot{N}$ be a left $\mathfrak{R}$-module. Let $\dot{x} \in Ṁ^{l\times ɱ}$,ṡ ∈ $\mathfrak{R}^{ɱ\times ɳ}$and ẏ ∈ $Ṁ^{ɳ\times k}$, with multiplication, ẋṡ (resp. ṡẏ) is good defined element in $Ṁ^{l\times ɱ}$(resp. $Ṅ^{ɳ\times k}$). "If $Ҳ \subseteq Ṁ^{l\times ɱ}$, $S \subseteq \mathfrak{R}^{m\times n}andy \subseteq Ṅ^{ɳ\times k}$ are define

$$\ell_{Ṁ^{l\times ɱ}}(Ṡ) = \{ừ \in Ṁ^{l\times ɱ} | ừṡ = 0\,;\ for\ all\ ṡ \in Ṡ\}$$
$$r_{Ṅ^{ɳ\times k}}(Ṡ) = \{ǖ \in Ṅ^{ɳ\times k} | ṡǖ = 0\,;\ for\ all\ ṡ \in Ṡ\}$$
$$\ell_{\mathfrak{R}^{ɱ\times ɳ}}(Y) = \{ṡ \in \mathfrak{R}^{ɱ\times ɳ} | ṡω = 0\,;\ for\ all\ ω \in Y\}$$
$$r_{\mathfrak{R}^{ɱ\times ɳ}}(Ҳ) = \{ṡ \in \mathfrak{R}^{ɱ\times ɳ} | ẋṡ = 0\,;\ for\ all\ ẋ \in Ҳ\}$$

Write $\dot{N}^{ɳ} = \dot{N}^{1\times ɳ}$, $\dot{N}_{ɳ} = N^{ɳ\times 1}$(3) . In our work for fixed positive integers ɳ,ɱ the concept of (ɱ,ɳ)-full stability Banach Algebra modules relative to an ideal have been introduced.

(ɱ,ɳ)-Strongly-Fully-Stable-Banach-Algebra Modules Related to ideal

A left B-algebra-module Ҳ is ɳ −generated where $ɳ \in N$ if there is exist $\dot{x}_1, \ldots, \dot{x}_n \in$ Ҳ such that for all $\dot{x} \in$ Ҳ can be represented $\dot{x} = \sum_{k=1} \mathring{a}_k . \dot{x}_k$ for some $\mathring{a}_1, \ldots, \mathring{a}_n$ in algebra. A module which is 1-generated is called a cyclic module (4) .A right module over ℜ, Ṁ is called strongly fully (ɱ,ɳ)-stable relative to an ideal A of $R^{ɳ\times ɱ}$, if $\dot{N} \cap \dot{M}^{ɱ}$ Ạ ⊇θ(Ṅ) for all ɳ-generated sub-module of $\dot{M}^{ɱ}$ and θ : Ṅ→Ṁ ℜ-homomorphism (5)

Definition 1: Let Ķ be B- Ạ-module , Ķ is called (ɱ,ɳ)-S-F-S-B-A-M-R to ideal Ḫ of $Ạ^{ɱ\times ɳ}$, if for everyɱ −generated sub-moduleĴof $Ķ^{ɳ}$ and for each multiplier $\theta : \hat{J} \rightarrow Ķ^{ɳ}$ which satisfies $\theta(\hat{J}) \subseteq \hat{J} \cap Ķ^{ɱ}Ḫ$ for two fixed positive integers ɳ,ɱ.

In (1) "Let Ṁ be nonempty subset of a left B- Ạ-module Ҳ, the annihilater $ann_{Ạ}$(Ṁ) of B-A-module Ṁ is $\{\mathring{a} \in Ạ ; \mathring{a}.\dot{x} = 0 \ for \ all \ \dot{x} \in \dot{M}\} = ann_{Ạ}(\dot{M})$.

Notation 1:

Suppose that Ҳ be a B-algebra-module

1) $Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ} = \{\oplus ṋ_{\dot{x}_i} | ṋ \in Ṋ, \ \dot{x}_i \in Ҳ, i = 1,2,\cdots,ɳ\}$

$\acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ} = \{\oplus \acute{m}_{\grave{y}_i} | \ \acute{m} \in \acute{M}, \ \grave{y}_i \in Ҳ, i = 1,2,\cdots,ɳ\}$

2) $\ell_{Ạ^{ɱ\times ɳ}} Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ} = \{\mathring{a} \in Ạ^{ɱ\times ɳ}, \ \mathring{a}.(\oplus ṋ_{\dot{x}_i}) = 0, \ \forall ṋ_{\dot{x}_i} \in Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$

$\ell_{Ạ^{ɱ\times ɳ}} \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ} = \{\mathring{a} \in Ạ^{ɱ\times ɳ}, \ \mathring{a}.(\oplus \acute{m}_{\grave{y}_i}) = 0, \ \forall m_{\grave{y}_i} \in \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$

Proposition 1: A B- Ạ-module Ҳ is (ɱ,ɳ)-S-F-S-B-A-M-R to ideal , if and only if for any two ɱ-element sub-sets $\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$and $\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$of $Ҳ^{ɳ}$, if $\beta_j \notin \sum_{i=1}^{n} \alpha_i A \cap Ҳ^{ɱ}Ḫ$, for each j = 1, …, ɱ, i = 1,…, ɳ$\alpha_i \in \{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$ and $\beta_j \in \{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$ implies $r_{Ạɳ}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \not\subseteq r_{Ạɳ}(\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\})$.

Proof: Presume that Ҳ is (ɱ,ɳ)-S-F-S-B-A-M-R to ideal and there exist two ɱ- element subsets $\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$ and $\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$of $\dot{M}_ɳ$ such that if $\acute{M}_{\grave{y}_j} \notin \sum_{i=1}^{n} Ạ\alpha_i \cap Ҳ^{ɱ}Ḫ$, for each j = 1, …, ɱ and$r_{Ạɳ}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \subseteq r_{Ạɳ}(\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\})$.Define f:$\sum_{i=1}^{ɳ} \alpha_i$ Ạ→$Ҳ^{ɳ}$ by f($\sum_{i=1}^{ɳ} a_i Ṋ_{\dot{x}_i}$) = $\sum_{i=1}^{ɳ} a_i \acute{M}_{\grave{y}_i}$. Let $Ṋ_{\dot{x}_i}$ = (m_{1i}, m_{2i}, …,$m_{ɳi}$). If $\sum_{i=1}^{ɳ} a_i Ṋ_{\dot{x}_i} = 0$, then $\sum_{i=1}^{ɳ} a_i k_{ij}$= 0, j = 1, 2, …,ɱ, implies that r$Ṋ_{\dot{x}_i}$= 0 where r = (r_1, …, $r_ɳ$) and hence $r^T \in r_{Ạɳ}\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$. By assumption r$K_{\grave{y}_j}$, = 0 where j = 1, …,ɱ, when $\sum_{i=1}^{n} r_i \acute{M}_{\grave{y}_i} = 0$. Thus f is well defined. Clearly that f is multiplier. (ɱ,ɳ)- strongly-fully-stable of Ҳ implies that there is t = (t_1,…,t_n) ∈$Ạ^{ɳ}$ such that f($\sum_{i=1}^{ɳ} r_i Ṋ_{\dot{x}_i}$) = $\sum_{k=1}^{ɳ} t_k(\sum_{i=1}^{ɳ} r_i Ṋ_{\dot{x}_i}) + b = \sum_{k=1}^{ɳ}\sum_{i=1}^{ɳ}(t_k r_i)Ṋ_{\dot{x}_i} + b$for each$\sum_{i=1}^{ɳ} r_i Ṋ_{\dot{x}_i} \in \sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}$ Ạ and b ∈ $Ҳ^{ɱ}I$ .Let r_i= (0, …,0,1, 0, …, 0) ∈$Ạ^{ɳ}$, have 1 in the position i-th and otherwise put 0. $\acute{M}_{\grave{y}_i}$= f($Ṋ_{\dot{x}_i}$) =$\sum_{k=1}^{ɳ} t_k Ṋ_{\dot{x}_i} + b \in \sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ \cap Ҳ^{ɱ}Ḫ$,this is contradiction. Conversely suppose that there exists ɱ-generated B-Ạ-sub-module of $Ҳ^{ɳ}$ and multiplier μ:$\sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ$→$Ҳ^{ɳ}$such that μ($\sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ$) ⊄ $\sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ \cap X^{ɱ}Ḫ$. Therefore there exists an element $\beta(=\sum_{i=1}^{ɳ} r_i Ṋ_{\dot{x}_i}) \in \sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ$such that μ($\acute{M}_{\grave{y}}$) ∉ $\sum_{i=1}^{ɳ} Ṋ_{\dot{x}_i}Ạ \cap Ҳ^{ɱ}Ḫ$.Take $\acute{M}_{\grave{y}_j}$=$\acute{M}_{\grave{y}}$, when j is 1, …, ɱ, hence own ɱ-element subset {μ($\acute{M}_{\grave{y}}$), …, μ($\acute{M}_{\grave{y}}$)},such that μ($\acute{M}_{\grave{y}}$) ∉ $\sum_{i=1}^{n} Ṋ_{\dot{x}_i}Ạ \cap Ҳ^{ɱ}Ḫ$, j = 1, …,ɱ. Let η= (t_1, …, t_n) ∈ $r_{Ạn}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\})$,then $\eta\alpha_j$= 0, i.e$\sum_{i=1}^{ɳ} t_i a_{ij}$= 0, for each j = 1, …, ɱ, $Ṋ_{\dot{x}_i}$= (a_{1j}, a_{2j}, …, a_{nj}) and{μ(M_y), …, μ(M_y)} $\eta = \sum_{k=1}^{ɳ} t_k\mu(M_y) = \sum_{k=1}^{n} t_k \mu(\sum_{i=1}^{n} r_i Ṋ_{\dot{x}_i}) = \sum_{k=1}^{n} \mu(\sum_{i=1}^{n} t_k r_i Ṋ_{\dot{x}_i}) = 0$ therefore

$r_{Ạɳ}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \subseteq r_{Ạɳ}(\{\mu(\acute{M}_{\grave{y}}), \ldots, \mu(\acute{M}_{\grave{y}})\})$, hence

$r_{Ạɳ}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \subseteq r_{Ạɳ}(\{\mu(\acute{M}_{\grave{y}_1}), \ldots, \mu(\acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ})\})$ this is a contradiction. Hence Ҳ is (ɱ,ɳ)-S-F-S-B-A-M-R to ideal Ḫ of $Ạ^{ɱ\times ɳ}$.

Corollary 1 : If Ҳ is (ɱ,ɳ)-S-F-S-B-A-M-R to ideal Ḫ of $A^{ɱ\times ɳ}$, therefore any two ɱ-element sub-sets $\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}$ and $\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\}$of $Ҳ^{ɳ}$, $r_{Ạɳ}(\{ Ṋ_{\dot{x}_1}, Ṋ_{\dot{x}_1,\dot{x}_2}, \cdots, Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}\}) \subseteq r_{Ạɳ}(\{\acute{M}_{\grave{y}_1}, \acute{M}_{\grave{y}_1,\grave{y}_2}, \cdots, \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}\})$ implies that $Ṋ_{\dot{x}_1}Ạ + Ṋ_{\dot{x}_1,\dot{x}_2}Ạ + \ldots + Ṋ_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_ɳ}Ạ \cap Ҳ^{ɱ}Ḫ = \acute{M}_{\grave{y}_1}Ạ + \acute{M}_{\grave{y}_1,\grave{y}_2}Ạ + \acute{M}_{\grave{y}_1,\grave{y}_2,\cdots,\grave{y}_ɳ}Ạ \cap Ҳ^{ɱ}Ḫ$.

Proof : The proof is clear

In (2), A B-Ạ- module Ҳ is called to holds Baer criterion (B-C) if all submodule of Ҳ holds Baer criterion, this mean that for every sub-module Ṅ in Ҳ and algebra− multiplier : Ṅ ⟶ Ҳ , so ∃å ∈ Ạ s.t $\theta(ṋ) = \mathring{a}ṋ \ \ \forall ṋ \in \dot{N}$".

Definition 2 :A B- algebra- module Ҳ is called hold Baer-(ɱ,ɳ)-criterion relates (B-(ɱ,ɳ)-C-R) to an ideal Ḫ if each sub-module of Ҳ satisfies B-(ɱ,ɳ)-C-R) to an ideal Ḫ ,this mean that, for every ɱ-

generated sub-module L of X^{η} and Ạ-multiplier $\theta: L \longrightarrow X^{\eta}$,there is å in Ạ such that $\theta(l) = ål \in X^{m}H$ for all $l \in L$.

Proposition 2 : If X satisfies B-(m,1)-C-R to ideal and $r_A(L \cap \dot{M}) = r_A(L) + r_A(\dot{M})$ for each m-generated sub-modules of X^{η}, then X satisfies B-(m,η)-C-R to an ideal.

Proof : Let P = $A\dot{x}_1 + A\dot{x}_2 + \ldots + A\dot{x}_m$ be *m*-generated sub-module of X^{η} , f : P → X^{η} multiplier . Now, by induction on *m*. Clearly that *X* holds B-(m,η)-C-R to an ideal, if *m*= 1. Suppose that *X* satisfies B-(m,η)-C-R to an ideal for each *k*-generated sub-module of X_{η}, for η-1≥ k. Write L= $A\dot{x}_1$, M= $A\dot{x}_2 + \ldots + A\dot{x}_m$, therefore for each $w_1 \in L$ and $w_2 \in \dot{M}$ $f|_L(w_1) = y_1 w_1$, $f|_M(w_2) = y_2 w_2$ for some $y_1, y_2 \in A$. It is clear $y_1 - y_2 \in r_A(L \cap \dot{M}) = r_A(L) + r_A(\dot{M})$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in r_A(L)$, $z_2 \in r_A(\dot{M})$ and let $y = y_1 - z_1 = y_2 + z_2$.Then for any $w = w_1 + w_2 \in P$ with $w_1 \in L$ and $w_2 \in \dot{M}$, $f(w) = f(w_1) + f(w_2) = w_1 y_1 + w_2 y_2 = w_1(y_1 - z_1) + w_2(y_2 + z_2) = w_1 y + w_2 y = (w_1 + w_2)y = wy$.

Proposition 3 : Suppose that X is a B-Ạ- module. Get X holds B-(m,η) – C-R to an ideal if and only if $\ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A) \subseteq N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \cap X^{m}H$ for any n-elements subset $\{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2}, \cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}\}$ of X^{η}.

Proof : Assume that B-(m,n)-C-R to an ideal holds for *m*-generated sub-module of X^{η}, let $N_{\dot{x}_i} = (k_{i1}; k_{i2}, \ldots, k_{im})$, for each i = 1, …, η and $\acute{M}_{\dot{y}} = \{\acute{M}_{\dot{y}_1}, \acute{M}_{\dot{y}_1,\dot{y}_2}, \cdots, \acute{M}_{\dot{y}_1,\dot{y}_2,\cdots,\dot{y}_\eta}\} \in \ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A)$, $\acute{M}_{\dot{y}_i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$. Define $\mu: N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \rightarrow X^{\eta}$ by $\mu(\sum_{i=1}^{\eta} N_{\dot{x}_i} a_i) = \sum_{i=1}^{\eta} \acute{M}_{\dot{y}_i} a_i$.If $\sum_{i=1}^{\eta} N_{\dot{x}_i} a_i$, then $\sum_{i=1}^{\eta} k_{ij}\, a_i = 0$ where j = 1, …,m, therefore $L_{x_i} r = 0$ and r = $(r_1, \ldots, r_\eta)$ and hence $r \in r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A)$. By assumption $rN_{\dot{x}_i} = 0$, i =1,…, η so $\sum_{i=1}^{\eta} \acute{M}_{\dot{y}_i} a_i = 0$. Therefore f is well defined and *μ* is an multiplier it is an easy. By assumption exist t ∈ Asuch that $\mu(\sum_{i=1}^{\eta} N_{\dot{x}_i} a_i) = t(\sum_{i=1}^{\eta} \acute{M}_{\dot{y}_i} a_i) = \sum_{i=1}^{\eta} \acute{M}_{\dot{y}_i}(t a_i)$ for each $\sum_{i=1}^{\eta} N_{\dot{x}_i} a_i \in \sum_{i=1}^{\eta} N_{\dot{x}_i}A$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^{\eta}$, in the i-the position is 1 and 0 otherwise.$M_{y_i} = \mu(\sum_{i=1}^{\eta} N_{\dot{x}_i}) = \sum_{i=1}^{\eta} N_{\dot{x}_i} t \in \sum_{i=1}^{\eta} N_{\dot{x}_i}A$which is contradiction. This implies that $\ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A) \subseteq N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \cap X^{m}H$. Conversely, suppose that $\ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A) \subseteq N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \cap X^{m}H$,for each $\{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2}, \cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}\}$in X^{η}. Then for each multiplier f : $N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2}, \cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta} \rightarrow X^{\eta}$ and s = $(s_1, \ldots, s_n) \in r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A)$, $\sum_{k=1}^{\eta} s_k(\sum_{i=1}^{\eta} N_{\dot{x}_i} t_i) = 0$, for each $\sum_{i=1}^{\eta} N_{\dot{x}_i} t_i \in \sum_{i=1}^{\eta} N_{\dot{x}_i}\, A$, thus $\sum_{k=1}^{\eta} s_k f(\sum_{i=1}^{\eta} N_{\dot{x}_i}\, t_i) = \sum_{k=1}^{\eta} f(\sum_{i=1}^{\eta} N_{\dot{x}_i}\, s_k t_i) = 0$, thus $f(\sum_{i=1}^{\eta} N_{\dot{x}_i} t_i) \in \ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A) = N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \cap X^{m}H$, for some t ∈A. Hence X satisfies B-(m, η)-C-R to an ideal.

Corollary 2 : Let X be a B-A- module. X is (m,η)-S-F-S-B-A-M-R to an ideal if and only if $\ell_X{}^{\eta} r_{A\eta}(N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A) = N_{\dot{x}_1}A + N_{\dot{x}_1,\dot{x}_2}A + \ldots + N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}A \cap X^{m}H$ for n-element subset $\{N_{\dot{x}_1}, N_{\dot{x}_1,\dot{x}_2}, \cdots, N_{\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_\eta}\}$ of X^{η}.

Following (1) "suppose that *A* is a unital B-Ạ and assume α > 1.Algebra-module X is said Quasi α-injective (Q-α-inj), if algebra-module homomorphism *φ* :$\dot{N} \longrightarrow X$ s.t ‖ *φ* ‖ ≤ 1and there is algebra-module homomorphism *θ: X⟶X,* s.t *θ o i = φ* and ‖ *θ* ‖ ≤ α, *i* is an isometry from submodule Ṅ of X. Call *X* is –inj, if it is Q- α - inj for some α".

Following (1), assume that *A* is unital B- Ạ and suppose that α >1. Algebra−module X is said to be Quasi-α-injective relate to an ideal H of algebra if,

φ : $\dot{N} \longrightarrow X$ is algebra-module homomorphism s.t 1 ≥ ‖*φ* ‖, and there is algebra-module homomorphism *θ:X⟶X,* s.t (*θ o i*)(n) –*φ*(n) ∈ X*H* and α ≥ ‖*θ* ‖ where *i* is an isometry from submodule Ṅ of X to X.

The concepts strongly Quasi-(m,η)- α-injective -B-A- module related to ideal for some α is introduced.

Definition 3 : Suppose that *A* is a unital B-A and 1< α. X is said to be strongly Quasi- (m,η)-α-injective relate to an ideal I of $A^{m\times n}$ if β: $\dot{N} \rightarrow X^{\eta}$ is algebra-module homomorphisms such that 1≥‖*β*‖ , there is *α* :$X^{\eta} \longrightarrow X^{\eta}$ algebra-module homomorphism, such that (*α o i*)(n) –*β*(n) ∈ $X^{m}I$and 1 ≥‖*θ* ‖ , *i* is an isometry from m-generated submodule $\dot{N}$ in *X*. *X* is strongly Quasi-(m,η)-injective relate to an ideal I ,if *X* is strongly – Quasi - (m,η)- α– injective relate to ideal for some α.

Proposition 4 : If X is (m,η)-S-F-S-B-A-M-R to I ideal of an algebra, then X is strongly Quasi (m,η)-inective B- algebra module relate to an ideal I.

Proof : set N = $\alpha_1 A + \ldots + \alpha_n A$, m-generated sub-module of X^{η}, $\alpha_i \in X^{\eta}$,let α be greater than 1and f be any algebra-modulehomomorphism from N to X^{η} such that ‖*f* ‖ ≤ 1. Since X(m,η)-S-F-S-R to ideal, therefore $f(\alpha_1 A + \ldots + \alpha_n A) \subseteq \alpha_1 A + \ldots + \alpha_n A \cap X^{m}I$,

thus there is t = (t_1, …, $t_{ɳ}$) ∈$Ạ_{ɳ}$ and w ∈ $Ẋ^{ɱ}I$. Let a_i = (0, …., 1, 0, …., 0) such that f($\sum_{i=1}^{n}\alpha_i$) = t ($\sum_{i=1}^{ɳ}\alpha_i$) + w. Define g :$Ẋ^{ɳ}$ → Ẋ as g (α_i) = $t^T\alpha_i$, clearly g is well defined algebra-module homomorphism. Now f($\sum_{i=1}^{ɳ}\alpha_i$) - g($\sum_{i=1}^{ɳ}\alpha_i$) = t ($\sum_{i=1}^{ɳ}\alpha_i$) + w - t ($\sum_{i=1}^{n}\alpha_i$) = w ∈ $X^m I$and since for all y ∈α_1Ạ + ….+ α_nẠ, y = $\sum_{i=1}^{ɳ}\alpha_i s_i$for some s = ($s_1$,…, s_n) ∈ A, f(y) − g(y) = f($\sum_{i=1}^{ɳ}\alpha_i s_i$) − $g(\sum_{i=1}^{ɳ}\alpha_i s_i)$ = f(($\sum_{i=1}^{ɳ}\alpha_i$) s) - g(($\sum_{i=1}^{ɳ}\alpha_i$)s) = (f ($\sum_{i=1}^{ɳ}\alpha_i$) − g($\sum_{i=1}^{ɳ}\alpha_i$))s ∈ $X^m I$, therefore X is strongly quasi (m,n)-banach algebra module relative to ideal.

Definition 4: A sub-module Ṅ of Banach Ạ-module is called pure-(ɱ, ɳ)- sub-module if IṄ = Ṅ∩$Ẋ^{ɱ}$ I ∀I of $Ạ^{ɱ×ɳ}$.

When the sub-module of (ɱ, ɳ)-S-F-S-B-A-M-R to ideal have been partial answer in the next proposition .

Proposition 5: Let Ẋ be a (ɱ, ɳ)-S-F-S-B-A-M-R to a non-zero ideal I of $Ạ^{ɱ×ɳ}$, then every (ɱ,ɳ)-pure sub-module is (ɱ, ɳ)-S-F-S-B-A-M-R to an ideal.

Proof: Assume that Ṅ is pure-(ɱ,ɳ)- sub-module of Ẋ. For every sub-module L of Ṅ and a multiplier f: L → Ṅ, put g=i o f: L → Ẋ(where i is the inclusion mapping of Ṅ to X),then by assumption f(L) = g(L) ⊆$Ẋ^{ɱ}$I , since f(L) ⊆ N. Hence f (L) ⊆L ∩ $Ẋ^{ɱ}$I ∩ Ṅ. Because Ṅ is pure (ɱ, ɳ)-sub-module of Ẋ then Ṅ ∩ $Ẋ^{ɱ}$I =$Ṅ^{ɱ}$I, for all ideal I of $A^{m×n}$, therefore f(L) ⊆L ∩$Ṅ^{ɱ}$I .Therefore N is (ɱ, ɳ)-S-F-S-B-A-M-R to I.

Conclusion:

In this work, the concept of (ɱ,ɳ) strong full stability B-Algebra-module related to a non-zero ideal I of $Ạ^{ɱ×ɳ}$ has been introduced and it is also easy to study its properties by linking it with other concepts. The relationship of (ɱ,ɳ) strong full stability B-Algebra-module related to an ideal that states, if Ẋ is (ɱ,ɳ)- strong full stability B-Algebra-module related to an ideal I of an algebra, then Ẋ is strongly Quasi (ɱ,ɳ)-inective B- algebra module relate to an ideal I have been proved, and show that every (ɱ,ɳ)-pure sub-module of (ɱ,ɳ) strong full stability B-Algebra-module related to a non-zero ideal I of $Ạ^{ɱ×ɳ}$ is (ɱ,ɳ) strong full stability B-Algebra-module related to a non-zero ideal I of $Ạ^{ɱ×ɳ}$

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

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حول مقاسات بناخ الاجبرا تامة الاستقرارية من النمط (ɱ, ɳ) بالنسبة الى مثالي $A^{m\times ɳ}$

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الخلاصة:

في هذا البحث تم دراسة مفهوم مقاسات بناخ الاجبرا تام الاستقراية من النمط (ɱ, ɳ) بالنسبة الى مثالي $A^{m\times ɳ}$ و دراسة بعض خواصه.قد تم برهنت العديد من العلاقات منها يكون المقاس X تام الاستقرارية من النمط (ɱ,ɳ) بالنسبة الى مثالي Ḥ اذا وفقط اذا لاي محموعتين جزئيتين من العناصر من النمط $\{ Ṋ_{ẋ_1}, Ṋ_{ẋ_1,ẋ_2}, \cdots, Ṋ_{ẋ_1,ẋ_2,\cdots,ẋ_ɳ}\}$ ɱ و $\{Ḿ_{ẏ_1}, Ḿ_{ẏ_1,ẏ_2}, \cdots, Ḿ_{ẏ_1,ẏ_2,\cdots,ẏ_ɳ}\}$ من $Ẋ^{ɳ}$ و ادا كان $\beta_j \notin \sum_{i=1}^{n}\alpha_i A \cap Ẋ^{ɱ}Ḥ$ لكل i = 1,…, ɳ و j = 1, …, ɱ, و $\alpha_i \in \{ Ṋ_{ẋ_1}, Ṋ_{ẋ_1,ẋ_2}, \cdots, Ṋ_{ẋ_1,ẋ_2,\cdots,ẋ_ɳ}\}$ و $\beta_j \in \{Ḿ_{ẏ_1}, Ḿ_{ẏ_1,ẏ_2}, \cdots, Ḿ_{ẏ_1,ẏ_2,\cdots,ẏ_ɳ}\}$ يؤدي الى $r_{Ạɳ}(\{Ḿ_{ẏ_1}, Ḿ_{ẏ_1,ẏ_2}, \cdots, Ḿ_{ẏ_1,ẏ_2,\cdots,ẏ_ɳ}\}) \not\subseteq r_{Ạɳ}(\{ Ṋ_{ẋ_1}, Ṋ_{ẋ_1,ẋ_2}, \cdots, Ṋ_{ẋ_1,ẋ_2,\cdots,ẋ_ɳ}\})$

الكلمات المفتاحية : مقاسات بناخ الاجبرا تامة الاستقراية بالنسبة الى مثالي, مقاسات بناخ الاجبرا جداء مباشر من النمط (ɱ, ɳ) بالنسبة الى مثالي, مقاسات بناخ الاجبرا تام الاستقراية من النمط (ɱ, ɳ) بالنسبة الى مثالي, مقاسات جزئية خالصة من مقاسات بناخ النمط (ɱ, ɳ).