Comparison of Some of Estimation methods of Stress-Strength Model: \( R = P(Y < X < Z) \)

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Abstract:

In this study, the stress-strength model \( R = P(Y < X < Z) \) is discussed as an important parts of reliability system by assuming that the random variables follow Invers Rayleigh Distribution. Some traditional estimation methods are used to estimate the parameters namely; Maximum Likelihood, Moment method, and Uniformly Minimum Variance Unbiased estimator and Shrinkage estimator using three types of shrinkage weight factors. As well as, Monte Carlo simulation are used to compare the estimation methods based on mean squared error criteria.

Key words: Stress-Strength Model, Inverse Rayleigh Distribution, Least Square Estimator, Maximum Likelihood Method Estimator, Moment method, Uniformly Minimum Variance Unbiased Method, Shrinkage Methods.

Introduction:

The many technological and mechanical malfunction problems that emerge due to the constant developments of life might not be the only reason behind the study of reliability and stress-strength models as a parts of reliability system since these models have several applications in different sciences areas, like reliability design of systems, quality control, physics, engineering and medicine. According to Rao ‘the reliability can be defined as ‘the probability of a device performing its function over a specified period of time and under specified operating conditions’(1). The probability equation: \( R = P(Y < X) \) of “stress-strength” reliability shows that if the component which has a random strength represented by \( X \) is greater than the stress represented by the random variables \( Y \) where both of \( X \) and \( Y \) are independent then the system works statistically. Statistical studies of stress-strength reliability system were introduced for the first time in 1956 by Birnbaum (2). This paper is interested in the model \( R = P(Y < X < Z) \) that discusses the case where the strength \( X \) should not only be greater than stress \( Y \) but also be smaller than stress \( Z \), for example person’s blood pressure has two limits systolic and diastolic and his/her blood pressure should lie within these limits (3). Singh in 1980, constructed maximum likelihood (ML), the minimum variance unbiased (MVU), and empirical estimators of stress-strength model \( R = P(Y < X < Z) \) by assuming that strength of a component lies in an interval where \( Y, X \) and \( Z \) are independent random variable based on normal distribution (4). After that in 2013 Wang et al. made statistical inference for \( R \) using nonparametric normal- approximations and the jackknife empirical likelihood under the assumption that the three samples were, independent, without ties among them (5). The Maximum likelihood estimator, moment estimator (ME) and mixture estimator (Mix) for the estimation of \( R=(Y < X < Z) \) and the stresses \( Y \) and \( Z \) and the strength \( X \) have Weibull distribution with common known shape and scale parameters and \( X, Y \) and \( Z \) are independent constructed in 2013 by Amal et al.(6). In 2016 Patowary et al. discussed the technique of Reliability estimation for \( P(Y < X < Z) \) of n-standby system (n=1, 2), through Monte-Carlo
system strength is given by (7). In this work, the estimation of the system reliability \( P(Y < X < Z) \) based on the Inverse Rayleigh Distribution is considered. In addition, Monte Carlo simulation is performed for comparing several methods.

Model Description
Inverse Rayleigh Distribution (IRD) is first suggested by Trayer in 1964 see (8) and it has many applications in the area of reliability studies. The probability density function (pdf) and the corresponding cumulative distribution function (CDF) of one parameter Inverse Rayleigh Distribution are respectively defined as:

\[
f(x; \gamma) = \frac{2y}{x^3} \exp \left( -\frac{y^2}{x^2} \right) \quad ; \quad x > 0 \quad , \quad \gamma > 0
\]

\[= f_0 \int_0^\infty F_y(x) \, f_2(x) \, f(x) \, dx = \int_0^\infty F_y(x) \left[ 1 - F_2(x) \right] \, f(x) \, dx
\]

\[= f_0 \int_0^\infty F_3(x) \, f(x) \, dx - \int_0^\infty F_3(x) \, F_2(x) \, f(x) \, dx
\]

\[= \int_0^\infty e^{-\left( \frac{y_2}{x} \right)^2} \, e^{-\left( \frac{y_1}{x} \right)^2} \, dx - \int_0^\infty e^{-\left( \frac{y_2}{x} \right)^2} \, e^{-\left( \frac{y_3}{x} \right)^2} \, dx
\]

\[= \frac{y_1}{y_1+y_2} \quad \frac{y_3}{y_1+y_2+y_3}
\]

So that:

\[
R = \frac{y_1 y_3}{(y_1+y_2)(y_1+y_2+y_3)}
\]

Maximum Likelihood Estimation of R
The Maximum likelihood method is an important and commonly, since it contained properties for good estimate and have invariant property . In this section, the MLE obtained for \( R = P(Y < X < Z) \) throw the derivation of scale parameters \( y_1, y_2, y_3 \) of the r.v. \( X, Y, Z \) as following: Let \( x_1, x_2, \ldots, x_n \) be a random strength sample of size n with pdf. as in (eq.3) then let \( y_1, y_2, \ldots, y_m \) and \( z_1, z_2, \ldots, z_m \) be the random stresses with pdf. as in eq.4 and eq.5 respectively, the Maximum Likelihood function of the observed sample is:

\[
L(y_1, y_2, y_3 | data) = \prod_{i=1}^n f(x_i) \cdot \prod_{j=1}^{m_1} f(y_j) \cdot \prod_{k=1}^{m_2} f(z_k)
\]

\[
= \prod_{i=1}^n \frac{2y_1}{x_i^3} e^{-\left( \frac{y_1}{x_i} \right)^2} \cdot \prod_{j=1}^{m_1} \frac{2y_2}{y_j^3} e^{-\left( \frac{y_2}{y_j} \right)^2} \cdot \prod_{k=1}^{m_2} \frac{2}{z_k^3} e^{-\left( \frac{y_3}{z_k} \right)^2}
\]

\[
= (2y_1)^n + \prod_{i=1}^n \frac{1}{x_i^3} e^{-\left( \frac{y_1}{x_i} \right)^2} \cdot (2y_2)^{m_1} \prod_{j=1}^{m_1} \frac{1}{y_j^3} e^{-\left( \frac{y_2}{y_j} \right)^2} \cdot (2y_3)^{m_2} \prod_{k=1}^{m_2} \frac{1}{z_k^3} e^{-\left( \frac{y_3}{z_k} \right)^2}
\]

Taking the logarithm of likelihood function eq.7 then differentiating the result partially with respect to \( y_1, y_2, y_3 \) and equalizing to zero respectively to get the estimated parameters \( \hat{y}_1, \hat{y}_2, \hat{y}_3 \) as follows:

\[
\frac{\partial \ln L}{\partial y_1} = \frac{n}{y_1} - \sum_{i=1}^n \frac{1}{x_i^2} = 0
\]

\[
\frac{\partial \ln L}{\partial y_2} = \frac{m_1}{y_2} - \sum_{j=1}^{m_1} \frac{1}{y_j^2} = 0
\]

\[
\frac{\partial \ln L}{\partial y_3} = \frac{m_2}{y_3} - \sum_{k=1}^{m_2} \frac{1}{z_k^2} = 0
\]
thus
\[ \hat{y}_{1MLE} = n \frac{1}{r_1} \]
\[ \hat{y}_{2MLE} = m_1 \frac{1}{r_2} \]
\[ \hat{y}_{3MLE} = m_2 \frac{1}{r_3} \]
\[ \hat{R}_{MLE} = \frac{(\hat{y}_{1MLE}+\hat{y}_{2MLE})(\hat{y}_{1MLE}+\hat{y}_{2MLE}+\hat{y}_{3MLE})}{(\hat{y}_{1MLE}+\hat{y}_{2MLE})+\hat{y}_{3MLE}} \]

When substituting the equations 8, 9 and 10 in \( \hat{R}_{MLE} \)
this leads to the estimation of the stress-strength model \( R = P(Y < X < Z) \) using Maximum
likelihood Estimator as below
\[ \hat{R}_{MLE} = \frac{n m_1 m_2}{n^2 + m_1 m_2 + m_3} \]

Moments Method Estimator
This method is one of conventional methods because its ease. The basic idea of this method is to
equate certain sample characteristics, such as the
mean, to the corresponding population expected
values. Then solving these equations for unknown
parameter values yields the estimators. The sample
mean of the r.v. \( X, Y, Z \) of IRD with unknown scale
parameters \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) respectively are given by
\[ \hat{X} = \frac{1}{n} \sum_{i=1}^{n} x_i \]
\[ \hat{Y} = \frac{1}{m_1} \sum_{j=1}^{m_1} y_j \]
\[ \hat{Z} = \frac{1}{m_2} \sum_{k=1}^{m_2} z_k \]
While \( \gamma^t \)th moments will be:
\[ E(x^\gamma) = \Gamma \left(1 - \frac{\gamma}{2}\right) y_1^\gamma \]
\[ E(y^\gamma) = \Gamma \left(1 - \frac{\gamma}{2}\right) y_2^\gamma \]
\[ E(z^\gamma) = \Gamma \left(1 - \frac{\gamma}{2}\right) z_3^\gamma \]
Equalize the sample mean with the first moment of \( X \)
, \( Y \) and \( Z \) the estimates of \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) become:
\[ \hat{\gamma}_{1MOM} = \frac{\frac{1}{n} \sum_{i=1}^{n} x_i}{\Gamma \left(1 \right)} \]
\[ \hat{\gamma}_{2MOM} = \frac{\frac{1}{m_1} \sum_{j=1}^{m_1} y_j}{\Gamma \left(1 \right)} \]
\[ \hat{\gamma}_{3MOM} = \frac{\frac{1}{m_2} \sum_{k=1}^{m_2} z_k}{\Gamma \left(1 \right)} \]
\[ \hat{R}_{MOM} = \frac{\hat{y}_{1 MOM} \hat{y}_{2 MOM}}{\hat{y}_{1 MOM} + \hat{y}_{2 MOM} + \hat{y}_{3 MOM}} \]

Substituted eq.12, 13 and eq.14 in eq.6 to obtain the
approximate moment estimator \( \hat{R}_{MOM} \) for stress-strength
reliability (\( R \)) as follows:
\[ \hat{R}_{MOM} = \frac{\hat{y}_{1 MOM} \hat{y}_{2 MOM}}{\hat{y}_{1 MOM} + \hat{y}_{2 MOM} + \hat{y}_{3 MOM}} \]

Uniformly Minimum Variance Unbiased Estimators.
An unbiased estimator \( \hat{\gamma} \) of \( \gamma \) is called the
uniformly minimum variance unbiased estimator
(UMVUE) if and only if \( \text{Var}(\hat{\gamma}) \leq \text{Var}(\hat{\gamma}_{ub}) \) for any
\( x \in X \) and any other unbiased estimator of \( \alpha \). (15).
The Uniformly Minimum Variance Unbiased Estimator
(UMVUE) of the scale parameters \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) of the random variables \( X, Y \)
and \( Z \) respectively of IRD will be discussed in this
section. The IRD belongs to the exponential family
with densities of the form
\[ f(x; \gamma) = a(y) b(x)e^{\sum_{j=1}^{n} \rho_j(y) k(x)} \]
where \( a(y), b(x) > 0, \gamma < x < \beta \)
and
\[ \gamma = \gamma_1 \gamma_2 ,... \gamma_k \]
with \( \alpha_j < \gamma_j < \delta_j \) and
each of \( \gamma, \beta, \alpha_j \) and \( \delta_j \) are constants
Put \( a(y) = 2y \); \( b(x) = 1/x^3 \); \( \rho_j(y) = -\gamma \)
\( k_j(x) = 1/x^2 \);
then \( T_i \) is a complete sufficient statistic for \( \gamma_i \) for
\( i=1, 2,3,.. \)
Where \( T_1 = \sum_{i=1}^{n} \frac{1}{x_i^2} \); \( T_2 = \sum_{j=1}^{m_1} \frac{1}{y_j^2} \) and \( T_3 = \sum_{k=1}^{m_2} \frac{1}{z_k^2} \)
the distribution of \( T_i \) can be found by following
supposition:
\[ q_1 = \frac{1}{x^2} , q_2 = \frac{1}{y^2} \]
and \( q_3 = \frac{1}{z^2} \)
consequently
\[ \xi(q_1) = \frac{1}{\sqrt{q_1}} \]
\[ \xi(q_2) = \frac{1}{\sqrt{q_2}} \]
\[ \xi(q_3) = \frac{1}{\sqrt{q_3}} \]
Substitute eq.3 in eq.16, eq.4 in eq.17 and eq.5 in
eq.18 to get
\[ \xi(q_1) = \gamma_1 e^{(-\gamma_1 q_1)} \]
\[ \xi(q_2) = \gamma_2 e^{(-\gamma_2 q_2)} \]
\[ \xi(q_3) = \gamma_3 e^{(-\gamma_3 q_3)} \]
So \( q_1 \sim \text{Exp}(\gamma_1) \), \( q_2 \sim \text{Exp}(\gamma_2) \) and \( q_3 \sim \text{Exp}(\gamma_3) \)
and \( T_1 \sim \Gamma(n, \gamma_1), T_2 \sim \Gamma(m_1, \gamma_2) \) and \( T_3 \sim \Gamma(m_2, \gamma_3) \)
the density function of $T_1$ is
$$U(t_1) = \frac{y_1}{\Gamma(n)} t^{n-1} e^{-y_1 t_1} \ ; \ t_1 > 0, \ y_1 > 0, \ n > 0$$
Thus
$$\frac{1}{T_1} = \frac{y_1}{\Gamma(n)}, \text{ and } \frac{n-1}{T_1} \text{ is an unbiased estimator of } (y_1), \text{then by Lehmann-Scheffe theorem the (UMVUE) of } (y_1) \text{ is giving by}$$
$$\hat{y}_1(\text{UMVUE}) = \frac{n-1}{T_1} \ ; \ldots(19)$$
Consequently obtain (UMVUE) of $\gamma_2, \gamma_3$ as follows
$$\hat{y}_2(\text{UMVUE}) = \frac{m_1-1}{T_2} \ ; \ldots(20)$$
$$\hat{y}_3(\text{UMVUE}) = \frac{m_2-1}{T_3} \ ; \ldots(21)$$
Substituted (eq.19,20 and 21) in (eq.6) to obtain the approximate UMVU($\hat{R}_{\text{UMVU}}$) estimator for stress-strength reliability $R$ as follows:
$$\hat{R}_{\text{UMVU}} = \frac{\hat{y}_1(\text{UMVUE}) \hat{y}_3(\text{UMVUE})}{(\hat{y}_1(\text{UMVUE}) + \hat{y}_2(\text{UMVUE}) + \hat{y}_2(\text{UMVUE}) + \hat{y}_3(\text{UMVUE}))} \ ; \ldots(22)$$

**Least Square Estimator (LS)**

The least squares method estimators depending on the idea of minimizing the sum of square error between the value and it’s expected value. Suppose $x_1, x_2, \ldots, x_n$ a random strength sample of size $n$ follows $\text{IRD}(x; y_1)$ and $y_1, y_2, \ldots, y_{m_1}$ and $z_1, z_2, \ldots, z_{m_2}$ are random stresses of size $m_1$ and $m_2$ respectively and both have pdf of IRD as in eq.4, and eq.5 then let :

$$Q_1 = \sum_{i=1}^{n} \left[ F(x_i, y_1) - E(F(x_i, y_1)) \right]^2 = \rho_i \ ; \ldots(23)$$
$$Q_2 = \sum_{i=1}^{m_1} \left[ F(y_j) - E(F(y_j)) \right]^2 = \rho_{1j} \ ; \ldots(24)$$
$$Q_3 = \sum_{j=1}^{m_2} \left[ F(z_j) - E(F(z_j)) \right]^2 = \rho_{2j} \ ; \ldots(25)$$

Where $\rho_i, \rho_{1j}$ and $\rho_{2j}$ are the plotting position such that:
$$\rho_i = \frac{i}{n+1} = E(F(x_i, y_1)) \ ; \ i = 1, 2, \ldots, n$$
$$\rho_{1j} = \frac{j}{m_1+1} = E(F(y_j)) \ ; \ j = 1, 2, \ldots m_1$$
And
$$\rho_{2j} = \frac{j}{m_2+1} = E(F(z_j)) \ ; \ j = 1, 2, \ldots m_1$$

Hence according to CDF formula in (eq. 2)

$$F_x(x_i, y_1) = e^{-\frac{y_1}{x_i^2}} \text{then } \left( F_x(x_i, y_1) \right)^{-1} = e^{\frac{x_i}{y_1}}$$
$$F_y(y_j, y_2) = e^{-\frac{y_2}{y_j^2}} \text{then } \left( F_y(y_j, y_2) \right)^{-1} = e^{\frac{y_j}{y_2}}$$
$$F_z(z_j, y_3) = e^{-\frac{y_3}{z_j^2}} \text{then } \left( F_z(z_j, y_3) \right)^{-1} = e^{\frac{z_j}{y_3}}$$
After simplification and replacing $F_x(x_i, y_1)$, $F_y(y_j, y_2)$ and $F_z(z_j, y_3)$ by plotting position $\rho_i$, $\rho_{1j}$ and $\rho_{2j}$ then equalized to zero, the following equations were obtained:

$$\ln(\rho_i)^{-1} - \frac{y_1}{x_i^2} = 0$$
$$\ldots(26)$$
$$\ln(\rho_{1j})^{-1} - \frac{y_2}{y_j^2} = 0$$
$$\ldots(27)$$
$$\ln(\rho_{2j})^{-1} - \frac{y_3}{z_j^2} = 0$$
$$\ldots(28)$$
Substitution eq.26 in eq.23 to get:
$$\sum_{i=1}^{n} \left[ \ln(\rho_i)^{-1} - \frac{y_1}{x_i^2} \right]^2 = \sum_{i=1}^{n} \left( \ln(\rho_i)^{-1} \right)^2 - 2 \frac{y_1}{x_i^2} \sum_{i=1}^{n} \ln(\rho_i/x_i^2) \ ; \ldots(29)$$
By the same way Substitution eq.27 in eq.24 and eq.28 in eq.25 for getting
$$\sum_{j=1}^{m_1} \left[ \ln(\rho_{1j})^{-1} - \frac{y_2}{y_j^2} \right]^2 = \sum_{j=1}^{m_1} \left( \ln(\rho_{1j})^{-1} \right)^2 - 2 \frac{y_2}{y_j^2} \sum_{j=1}^{m_1} \ln(\rho_{1j}/y_j^2) \ ; \ldots(30)$$
$$\sum_{j=1}^{m_2} \left[ \ln(\rho_{2j})^{-1} - \frac{y_3}{z_j^2} \right]^2 = \sum_{j=1}^{m_2} \left( \ln(\rho_{2j})^{-1} \right)^2 - 2 \frac{y_3}{z_j^2} \sum_{j=1}^{m_2} \ln(\rho_{2j}/z_j^2) \ ; \ldots(31)$$
Substitute eq.29, 30 and 31 in eq.6 to obtain the approximate LS estimator ($\hat{R}_{\text{LS}}$) for stress-strength reliability $R$ as follows:
$$\hat{R}_{\text{LS}} = \frac{\hat{y}_1(\text{LS}) \hat{y}_3(\text{LS})}{(\hat{y}_1(\text{LS}) + \hat{y}_2(\text{LS}) + \hat{y}_2(\text{LS}) + \hat{y}_3(\text{LS}))} \ ; \ldots(32)$$

**Shrinkage Estimation Methods (Sh)**

Thompson in 1968, suggested a new idea for estimate the unknown parameter by shrinking the classical estimator($\hat{y}$) to a prior information(prior estimate) $\alpha_0$ due the past studies or previous experiences using shrinkage weight factor $\Theta(\hat{y})$ as below(16):

$$\hat{y}_{sh} = \Theta(\hat{y}) \hat{y}_0 + (1 - \Theta(\hat{y})) \alpha_0 \ ; \quad 0 \leq \Theta(\hat{y}) \leq 1$$
Where $\hat{y}_{sh}$ is the shrinkage estimator of $\gamma$ and $\hat{y}_{ub}$ is unbiased estimator of $\gamma$ as it is defined previously where $\hat{y}_{ub} = \bar{y}(UMU)$. The factor $\Theta(\hat{y})$ \((0 \leq \Theta(\hat{y}) \leq 1)\), represent a shrinkage weight factor which can be obtained by minimizing the mean squared error of $\hat{y}_{sh}$ or it can be considered as a constant, a function of sample size or a function of $\hat{y}_{ub}$ (ad hoc basis). Noted $\Theta(\hat{y})$ refers to the believe of unbiased estimator $\hat{y}_{ub}$, and $(1 - \Theta(\hat{y}))$ represents to believe of $\gamma_0$. Although the shrinkage estimator is biased, it is well known that it has minimum quadratic risk compared to classical estimators (mostly the maximum likelihood estimator or unbiased estimator); (17). In this section shrinkage estimator obtained according to Thompson’s shrinkage technique also for more details see (18,19 and 20).

Noted that, $E(\hat{y}_{iub}) = \frac{v-1}{T_i}$ and $\text{var}(\hat{y}_{iub}) = \frac{v-2}{T_i}$

Where, $i=1, 2, 3$ and $v$ refer to $n$, $m_1$, $m_2$ or respectively depends on $i$.

Thus, the shrinkage estimator of the scale parameters $\gamma_1, \gamma_2, \gamma_3$ of the random variables $X, Y, Z$ that follows IRD will be as follows:

$$\hat{y}_{sh} = \Theta_1(\hat{y}_i)\hat{y}_{iub} + (1 - \Theta_1(\hat{y}_i))\gamma_{i0}, \quad i=1, 2, 3$$

...(33)

**Constant Shrinkage Weight Function (Sh1)**

Here assume that the constant shrinkage weight factor is equal to the following:

$\Theta_1(\hat{y}_i) = 0.03$ \(:= i=1,2,3$

Then applying in eq. 32 to obtain the following shrinkage estimators

$$\hat{y}_{sh1} = \Theta_1(\hat{y}_1)\hat{y}_{1ub} + (1 - \Theta_1(\hat{y}_1))\gamma_{10}$$

...(34)

$$\hat{y}_{sh2} = \Theta_2(\hat{y}_2)\hat{y}_{2ub} + (1 - \Theta_2(\hat{y}_2))\gamma_{20}$$

...(35)

$$\hat{y}_{sh3} = \Theta_3(\hat{y}_3)\hat{y}_{3ub} + (1 - \Theta_3(\hat{y}_3))\gamma_{30}$$

...(36)

where, $\gamma_{i0}$ \((i=1,2,3)\) are prior information of $\gamma_i$.

Substitute equations 34, 35 and 36 in equation 6 to get the constant shrinkage estimator $\hat{R}_{sh1}$ of S-S reliability ($\hat{R}$) using shrinkage estimator $\hat{R}_{sh}$ as

$$\hat{R}_{sh1} = \frac{\hat{y}_{sh1} \hat{y}_{sh2} \hat{y}_{sh3}}{\hat{y}_{1sh1} \hat{y}_{2sh2} \hat{y}_{3sh3}}$$

...(37)

**Shrinkage weight function (Sh0)***

In this subsection, the shrinkage weight factor has been suggested to be a function of $n, m_1$ and $m_2$ respectively in eq.33 as bellow

$$\Theta_1(\hat{y}_1) = e^{-\frac{n}{n}}$$

$\Theta_2(\hat{y}_2) = e^{m_1}/m_1$

And $\Theta_3(\hat{y}_3) = e^{-m_2}/m_2$

So

$$\hat{y}_{1sh2} = e^{-n}/n \gamma_{1ub} + (1 - e^{-n}/n)\gamma_{10}$$

...(38)

$$\hat{y}_{2sh2} = e^{m_1}/m_1 \gamma_{2ub} + (1 - e^{m_1}/m_1)\gamma_{20}$$

...(39)

$$\hat{y}_{3sh2} = e^{-m_2}/m_2 \gamma_{3ub} + (1 - e^{-m_2}/m_2)\gamma_{30}$$

...(40)

Substitute equations 38, 39 and 40 in equation 6 to get the estimation of reliability models ($\hat{R}_{sh2}$) using shrinkage function estimator as following:

$$\hat{R}_{sh2} = \frac{\hat{y}_{1sh2} \hat{y}_{2sh2} \hat{y}_{3sh2}}{\hat{y}_{1sh1} \hat{y}_{2sh1} \hat{y}_{3sh3}}$$

...(41)

**Modified Thompson Type Shrinkage Weight Function (MThShw)**

In this subsection, the modification to the shrinkage weight factor of Thompson type estimator has been suggested as the following equation:

$$\psi(\hat{y}_{iub}) = \frac{(\hat{y}_{iub} - \gamma_{i0})^2}{(\hat{y}_{iub} - \gamma_{i0}) + \text{var}(\hat{y}_{iub}) (0.03)}$$

for $i = 1,2,3$

Therefore, the modified Thompson type shrinkage estimator will be

$$\hat{y}_{TH} = \psi(\hat{y}_i)\hat{y}_{iub} + (1 - \psi(\hat{y}_i))\gamma_{i0} \text{ for } i = 1,2,3$$

...(42)

Substitute eq.40 in the eq.5 when $i=1,2,3$ then the modified Thompson type shrinkage estimation of the (S-S) reliability is as below

$$\hat{R}_{TH} = \frac{\hat{y}_{1TH} \hat{y}_{2TH} \hat{y}_{3TH}}{\hat{y}_{1TH} + \hat{y}_{2TH} + \hat{y}_{3TH}}$$

...(43)

**Simulation**

Monte Carlo simulation is used to compare the estimators obtained in this study. Samples of different sizes, $n = 10, 35, \text{ and } 75$ has been generated from one parameter inverse Rayleigh distribution based on MSE criteria, with 1000 replicates. The steps of Simulation for Mote Carlo as follows;

Step1: Generate random samples as $u_{i1}, u_{i2}, ..., u_{in_i}$, $v_1, v_2, ..., v_{m_1}$, and $w_1, w_2, ..., w_m$, for all $i = 1,2,...,k$.

Respectively which follow the continuous uniform distribution defined on the interval $(0,1)$. Step2: Transform the above uniform random m samples to random samples follows IRD using the cumulative distribution function (CDF) as follow:
Step 4: Compute the R from equation (6).

Step 5: Form the moment estimator for reliability by using equation (16).

Step 5: Form the least square estimator for reliability by using equation (32). Step 6: Find the uniformly minimum variance unbiased method of R using equation (22).

Step 6: Apply Shrinkage estimators of reliability using equations (37, 41, and 43).

### Table 1. Estimation value of $R = 0.098039$, when $y_1 = 2$, $y_2 = 4$ and $y_3 = 2.5$

<table>
<thead>
<tr>
<th>$(n,m_1,m_2)$</th>
<th>$R_{MLE}$</th>
<th>$R_{Ub}$</th>
<th>$R_{MOM}$</th>
<th>$R_{LS}$</th>
<th>$R_{Sh1}$</th>
<th>$R_{Sh2}$</th>
<th>$R_{Th}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,50,50)</td>
<td>7.3768e-02</td>
<td>7.5943e-02</td>
<td>1.8407e-01</td>
<td>7.7321e-02</td>
<td>9.4103e-02</td>
<td>9.8018e-02</td>
<td>9.7758e-02</td>
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### Table 2. MSE value of $R = 0.098039$, when $y_1 = 2$, $y_2 = 4$ and $y_3 = 2.5$

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<th>$R_{Sh1}$</th>
<th>$R_{Sh2}$</th>
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Table 3. Estimation value of $R = 0.19780$, when $\gamma_1 = 1.5$, $\gamma_2 = 2$ and $\gamma_3 = 3$

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<th>$R_{Sh1}$</th>
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Table 4. MSE value of $R = 0.19780$ when $\gamma_1 = 1.5$, $\gamma_2 = 2$ and $\gamma_3 = 3$

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<th>$R_{Sh1}$</th>
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Table 5. Estimation value of $R = 0.1684$, when $\gamma_1 = 2.2$, $\gamma_2 = 3.3$ and $\gamma_3 = 4$

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Conclusion:
The estimation of S-S reliability for Inverse Rayleigh distribution was introduced in this paper using different methods as; Maximum Likelihood, Moment method, Uniformly Minimum Variance Unbiased, Constant Shrinkage Estimation Method, Shrinkage Function Estimator, Modified Thompson Type Shrinkage Estimator. Monte Simulation was exhibited. Based on the results, the performance of shrinkage weight function ($\hat{R}_{Sh2}$) was appropriate behavior and it is efficient estimator than the others in the sense of MSE based on four parameters ($\beta_1$, $\beta_2$, $\beta_3$). While $\hat{R}_{Th}$ had the second rank.

Authors' declaration:
- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Garmian.

References:
8. Mohsin M, Shahbaz, M Q . Comparison of negative moment estimator with maximum likelihood estimator of inverse Rayleigh distribution. PJSOR.2005;1,45-48
مقارنة بعض طرق التقدير لنموذج الأجهاد–المتانة

$$R = P(Y < X < Z)$$

سيران حمزة رحيم 1
بيداء عطية خلف 2
عباس نجم سلمان 2

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2 قسم الرياضيات، كلية التربية للعلوم الصرفة (ابن الهيثم)، جامعة بغداد، بغداد، العراق.

الخلاصة:
كجزء مهم من نظام المعولية بفرض أن $R = P(Y < X < Z)$ في هذا البحث تم مناقشة نموذج الأجهاد–المتانة المتغيرات العشوائية تتبع توزيع معلومات رالي. بعض طرق التقدير التقليدية استخدمت لتخمين المعلمات وهي: طريقة الامكان الاعظم، طريقة العزوم، طريقة المقدر غير المتتبع ذي اقل تباين وثلاث طرائق من عوامل الوزن المقلصة. بالإضافة إلى ذلك تم استخدام محاكاة مونت كارلو للمقارنة بين طرق التخمين المستخدمة بناءً على متوسط مربعات الخطأ.

الكلمات المفتاحية: نموذج الأجهاد–المتانة، توزيع معلومات رالي، مقدر المربعات الصغرى، طريقة الامكان الاعظم، طريقة الحد الأدنى من التباين غير المتبع، طريقة التقلص.