On (m,n)-Strongly Fully Stably Banach Algebra Modules Related to an Ideal of $A^{m \times R}$

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Received 3/2/2020, Accepted 27/9/2020, Published Online First 30/4/2021

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Abstract:

The aim of this paper is introducing the concept of (m,n) strongly full stability B-Algebra-module related to an ideal. Some properties of (m,n)-strong full stability B-Algebra-module related to an ideal have been studied and another characterizations have been given. The relationship of (m,n) strongly full stability B-Algebra-module related to an ideal that states, a $B$-module $X$ is $(m,n)$-strong full stability $B$-Algebra-module related to an ideal $H$, if and only if for any two $m$-element sub-sets $\{N_1, N_2, N_3, \ldots, N_m\}$ and $\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \ldots, \tilde{N}_m\}$ of $X^n$, if $\beta_j \notin \sum_{i=1}^{m} \alpha_i A \cap X^n H$, for each $j = 1, \ldots, m$, $i = 1, \ldots, n\alpha_i$ $N_i \in \{N_1, N_2, N_3, \ldots, N_m\}$ and $\tilde{\beta}_j \in \{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \ldots, \tilde{N}_m\}$ implies $\sigma(A)(\{N_1, N_2, N_3, \ldots, N_m\}) \subseteq \sigma(A)(\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \ldots, \tilde{N}_m\})$ have been proved.

Keywords: Baer-(m,n)-criterion related to a ideal, F-S-B-A-module related to an ideal, (m,n)-full-stable-B-A-module related to ideal, Multiplication-(m,n)-B-A-module related to ideal, Pure-(m,n)-sub-module.

Introduction:

An algebra is a set $A \neq \emptyset$ and if the following conditions are satisfied, the set $A$ with addition and multiplication are satisfied by a domain $F$ a space of vectors. A $\alpha (a \delta) ^{\alpha \delta} a \delta = \alpha a \delta$ $a \delta = \alpha (a \delta)$ for all $a \in F$, $\alpha a \delta$ is defined, and the set $A$ with $\alpha$ and $\delta$ forms a ring by $\alpha -$. $\mathcal{R}$ is called an algebra where $\mathcal{R}$ is a ring, $[\mathcal{R}, +, -, 0]$ such that $\alpha$ and $\delta$ are binary operations, and the nullary element is 0 satisfying, $[\mathcal{R}, +, -, 0]$ group which is commutative, $[\mathcal{R}, \cdot]$ which is a semi-group and $\alpha (\delta + \delta) = (\alpha \delta) + (\delta \alpha)$ and $\alpha (\delta + \delta) = (\delta \alpha) + (\delta \alpha)$. Suppose that $A$ is an algebra, recall that a $B$-algebra left module ($B$-A-left module) is a B-space $E$ isomorphic to $E$ is an algebra-left module, and $\|a\| \|x\| \geq \|a \cdot x\|$ $(a \in A, x \in E)$ according to (1). Following (2) a map from a B-algebra left module $X$ into a $B$-algebra left module $\tilde{Y}$ (algebra $\tilde{A}$ is not necessary abelian) is called a $A$-multiplier (homorphism) if it satisfies $\forall b \in A, x \in X, \tilde{b}(a, x) = a \cdot x$. In (1), a sub-module $N$ in $M$ is said to be stable, if $N \supseteq f(N) \forall R$ -homomorphism f from sub-module $N$ into module $M$. $M$ is called full stability $\mathcal{R}$-module, if each sub-module in $M$ is stable. Assume that $X$ is a $B$-algebra module, $Y$ is called F-S-B-algebra module related to an ideal $K$ of algebra $A$, if $\forall$ sub-module $N$ in $X$ and $\forall$ multiplier $\theta : N \rightarrow X$ holds $N + \theta(N) \subseteq \theta(N)^{(1)}$. Let $\mathbb{F}^{m \times n}$ be the collection of every matrices $m \times n$ over a ring $\mathcal{R}$. $\mathcal{A} \in \mathbb{F}^{m \times n}$, denote $\tilde{A}$ is transpose of $\mathcal{A}$. In general, write $\tilde{N}^{m \times n}$ for an $\mathcal{R}$-module $N$, $\tilde{N}$ is the collection of all matrices $m \times n$ where all elements in $N$. Suppose that $M$ a right Banach Algebra-module and let $\mathcal{N}$ be a left $\mathcal{R}$-module. Let $\tilde{\mathcal{A}} \in M^{(x \times n)}$, $\mathcal{S} \in \tilde{\mathcal{N}}^{m \times k}$ and $\tilde{Y} \in M^{(k \times x)}$, with multiplication, $\tilde{X} \tilde{S}$ (resp. $\tilde{Y}$) is a good defined element in $M^{(x \times n)}$ (resp. $N^{m \times k}$). If $\tilde{\mathcal{A}} \in M^{(x \times n)}$, $\mathcal{S} \subseteq \tilde{\mathcal{N}}^{m \times n}$ and $\tilde{\mathcal{Y}} \in \mathcal{N}^{m \times k}$ are define $\mathcal{S} = \{\tilde{\mathcal{S}} : \tilde{\mathcal{S}} \in \tilde{\mathcal{N}}^{m \times k} | x \mathcal{S} = 0 \}$ for all $\tilde{\mathcal{S}} \in \tilde{S}$ $\mathcal{r}_{\tilde{\mathcal{N}}^{m \times k}}(\mathcal{S}) = \{\tilde{\mathcal{S}} \in \tilde{\mathcal{N}}^{m \times k} | x \tilde{\mathcal{S}} = 0 \}$ for all $\tilde{S} \in \tilde{\mathcal{S}}$ $\mathcal{r}_{\tilde{\mathcal{N}}^{m \times k}}(\mathcal{S}) = \{\tilde{\mathcal{S}} \in \tilde{\mathcal{N}}^{m \times k} | x \tilde{\mathcal{S}} = 0 \}$ for all $\tilde{\mathcal{S}} \in \tilde{\mathcal{S}}$ $\tilde{\mathcal{Y}} \in \mathcal{N}^{m \times k}$ and $\tilde{X} \in \mathcal{N}^{m \times k}$ are define $\mathcal{S} = \{\tilde{\mathcal{S}} : \tilde{\mathcal{S}} \in \tilde{\mathcal{N}}^{m \times k} | x \mathcal{S} = 0 \}$ for all $\tilde{X} \in \tilde{X}$ $\tilde{\mathcal{X}} \tilde{\mathcal{Y}} = \{\tilde{\mathcal{S}} \in \tilde{\mathcal{N}}^{m \times k} | x \tilde{\mathcal{S}} = 0 \}$ for all $\tilde{X} \in \tilde{X}$ $\mathcal{Y} = \{\tilde{\mathcal{Y}} : \tilde{\mathcal{Y}} \in \mathcal{N}^{m \times k} | x \tilde{\mathcal{Y}} = 0 \}$ for all $\tilde{X} \in \tilde{X}$ Write $N^{m \times n}$, $\tilde{N}^{m \times n}$, $\tilde{N}^{m \times k}$ $\tilde{N}^{m \times k}$ in (3). In our work for fixed positive integers $m,n$ the concept of (m,n)-full stability Banach Algebra modules relative to an ideal have been introduced.
(m, n)-Strongly-Fully-Stable-Banach-Algebra Modules Related to ideal
A left B-algebra-module $X$ is $n$-generated where $n \in N$ if there is exist $x_1, \ldots, x_n \in X$ such that for all $\lambda \in X$ can be represented $\lambda = \sum_{k=1}^{n} \lambda_k x_k$ for some $\lambda_1, \ldots, \lambda_n$ in an algebra. A module which is $1$-generated is called a cyclic module (4). A right module over $R$, $M$ is called strongly fully $(m, n)$-stable relative to an ideal $A$ of $R^{m,n}$, if $\forall \eta \in M^n$ $A \supseteq \theta(\eta)$ for all $n$-generated sub-module of $M^n$ and $\theta : N \to M$ $R$-homomorphism (5).

Definition 1: Let $K$ be B-A-module , $K$ is called $(m, n)$-S-F-S-B-A-M-R to ideal $H$ of $A^{m,n}$, if for every $m$-generated sub-module of $K^n$ and for each multiplier $\theta : J \to K^n$ which satisfies $\theta(\eta) \subseteq \eta \cap H$ for two fixed positive integers $m, n$.

In (1) "Let $M$ be nonempty subset of a left B-A-module $X$, the annihilator $ann_A(M)$ of B-A-module $M$ is $\{a \in A : \forall \lambda \in X, \; \lambda a = 0 \}$ for all $\lambda \in X \in M\} = ann_A(M)$.

Notation 1:
Suppose that $X$ be a B-algebra-module
1) $\mathcal{N}_{x_1, x_2, \ldots, x_n} = (\theta(x_i))_i \in N$, $x_i \in X, i = 1, 2, \ldots, n$.
2) $\mathcal{N}_{x_1, x_2, \ldots, x_n} = \{m \in M, \; \gamma_i \in \gamma, \; i = 1, 2, \ldots, n \}$

Definition 2: A B-A-module $X$ is hold $\theta(\eta)$ is $n$-generated in algebra. A module which is $1$-generated relative to ideal $A$ of $R^{m,n}$, if for all $n$-generated sub-module of $X$ such that $\theta(\eta)$ is $1$-generated relative to $A$. Moreover, $\theta(\eta)$ is $1$-generated relative to $A$ is called hold $\theta(\eta)$ is $1$-generated relative to $A$.

Corollary 1: If $X$ is $(m, n)$-S-F-S-B-A-M-R to ideal $H$ of $A^{m,n}$, then $\forall \eta \in X^n$ such that $\theta(\eta)$ is $1$-generated relative to $A$.

Proof: The proof is clear in (2). A B-A-module $X$ is hold Baer criterion (B-C) if all submodule of $X$ holds Baer criterion, this mean that for every submodule $N$ in $X$ and algebra multiplior $N \to X$, so $\exists \beta \in A$ s.t $\theta(\eta) = \beta \forall \eta \in N$.

Definition 2: A B-algebra module $X$ is called hold Baer-(m, n)-criterion relates (B-(m, n)-C-R) to an ideal $H$ if each sub-module of $X$ satisfies $B-(m, n)-C-R$ to an ideal $H$.
generated sub-module of \( X^n \) and \( A \)-multiplier \( \theta : L \to X^n \), there is \( \theta \) such that \( \theta(l) = \delta l \in X^n \) for all \( l \in L \).

**Proposition 2**: If \( X \) satisfies \( B-(m,n) \)-C-R to ideal and \( \sigma(M) = \sigma_{L}(L \cap M) + \sigma_{R}(M) \) for each \( m \)-generated sub-modules of \( X^n \), then \( X \) satisfies \( B-(m,n) \)-C-R to ideal.

**Proof**: Let \( P = A\delta x_{1} + A\delta x_{2} + \ldots + A\delta y_{m} \) be \( m \)-generated sub-module of \( X^n \), \( f : P \to X^{n} \) multiplier. Now, by induction on \( m \). Clearly that \( \chi \) holds \( B-(m,n) \)-C-R to ideal, if \( m = 1 \). Suppose that \( \chi \) satisfies \( B-(m,n) \)-C-R to ideal for each \( k \)-generated sub-module of \( X^n \), for \( n \geq k \). Write \( L = A\delta x_{1} + A\delta x_{2} + \ldots + A\delta y_{m} \), therefore for each \( w_{i} \in W \) and \( w_{j} \in M \) \( f \ket{(w_{i})} = y_{1}w_{i} \), \( f \ket{(w_{j})} = y_{2}w_{j} \) for some \( y_{1}, y_{2} \in A \). It is clear \( y_{1}y_{2} \in \sigma_{L}(L \cap M) + \sigma_{R}(M) \). Suppose that \( y_{1}y_{2} = z_{1} + z_{2} \), \( z_{1} \in \sigma_{L}(L), z_{2} \in \sigma_{R}(M) \) and let \( y = x_{1} = x_{2} + x_{3} \), then for any \( w = w_{i} + v_{2} + P \in W \) and \( w_{2} \in M \), \( f(w) = f(w_{1}) + f(v_{2}) = w_{1} + w_{2}y = w_{1}y_{1}x_{1} + w_{2}y_{2}x_{3} = w_{1}(y_{1}x_{1}) + w_{2}(y_{2}x_{2}) \). Hence \( \chi \) satisfies \( B-(m,n) \)-C-R to ideal.

**Corollary 2**: Let \( \chi \) be a \( B \)-A-module. \( \chi \) is \( (m,n) \)-S-F-S-B-A-M-R to ideal if and only if \( \sigma(M) = \sigma_{L}(M) + \sigma_{R}(M) \). The concepts strongly quasi-\( (m,n) \)-\( \alpha \)-injective module related to ideal \( H \) of algebra if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

Following (1) "suppose that \( \alpha \) is a unital \( B \)-A and assume \( \alpha > 1 \). Algebra-module \( \chi \) is said to be quasi-\( \alpha \)-injective relate to ideal \( H \) of algebra if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

Thus \( \chi \) is quasi-\( (m,n) \)-\( \alpha \)-injective -B-A-module related to ideal for some \( \alpha \) is introduced.

**Definition 3**: Suppose that \( \alpha \) is a unital \( B \)-A and \( 1 \leq \alpha \). \( \chi \) is said to be strongly quasi-\( (m,n) \)-\( \alpha \)-injective relate to an ideal \( I \) of \( A \)-module if \( \sigma(E) = \sigma_{L}(E) + \sigma_{R}(E) \). Then \( \chi \) is quasi-\( (m,n) \)-\( \alpha \)-injective module related to ideal if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

In ideal for some \( \alpha \) if and only if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

The concepts strongly quasi-\( (m,n) \)-\( \alpha \)-injective relate to ideal \( 1 \) if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

Proposition 4**: If \( \chi \) is \( (m,n) \)-S-F-S-B-A-M-R to I ideal of an algebra, then \( \chi \) is strongly quasi-\( (m,n) \)-\( \alpha \)-injective relate to ideal I if

\[
\begin{align*}
\sigma_{L}(L \cap M) + \sigma_{R}(M) = \sigma(M) = \sigma_{L}(L) + \sigma_{R}(M)
\end{align*}
\]

Proof**: Set \( N = a_{1}A + \ldots + a_{n}A \), \( m \)-generated sub-module of \( X^{n} \), \( a \in X^{n} \), \( \alpha \) be greater than \( 1 \) and \( f \) be any algebra-module-homomorphism from \( N \) to \( X^{n} \) such that \( ||f|| \leq 1 \). Since \( (m,n) \)-S-F-S-R to ideal, therefore \( f(\alpha_{1}A + \ldots + a_{n}A) \leq a_{1}A + \ldots + a_{n}A \cap X^{n} \),
thus there is \( t = ( t_1, \ldots, t_n) \in \mathbb{A}_n \) and \( w \in X^n I \). Let \( a_i = (0, 0, \ldots, 1, 0, \ldots, 0) \) such that \( f(\sum_{i=1}^n a_i) = t \). Define \( g : X^n \rightarrow X \) as \( g(a_i) = t^i a_i \), clearly \( g \) is well defined algebra-module homomorphism. Now \( f(\sum_{i=1}^n a_i) - g(\sum_{i=1}^n a_i) = t - w \). Since \( f(\sum_{i=1}^n a_i) - g(\sum_{i=1}^n a_i) = w \in X^m I \) and since for all \( x \in \mathbb{A}_n \), \( f(x) - g(x) = f(\sum_{i=1}^n a_i) - g(\sum_{i=1}^n a_i) = (f - g)(\sum_{i=1}^n a_i) = w = x \). Therefore \( X \) is strongly quasi \((m,n)\)-banach algebra module relative to ideal.

**Definition 4**: A sub-module \( N \) of Banach \( A \)-module is called pure-(\( m,n \))-sub-module if \( N = N \cap X^n I \cap I \) for some \( s = \mathbb{A}_n \).

When the sub-module of \((m,n)\)-S-F-S-B-A-M-R to ideal have been partial answer in the next proposition.

**Proposition 5**: Let \( X \) be a \((m,n)\)-S-F-S-B-A-M-R to a non-zero ideal \( I \) of \( A^{m,n} \), then every \((m,n)\)-pure sub-module is \((m,n)\)-S-F-S-B-A-M-R to ideal.

**Proof**: Assume that \( N \) is pure-(\( m,n \))-sub-module of \( X \). For every sub-module \( L \) of \( N \) and a multiplier \( f: L \rightarrow N \), put \( g = f o i: L \rightarrow X \) where \( i \) is the inclusion mapping of \( N \) to \( X \), then by assumption \( f(L) = g(L) \subseteq X^n I \), since \( f(L) \subseteq N \). Hence \( f(L) \subseteq L \cap X^n I \cap N \). Because \( N \) is pure-(\( m,n \))-sub-module of \( X \) then \( N \cap X^n I = N^m I \), for all ideal \( I \) of \( A^{m,n} \), therefore \( f(L) \subseteq L \cap N^m I \). Therefore \( N \) is \((m,n)\)-S-F-S-B-A-M-R to \( I \).

**Conclusion**:

In this work, the concept of \((m,n)\) strong full stability \( B\)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m,n} \) has been introduced and it is also easy to study its properties by linking it with other concepts. The relationship of \((m,n)\) strong full stability \( B\)-Algebra-module related to an ideal that states, if \( X \) is \((m,n)\)-strong full stability \( B\)-Algebra-module related to an ideal \( I \) of an algebra, then \( X \) is strongly Quasi \((m,n)\)-inactive \( B\)-algebra module relative to an ideal I have been proved, and show that every \((m,n)\)-pure sub-module of \((m,n)\) strong full stability \( B\)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m,n} \) is \((m,n)\) strong full stability \( B\)-Algebra-module related to a non-zero ideal \( I \) of \( A^{m,n} \).

**Authors' declaration**:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

**References**: