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New White Method of Parameters and Reliability Estimation for Transmuted Power Function Distribution

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Abstract:

In this paper, an estimate has been made for parameters and the reliability function for Transmuted power function (TPF) distribution through using some estimation methods as proposed new technique for white, percentile, least square, weighted least square and modification moment methods. A simulation was used to generate random data that follow the (TPF) distribution on three experiments (E_1, E_2, E_3) of the real values of the parameters, and with sample size (n=10,25,50 and 100) and iteration samples (N=1000), and taking reliability times $(0 < t < \theta)$. Comparisons have been made between the obtained results from the estimators using mean square error (MSE). The results showed the percentile estimator is the best in (E_1, E_2) but modification moment is the best in (E_3).

Keywords: Estimation methods, Mean square error, Percentiles, Quadratic rank transmutation map, Transmuted power function distribution.

Introduction:

The power function (PF) distribution is one of the distributions derived from the beta distribution, many researchers have studied this distribution by estimating its parameters and the reliability function because it is one of the distributions used in reliability .The probability density function (pdf) and the cumulative distribution function (cdf) with scale parameter $\theta > 0$ and shape parameter $\alpha > 0$ are given, respectively, as follows [1]:

$$g(x;\alpha,\theta) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha}, 0 < x < \theta$$

$$G(x;\alpha,\theta) = \frac{x^\alpha}{\theta^\alpha} \quad , 0 < x < \theta$$

Muhammad and others (2016) [2], suggested transmuted power function (T-PF) distribution by using the quadratic rank transmutation map [3]:

$$f(x) = (1+\lambda)G(x) - \lambda\left(G(x)\right)^2, \ |\lambda| \le 1$$

$$F(x) = g(x)\left((1+\lambda) - 2\lambda G(x)\right)$$

Where $g(x)$ and $G(x)$ are (pdf) and (cdf) for (PF) distribution, respectively, $f(x)$and $F(x)$ are (pdf) and (cdf) for (TPF) distribution, respectively, and λ is transmuted parameter$|\lambda| \le 1$, as follows [2]:

$$f(x;\alpha,\theta,\lambda) = \frac{\alpha x^{\alpha-1}}{\theta^\alpha}\left(1+\lambda - 2\lambda\left(\frac{x}{\theta}\right)^\alpha\right) \quad \dots(1)$$

$$F(x;\alpha,\theta,\lambda) = \left(\frac{x}{\theta}\right)^\alpha\left(1+\lambda-\lambda\left(\frac{x}{\theta}\right)^\alpha\right) \quad \dots(2)$$

The reliability function for (TPF) distribution is

$$R(t;\alpha,\theta,\lambda) = 1 - \left(\frac{t}{\theta}\right)^\alpha\left(1+\lambda - \lambda\left(\frac{t}{\theta}\right)^\alpha\right) \quad \dots(3)$$

The quantile function for (TPF) distribution is

$$x_F = \left(\frac{\theta^\alpha}{\sqrt{\lambda}}\left(\left(\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 - F\right)^{\frac{1}{2}} + \frac{1+\lambda}{2\sqrt{\lambda}}\right)\right)^{\frac{1}{\alpha}} \quad \dots(4)$$

Estimation Method:

In this item methods for estimating parameters and reliability function are presented.

- **New Technique White Method**

White estimation method depends mainly on its application on the reliability function of distribution [4], a new technique is proposed for white method which depends on the cumulative distribution function. for estimating its parameters and format conversion function to formulate similar linear regression equation, which is characterized by the use of its estimate as an initial value of the other estimation methods, from (1), getting :

$$x_{(i)}=\left(\frac{\theta^{\alpha}}{\sqrt{\lambda}}\left(\left(\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^{2}-P_i\right)^{\frac{1}{2}}+\frac{1+\lambda}{2\sqrt{\lambda}}\right)\right)^{\frac{1}{\alpha}} \quad \dots(5)$$

Where

$$P_i=\frac{i}{n+1}=F\left(x_{(i)};\alpha,\theta,\lambda\right) \quad ,i=1,2,\dots,n \quad \dots(6)$$

Let

$$k_i=\left(\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^{2}-P_i\right)^{\frac{1}{2}}+\frac{1+\lambda}{2\sqrt{\lambda}} \quad \dots(7)$$

Then (5) became

$$x_{(i)}^{\alpha}=\frac{\theta^{\alpha}}{\sqrt{\lambda}}k_i \quad \dots(8)$$

$$ln\left(x_{(i)}^{\alpha}\right)=ln\left(\frac{\theta^{\alpha}}{\sqrt{\lambda}}k_i\right)$$

$$\alpha ln\left(x_{(i)}\right)=ln(\theta^{\alpha})-ln\sqrt{\lambda}+ln(k_i)$$

$$ln(k_i)-ln\sqrt{\lambda}=\alpha ln\left(x_{(i)}\right)-ln(\theta^{\alpha}) \dots(9)$$

The linear equation

$$z=\varphi y+\psi \quad \dots(10)$$

Compare (9) with (10), getting:

$$z_i=\ ln(k)-ln\sqrt{\lambda}\ ,\varphi=\alpha\ ,y_i=ln\left(x_{(i)}\right)$$

$$\psi=-ln(\theta^{\alpha}) \quad \dots(11)$$

Since

$$\hat{\varphi}=\frac{\sum_{i=1}^{n}(y_i-\bar{y})(z_i-\bar{z})}{\sum_{i=1}^{n}(y_i-\bar{y})^2}$$

$$\therefore\ \hat{\alpha}_{NW}=\hat{\varphi} \quad \dots(12)$$

From (10), getting:

$$\hat{\psi}=\bar{z}-\hat{\varphi}\bar{y} \quad \dots(13)$$

From (11) and (12), getting:

$$\hat{\theta}_{NW}=exp\left(-\frac{\hat{\psi}}{\hat{\alpha}_W}\right) \quad \dots(14)$$

And (12) and (14) are the initial value (α_0,θ_0) for the estimators in the other methods.

Substitute (12) and (14) in (3), getting:

$$\hat{R}_{NW}(t;\alpha,\theta,\lambda)=1-\left(\frac{t}{\hat{\theta}_W}\right)^{\alpha}\left(1+\lambda-\lambda\left(\frac{t}{\hat{\theta}_W}\right)^{\alpha}\right) \dots(15)$$

- **Percentile Method**

Parameters estimation for this method depends on inverse distribution function for any distribution. Is depends on minimizing value in (5), by equating with zero and square this value, getting [5]:

$$s=\sum_{i=1}^{n}\left(x_{(i)}-\left(\frac{\theta^{\alpha}}{\sqrt{\lambda}}\left(\left(\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^{2}-P_i\right)^{\frac{1}{2}}+\frac{1+\lambda}{2\sqrt{\lambda}}\right)\right)^{\frac{1}{\alpha}}\right)^{2} \quad \dots(16)$$

Substitute (7) in (16), getting :

$$s=\sum_{i=1}^{n}\left(x_{(i)}-\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\right)^{2} \quad \dots(17)$$

To find estimation formulas for (θ) and (α), taking the partial derivative of (17) w.r.t (θ) and (α) and equating to zero, respectively, getting:

$$\frac{\partial s}{\partial\theta}=\sum_{i=1}^{n}2\left(x_{(i)}-\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\right)\left(-\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\right)=0$$

$$\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\sum_{i=1}^{n}x_{(i)}k_i^{\frac{1}{\alpha}}-\theta\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{2}{\alpha}}\sum_{i=1}^{n}k_i^{\frac{2}{\alpha}}=0$$

$$\hat{\theta}_{PER}=\frac{\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\sum_{i=1}^{n}x_{(i)}k_i^{\frac{1}{\alpha}}}{\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{2}{\alpha}}\sum_{i=1}^{n}k_i^{\frac{2}{\alpha}}} \quad \dots(18)$$

$$\frac{\partial s}{\partial\alpha}=\sum_{i=1}^{n}2\left(x_{(i)}-\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\right)\left(-\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\left(-\frac{1}{\alpha^2}\right)\right)=0$$

$$\sum_{i=1}^{n}\left(x_{(i)}\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}ln\left(\frac{k_i}{\sqrt{\lambda}}\right)-\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{2}{\alpha}}\ ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\right)=0 \quad \dots(19)$$

Now, solving (16) numerically by Newton Raphson method

Let

$$f(\alpha)=\sum_{i=1}^{n}\left(x_{(i)}k_i^{\frac{1}{\alpha}}ln\left(\frac{k_i}{\sqrt{\lambda}}\right)-\theta\frac{k_i^{\frac{2}{\alpha}}}{(\sqrt{\lambda})^{\frac{1}{\alpha}}}ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\right)$$

Then

$$f'(\alpha)=\sum_{i=1}^{n}\left(x_{(i)}k_i^{\frac{1}{\alpha}}\ln(k_i)\left(\frac{-1}{\alpha^2}\right)\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)+\frac{\theta}{(\sqrt{\lambda})^{\frac{1}{\alpha}}}\frac{k_i^{\frac{2}{\alpha}}}{\alpha^2}\left(\frac{\ln(k_i)}{\alpha^2}-\ln(\sqrt{\lambda})\right)\right)$$

So

$$\hat{\alpha}_{PER}=\alpha_0-\frac{f(\alpha)}{f'(\alpha)} \quad \dots(20)$$

Substitute (18) and (20) in (3), getting:

$$\hat{R}_{PER}(t;\alpha,\theta,\lambda)=1-\left(\frac{t}{\hat{\theta}_{PER}}\right)^{\hat{\alpha}_{PER}}\left(1+\lambda-\lambda\left(\frac{t}{\hat{\theta}_{PER}}\right)^{\hat{\alpha}_{PER}}\right)\dots(21)$$

- **Least Square Method**

The idea of this method is to minimize sum of squared differences between observed sample values and the estimated expected values by linear approximation. A technique is used which is proposed by (Kindermann ,Lariccia) (1983) to converse of non-linear model to linear model and useing (Parzen) (1979) technique by inverse distribution [6].

By using (8), getting:

$$x(P_i)=\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}} \quad \dots(22)$$

Now, taking partial derivatives of (22) w.r.t (θ) and (α) respectively:

$$\frac{\partial x(P_i)}{\partial\theta}=\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}$$

$$\frac{\partial x(P_i)}{\partial\alpha}=\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha}}\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\left(\frac{-1}{\alpha^2}\right)$$

By estimating the first order Taylor series

$$f(\tau+h)=f(\tau)+h_1f_1'(\tau)+h_2f_2'(\tau) \quad \dots(23)$$

Where

$$\left.\begin{aligned}f(\tau+h)&=E\big(x(P_i)\big)\\ f(\tau)&=\theta_0\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\\ f_1'(\tau)&=\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\\ f_2'(\tau)&=\left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\\ h_1&=\theta-\theta_0\ ,h_2=\alpha-\alpha_0\end{aligned}\right\}\quad \dots(24)$$

Substitute (24) in (23), getting:

$$E\big(x(P_i)\big)=\theta_0\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}+(\theta-\theta_0)\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}+(\alpha-\alpha_0)\left(\left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\right)$$

$$E\big(x(P_i)\big)=\theta\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}+(\alpha-\alpha_0)\left(\left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)\right)$$

Let

$$H=\begin{bmatrix}\left(\frac{k_1}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}} & \left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_1}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_1}{\sqrt{\lambda}}\right)\\ \left(\frac{k_2}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}} & \left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_2}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_2}{\sqrt{\lambda}}\right)\\ & \vdots\\ \left(\frac{k_n}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}} & \left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_n}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_n}{\sqrt{\lambda}}\right)\end{bmatrix}$$

Let

$$\varpi(P_i)=\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}$$

$$\varphi(P_i)=\left(\frac{-\theta_0}{\alpha_0^2}\right)\left(\frac{k_i}{\sqrt{\lambda}}\right)^{\frac{1}{\alpha_0}}\ln\left(\frac{k_i}{\sqrt{\lambda}}\right)$$

Then the matrix (H) , becomes as following

$$H=\begin{bmatrix}\varpi(P_1) & \varphi(P_1)\\ \varpi(P_2) & \varphi(P_2)\\ & \vdots\\ \varpi(P_n) & \varphi(P_n)\end{bmatrix}$$

Then the general formula for estimating the parameters is

$$Y^T=(H^TH)^{-1}H^TX_i$$

Where

$$Y^T=\begin{pmatrix}\hat{\theta}_{LS}\\ \hat{\alpha}_{LS}-\alpha_0\end{pmatrix} \text{ and } X_i=\begin{pmatrix}x_{(1)}\\ x_{(2)}\\ \vdots\\ x_{(n)}\end{pmatrix}$$

$$\begin{pmatrix}\hat{\theta}_{LS}\\ \hat{\alpha}_{LS}-\alpha_0\end{pmatrix}=\begin{pmatrix}\xi_1(P)\\ \xi_2(p)\end{pmatrix}$$

Getting:

$$\left.\begin{aligned}\hat{\theta}_{LS}&=\xi_1(P)\\ \hat{\alpha}_{LS}&=\alpha_0+\xi_2(p)\end{aligned}\right\}\quad \dots(25)$$

Substitute $\hat{\theta}_{LS}$ and $\hat{\alpha}_{LS}$ from (25) in (3), getting approximate reliability estimators as follows

$$\hat{R}_{LS}(t;\alpha,\theta,\lambda) = 1 - \left(\frac{t}{\hat{\theta}_{LS}}\right)^{\hat{\alpha}_{LS}}\left(1+\lambda - \lambda\left(\frac{t}{\hat{\theta}_{LS}}\right)^{\hat{\alpha}_{LS}}\right) \quad \dots (26)$$

- **Weighted Least Square Method**

This method is used to find estimation of α , θ on order and this idea based on finding inverse function of distribution and compare it with the formula of model sample linear regression as the following formula [7]:

$$Y_i = a + b\Psi_i + \epsilon \quad \dots (27)$$

From (22), getting:

$$ln(x_{(i)}) = ln(\theta) + \frac{1}{\alpha} ln\left(\frac{k_i}{\sqrt{\lambda}}\right) \quad \dots (28)$$

And comparing (28) with (27), getting:

$$Y_i = ln(x_{(i)})$$

$$a = ln(\theta) \quad \dots (29)$$

$$b = \frac{1}{\alpha} \quad \dots (30)$$

$$\Psi_i = ln\left(\frac{k_i}{\sqrt{\lambda}}\right)$$

From (27), getting:

$$\epsilon = Y_i - a - b\Psi_i \quad \dots (31)$$

By multiplication (31) by $\left(\frac{1}{Y_i}\right)$ and taking the sum of it, after that squared the two sides, getting:

$$\delta = \sum_{i=1}^{n}\left(\frac{\epsilon}{Y_i}\right)^2 = \sum_{i=1}^{n}\left(1 - \frac{1}{Y_i}a - \frac{\Psi_i}{Y_i}b\right)^2 \dots (32)$$

This method based on making(δ) , which called weighted least squares function, to be less:

Let

$$\sigma_i = \frac{1}{Y_i}, \gamma_i = \frac{\Psi_i}{Y_i}$$

Then (32) became:

$$\delta = \sum_{i=1}^{n}(1 - a\sigma_i - b\gamma_i)^2 \quad \dots (33)$$

Now, driving the equation (33) (w.r.t) (a) and(b) , and equaling with zero, getting:

$$\frac{\partial\delta}{\partial a} = \sum_{i=1}^{n} -2(1 - a\sigma_i - b\gamma_i)\sigma_i = 0$$

$$\sum_{i=1}^{n}(\sigma_i - a\sigma_i^2 - b\gamma_i\sigma_i) = 0$$

$$a = \frac{\sum_{i=1}^{n}\sigma_i - b\sum_{i=1}^{n}\gamma_i\sigma_i}{\sum_{i=1}^{n}\sigma_i^2} \quad \dots (34)$$

$$\frac{\partial\delta}{\partial b} = \sum_{i=1}^{n} -2(1 - a\sigma_i - b\gamma_i)(\gamma_i) = 0$$

$$\sum_{i=1}^{n}\gamma_i - a\sum_{i=1}^{n}\sigma_i\gamma_i - b\sum_{i=1}^{n}\gamma_i^2 = 0$$

$$a = \frac{\sum_{i=1}^{n}\gamma_i - b\sum_{i=1}^{n}\gamma_i^2}{\sum_{i=1}^{n}\sigma_i\gamma_i} \quad \dots (35)$$

By equating (34) with (35), getting:

$$b\sum_{i=1}^{n}\gamma_i^2\sum_{i=1}^{n}\sigma_i^2 - b\left(\sum_{i=1}^{n}\gamma_i\sigma_i\right)^2 = \sum_{i=1}^{n}\gamma_i\sum_{i=1}^{n}\sigma_i^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\gamma_i$$

$$b = \frac{\sum_{i=1}^{n}\gamma_i\sum_{i=1}^{n}\sigma_i^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\gamma_i}{\sum_{i=1}^{n}\gamma_i^2\sum_{i=1}^{n}\sigma_i^2 - \left(\sum_{i=1}^{n}\gamma_i\sigma_i\right)^2} \quad \dots (36)$$

From equations (30) and (36), getting:

$$\hat{\alpha}_{WLS} = \frac{1}{b} \quad \dots (37)$$

Now, substituted (36) in (34), to get(a) , and from (29) , getting :

$$\hat{\theta}_{WLS} = exp(a) \quad \dots (38)$$

Substitute (37) and (38) in equation (3), getting:

$$\hat{R}_{WLS}(t;\alpha,\theta,\lambda) = 1 - \left(\frac{t}{\hat{\theta}_{WLS}}\right)^{\hat{\alpha}_{WLS}}\left(1+\lambda - \lambda\left(\frac{t}{\hat{\theta}_{WLS}}\right)^{\hat{\alpha}_{WLS}}\right) \quad \dots (39)$$

- **Modification Moment Method**

The idea of this method, for estimating the first parameter, is based on equating the approximate expected value of (cdf) with the formula of (cdf) in $x_{(1)}$, and on the variance of the distribution to estimate the second parameter [8]

$$E\left(\hat{F}\left(x_{(1)}\right)\right) = F\left(x_{(1)}\right) \quad \dots (40)$$

From (6), $E\left(\hat{F}\left(x_{(1)}\right)\right) = P_1 = \frac{1}{n+1} \quad \dots (41)$

Rewrite (4), for the first observation $\left(x_{(1)}\right)$ as follows

$$\left(x_{(1)}^{\alpha}\frac{\sqrt{\lambda}}{\theta^{\alpha}} - \frac{1+\lambda}{2\sqrt{\lambda}}\right) = \left(\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 - F\left(x_{(1)}\right)\right)^{\frac{1}{2}}$$

$$F\left(x_{(1)}\right) = \left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 - \left(x_{(1)}^{\alpha}\frac{\sqrt{\lambda}}{\theta^{\alpha}} - \frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 \quad \dots (42)$$

By substituting (41) and (42) in (40), getting:

$$x_{(1)}^{\alpha}\frac{\sqrt{\lambda}}{\theta^{\alpha}} - \frac{1+\lambda}{2\sqrt{\lambda}} = \sqrt{\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 - \frac{1}{n+1}}$$

$$\hat{\theta}_{MM} = \left(\frac{x_{(1)}^{\alpha}\sqrt{\lambda}}{\sqrt{\left(\frac{1+\lambda}{2\sqrt{\lambda}}\right)^2 - \frac{1}{n+1}} + \frac{1+\lambda}{2\sqrt{\lambda}}}\right)^{\frac{1}{\alpha}} \quad \dots (43)$$

To estimate the parameter(α), equate the sample variance with a population variance as follows:

$$\alpha\theta^2\left(\frac{2\alpha+2(1-\lambda)}{(\alpha+2)(2\alpha+2)}\right) - \alpha^2\theta^2\left(\frac{2\alpha+1-\lambda}{(\alpha+1)(2\alpha+1)}\right)^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

$$\hat{\alpha}_{MM} = \frac{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1} + \alpha^2\theta^2\left(\frac{2\alpha+1-\lambda}{(\alpha+1)(2\alpha+1)}\right)^2}{\theta^2\left(\frac{2\alpha+2(1-\lambda)}{(\alpha+2)(2\alpha+2)}\right)} \quad \ldots(44)$$

Substitute (43) and (44) in equation (3) , getting:

$$\hat{R}_{MM}(t;\alpha,\theta,\lambda) = 1 - \left(\frac{t}{\hat{\theta}_{MM}}\right)^{\hat{\alpha}_{MM}}\left(1+\lambda - \lambda\left(\frac{t}{\hat{\theta}_{MM}}\right)^{\hat{\alpha}_{MM}}\right) \quad \ldots(45)$$

Simulation Experiments and Results:

In This section estimation of the parameters is presented by NW, PER, LS, WLS and MM for multiple parameters values and sample sizes, the building simulation experiments include stages as follows

1. Take sample size n: (n=10, 25, 50 and 100) and replicated sample (N=1000).
2. Take several values for the parameters (α,θ),as shown in Table 1:

Table 1. Default value for parameters

Experiment		E_1	E_2	E_3
Parameters	$\boldsymbol{\alpha}$	2.5	4	1
	$\boldsymbol{\theta}$	2.5	1.5	3
	$\boldsymbol{\lambda}$	0.1	0.5	0.9

3. Choose life time for estimating reliability
   - In case $(\boldsymbol{E_1})$, have chosen $0 < t < 2.5$.
   - In case $(\boldsymbol{E_2})$, have chosen $0 < t < 1.5$.
   - In case $(\boldsymbol{E_3})$, have chosen $0 < t < 3$.

   Such that $t_{(1)} = 0.1$, and $t_{(i+1)} = t_{(i)} + 0.1,\ i = 1,2,\ldots$

4. At this stage, random data is generated by (4) and using MATLAB language version R2015a.
5. After stage -4- , finding estimators and reliability function, and finding $MSE(\hat{\theta}) = \frac{\sum_{i=1}^{n}(\hat{\theta}_i-\theta)^2}{N}$, where $(\hat{\theta})$ is an estimator of parameter (θ).

Tables 2, 3 and 4 show the values of the mean and MSE for the parameter estimators and the reliability function, as follows

Table 2. Estimated values for $(\boldsymbol{\alpha},\boldsymbol{\theta})$and $(\boldsymbol{R})$ using$(\boldsymbol{E_1}: \boldsymbol{R = 0.71050})$

Methods	n	Mean Estimated Values			MSE		
		$\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$
NW	10	2.47943	2.43938	0.69027	0.22949	0.17207	0.02899
PER		2.47946	2.43935	0.69026	0.22947	0.17208	0.02899
LS		2.47946	2.38803	0.66974	0.21869	0.17828	0.03136
WLS		2.65771	2.59729	0.74608	0.30821	0.18579	0.02921
MM		2.26733	2.36578	0.64641	0.29698	0.18564	0.03645
NW	25	2.47190	2.46180	0.69961	0.05976	0.05255	0.01006
PER		2.47191	2.46178	0.69960	0.05975	0.05256	0.01006
LS		2.44625	2.46178	0.69034	0.05951	0.05428	0.01047
WLS		2.54951	2.53344	0.72565	0.06966	0.05468	0.01047
MM		2.25982	2.37663	0.653939	0.12042	0.06516	0.01492
NW	50	2.48160	2.47738	0.70337	0.02662	0.02381	0.00473
PER		2.48161	2.47738	0.70336	0.02662	0.02381	0.00473
LS		2.46828	2.46485	0.69868	0.02663	0.02427	0.00484
WLS		2.52091	2.51406	0.71672	0.02881	0.02424	0.00480
MM		2.26991	2.38809	0.65726	0.08092	0.03506	0.00885
NW	100	2.49234	2.49016	0.70787	0.01326	0.01181	0.00241
PER		2.49234	2.49015	0.70787	0.01326	0.01182	0.00241
LS		2.48553	2.48375	0.70548	0.01323	0.01190	0.00242
WLS		2.51221	2.50876	0.71471	0.01386	0.01197	0.00246
MM		2.28103	2.39874	0.66175	0.06200	0.02152	0.00583

Table 3. Estimated values for $(\alpha,\theta)$and (R)using(E_2: $R = 0.77301$)

Methods	n	Mean Estimated Values $\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$	MSE $\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$
NW	10	3.92861	1.47751	0.75758	0.75758	0.00738	0.01317
PER		3.93304	1.47741	0.75782	0.93220	0.00739	0.01312
LS		3.76691	1.46196	0.73987	0.94429	0.00885	0.01510
WLS		4.21068	1.50183	0.78437	1.15069	0.00630	0.01287
MM		3.73664	1.36641	0.68264	1.04280	0.02722	0.02852
NW	25	3.95035	1.49080	0.76830	0.24990	0.00234	0.00442
PER		3.95262	1.49075	0.76840	0.24937	0.00234	0.00441
LS		3.87060	1.48356	0.76027	0.25690	0.00253	0.00461
WLS		4.07582	1.50172	0.78035	0.27801	0.00215	0.00454
MM		3.76145	1.36144	0.68391	0.31708	0.02202	0.01844
NW	50	3.98683	1.49660	0.77346	0.12409	0.00116	0.00237
PER		3.98806	1.49657	0.77351	0.12402	0.00116	0.00236
LS		3.94363	1.49278	0.76919	0.12384	0.00112	0.00232
WLS		4.05191	1.50226	0.77972	0.13245	0.00111	0.00245
MM		3.79954	1.36011	0.68547	0.17054	0.02086	0.01592
NW	100	4.00761	1.49961	0.77624	0.06647	0.00060	0.00141
PER		4.00825	1.49960	0.77627	0.06649	0.00061	0.00141
LS		3.9854	1.49767	0.77409	0.06537	0.00059	0.00135
WLS		4.04043	1.50247	0.77943	0.06921	0.00059	0.00137
MM		3.82419	1.35925	0.68624	0.10183	0.02045	0.01488

Table 4. Estimated values for $(\alpha,\theta)$and (R)using(E_3: $R = 0.34500$)

Methods	n	Mean Estimated Values $\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$	MSE $\hat{\alpha}$	$\hat{\theta}$	$\hat{R}$
NW	10	0.97988	2.87867	0.31882	0.08647	0.30109	0.02352
PER		0.98877	2.87366	0.31986	0.11098	0.30392	0.02410
LS		0.95199	2.82495	0.30500	0.08535	0.32185	0.02479
WLS		1.04868	2.98115	0.34951	0.10968	0.28692	0.02411
MM		1.09842	2.47063	0.31157	0.07684	0.66741	0.02244
NW	25	0.97074	2.92156	0.32416	0.02326	0.10005	0.00749
PER		0.97637	2.91862	0.32510	0.03183	0.10062	0.00772
LS		0.95526	2.89158	0.31626	0.02456	0.10513	0.00803
WLS		0.99947	2.96640	0.33750	0.02520	0.09416	0.00729
MM		1.01674	2.45790	0.29029	0.00767	0.50386	0.01171
NW	50	0.98280	2.95787	0.33318	0.01025	0.04542	0.00341
PER		0.98557	2.95619	0.33371	0.01059	0.04551	0.00348
LS		0.97384	2.94094	0.32864	0.01080	0.04627	0.00357
WLS		0.99719	2.98025	0.33994	0.01055	0.04267	0.00328
MM		1.00354	2.47514	0.28930	0.00154	0.44015	0.00919
NW	100	0.98909	2.97477	0.33817	0.00479	0.02161	0.00159
PER		0.99053	2.97386	0.33847	0.00485	0.02160	0.00160
LS		0.98424	2.9657	0.33573	0.00502	0.02160	0.00164
WLS		0.99614	2.98573	0.34149	0.00475	0.02009	0.00151
MM		1.00053	2.48696	0.29186	0.00035	0.40844	0.00850

Conclusions:

- A new technique is proposed for white estimation method, and it has presented good estimators for scale parameters and reliability function.
- Taken, $0 \le \lambda \le 1$, because some estimation formula like (PER, LS, MM) included $(\sqrt{\lambda})$ then the results are complex number, If it is taken $(\lambda < 0)$.
- The results in tables (2,3 and 4) indicate that: -
  - In experiment $(\boldsymbol{E_1})$ appeared $(\hat{\alpha}_{LS}, \hat{\theta}_{NW})$ and $(\hat{R}_{NW}, \hat{R}_{PER})$ are better than other estimators.
  - In experiment $(\boldsymbol{E_2})$, $(\hat{\alpha}_{NW}, \hat{\alpha}_{PER})$ are better in $(n = 10,25)$ respectively, but $(\hat{\alpha}_{LS})$ is better in $(n = 50,100)$ and $(\hat{\theta}_{WLS})$ is better than other estimators. $(\hat{R}_{WLS}, \hat{R}_{PER})$ are better in $(n = 10,25)$ but $(\hat{R}_{LS})$ is better in$(n = 50,100)$.
  - In experiment $(\boldsymbol{E_3})$ appeared $(\hat{\alpha}_{MM})$ and $(\hat{\theta}_{WLS})$ are better than other estimators.

$(\hat{R}_{MM})$ is better in $(n = 10)$ but $(\hat{R}_{MM})$ is better in other samples.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

Authors' contributions statement:

M. A. derives the mathematical equations related to finding estimates of parameters and reliability of the distribution and analyzing the obtained data. J. H. wrote the computer programs required to complete the research and drafting the manuscript, revision and proofreading.

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طريقة وايت الجديدة لتقدير معلمات ومعولية توزيع تحويل دالة القوة

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الخلاصة:

في هذا البحث تم تقدير معلمات توزيع تحويل دالة القوة (TPF)، ودالة المعولية له من خلال بعض طرائق التقدير وهي مقترح للطريقة وايت،النسب المئينية ، المربعات الصغرى ، المربعات الصغرى الموزونة وطريقة العزوم المطورة. استخدمت المحاكاة في توليد بيانات عشوائية تتبع توزيع (TPF) على ثلاث تجارب (E_1, E_2, E_3) من القيم الحقيقية للمعلمات ، ومع حجم العينة (n=10, 25, 50, 100) وتكرار العينة $(N = 1000)$، وأخذت قيم اوقات المعولية $(0 < t < \theta)$ التي من خلالها تم تقدير دالة المعولية لكل حالة من التجارب الثلاثة. تم إجراء مقارنات بين النتائج التي تم الحصول عليها من مقدرات دالة المعولية باستخدام متوسط مربع الخطأ (MSE) وعرضت النتائج في جداول خاصة بها لغرض المقارنة.

الكلمات المفتاحية: توزيع تحويل دالة القوة ، طرائق التقدير، المئينات ،تطبيق تحويلات الدرجة التربيعية ، متوسط مربع الخطأ.