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Study of Second Hankel Determinant for Certain Subclasses of Functions Defined by Al-Oboudi Differential Operator

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Abstract:

The concern of this article is the calculation of an upper bound of second Hankel determinant for the subclasses of functions defined by Al-Oboudi differential operator in the unit disc. To study special cases of the results of this article, we give particular values to the parameters A, B and λ

Key words: Al-Oboudi differential operator, Convex functions, Second Hankel determinant, Starlike functions, Subordination property.

Introduction:

Let A be the class of analytic functions in the unit disk U, normalised by $f(0)=f'(0)-1=0$.

If K is the class of functions $F_\lambda(z)$ defined as

$$F_\lambda(z)=(1-\lambda)f(z)+\lambda zf'(z)=z+\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]a_k z^k,\ \lambda\in R,\lambda\in(0,1],$$

then $F_\lambda(z)$ can be considered as the analytic function in U.

This function studied by Challab et.al (see (1))

Furthermore, let S be defined as the class of functions $F_\lambda(z)\in K$, which is univalent in U. Using a Schwarzian function w with the conditions $w(0)=0$ and $|w(z)|<1$ and analytic in U, with f and g as its analytic functions in U. Subsequently, f is a subordinate to g (symbolically $f\prec g$) if $f(z)=g(w(z))$ is satisfied.

Let $S_\lambda^*(A,B)$ be a subclass of functions $F_\lambda(z)\in K$ with the condition

$$\frac{zF_\lambda'(z)}{F_\lambda(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,\lambda\in\mathbb{R},\lambda\in(0,1]$$

where $S_\lambda^*(A,B)$ is the subclass of starlike functions, whereby $S_0^*(A,B)\equiv S^*(A,B)$ has been investigated specifically by Goel and Mehrok(2). Note that $S_0^*(1,-1)\equiv S^*$, is the class of starlike functions (3).

Meanwhile, $K_\lambda(A,B)$ is the subclass of functions $F_\lambda(z)\in K$ with the condition

$$\frac{(zF_\lambda'(z))'}{F_\lambda'(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,\lambda\in R,\lambda\in(0,1]$$

with $K_0(A,B)\equiv K(A,B)$ being a subclass of convex functions. Therefore, $K_0(1,-1)\equiv K$ in particular is also a class of convex functions (3).

For a sequence $a_n, a_{n+1}, a_{n+2},\ldots$ of complex number, it is referred to as a Hankel matrix, which is the infinite matrix whose $(i,j)^{th}$ entry a_{ij} can be defined by $a_{ij}=a_{n+i+j-2}$, $(i,j,n\in N)$ as per (3).

According to its definition, the q^{th} Hankel matrix $(q\in N,\ \{1\})$ can be assumed to follow the $q\times q$ square submatrix

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$$\begin{bmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{bmatrix}.$$

We note that the sloping diagonals in Hankel matrix are constants, that is
$a_{ij} = a_{i-1,j+1}(i \in N\ ,\ \{1\}, j \in N)$.

Such condition is also an acceptable description of the Hankel matrix without specific reference to any special sequence. The norm indicates the determinant of the q^{th} Hankel matrix to be denoted by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}$$

which is also sometimes referred to as the q^{th} Hankel determinant. In case of special circumstances $q=2, n=1, a_1=1$ and $q=2, n=2,$ the Hankel determinant may be simplified accordingly and written as

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = |a_3 - a_2^2|$$ and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|$$

where $|a_2a_4 - a_3^2|$ is the second Hankel determinant. Besides, Noonan and Thomas(4) stated the q^{th} Hankel determinant of $f(z)$ for $q \ge 1$ and $n \ge 1$. Furthermore, Janteng et al.((5), (6), (7)), Singh and Singh(3), and various other researchers also yielded sharp upper bounds of $H_2(2)$ in their respective work according to different classes of analytic functions. Yavuz (8) in particular achieved an upper bound for the second Hankel determinant $|a_2a_4 - a_3^2|$ for the analytic functions, as per the definition of Ruscheweyh derivative in the unit disk. Meanwhile, Kund and Mishra(9) focused on the second Hankel determinant, involving the Carlson-Shaffer operator in the unit disk. Moreover, the work by Singh and Mehrok(10) had successfully revealed the sharp upper bound of $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for $p-$valent α-convex functions. In contrast, Deniz et al.(11) projected the second Hankel determinant $H_2(2)$ of bi-starlike and bi-convex functions of order β, which had been acknowledged as vital subclasses of bi-univalent functions. Similarly, another work by Deekonda and Thoutreddy(12) on the second Hankel determinant $|a_2a_4 - a_3^2|$ is by utilizing Toeplitz determinants. For other recently work (see (13), (14)).

Very recent Challab et. al. studied third Hankel determinant using Srivastava-Attiya integral operator (15).

Preliminary Results

To elucidate the achieved findings, T can be described as the cluster of all functions t analytic in U, for which $\Re(t(z)) > 0$ and $t(z)=1+t_1z+t_2z+...$ for $z \in U$.

Lemma2.1. (16)If $t \in T$, then $|t_k| \le 2\ (k = 1,2,3,...)$.

Lemma 2.2. (17),(18) If $t \in T$, then

$$2t_2 = t_1^2 + (4 - t_1^2)x$$

$$4t_3 = t_1^3 + 2t_1(4 - t_1^2)x - t_1(4 - t_1^2)x^2 + 2(4 - t_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $t_1 \in [0,2]$.

Main Results

Theorem 3.1. If $F_\lambda(z) \in S_\lambda^*(A,B)$, then

$$|a_2a_4 - a_3^2| \le \frac{(A-B)^2}{4(1+2\lambda)^2}. \quad (3.1)$$

Proof. If $F_\lambda(z) \in S_\lambda^*(A,B)$, then a Schwarz function $w(z)$ is undeniably existing, such that

$$\frac{zF_\lambda'(z)}{F_\lambda(z)} = \phi(w(z)), \quad (3.2)$$

where

$$\phi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + ...$$
$$= 1 + A_1z + A_2z^2 + A_3z^3 + ... \quad (3.3)$$

Furthermore, the function $t_1(z)$ can be defined as follows:-

$$t_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + b_1z + b_2z^2 + b_3z^3 + ... \quad (3.4)$$

Since $w(z)$ is a Schwarz function, we can write, $\Re(t_1(z)) > 0$ and $t_1(0)=1$. The function $h(z)$ can be defined by

$$h(z)=\frac{zF'_{\lambda}(z)}{F_{\lambda}(z)}=1+c_1z+c_2z^2+c_3z^3+\dots .$$

(3.5)

From the equations (3.2), (3.4) and (3.5), it can be shown that

$$h(z)=\phi\left(\frac{t_1(z)-1}{t_1(z)+1}\right)=\phi\left(\frac{b_1z+b_2z^2+b_3z^3+\dots}{2+b_1z+b_2z^2+b_3z^3+\dots}\right)$$

$$=\phi\left(\frac{1}{2}b_1z+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)z^2+\frac{1}{2}\left(b_3-b_1b_2-\frac{b_1^3}{4}\right)z^3+\dots\right)$$

$$=1+\frac{A_1b_1}{2}z+\left[\frac{A_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_2b_1^2}{4}\right]z^2+$$

$$\left[\frac{A_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{A_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_3b_1^3}{8}\right]z^3+\dots .$$

Thus,

$$c_1=\frac{A_1b_1}{2};c_2=\frac{A_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_2b_1^2}{4}$$

and

$$c_3=\frac{A_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{A_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_3b_1^3}{8}.$$

(3.6)

Moreover, by employing (3.3) and (3.5) in (3.6), we have

$$a_2=\frac{(A-B)b_1}{2(1+\lambda)},$$

(3.7)

$$a_3=\frac{A-B}{8(1+2\lambda)}\left[2b_2+(A-2B-1)b_1^2\right],$$

(3.8)

and

$$a_4=\frac{A-B}{48(1+3\lambda)}\left[8b_3+(6A-14B-8)b_1b_2+\left(A^2+6B^2-5AB-3A+7B+2\right)b_1^3\right].$$

(3.9)

Additionally, (3.7), (3.8), (3.9) shall give the following:

$$a_2a_4-a_3^2=\frac{(A-B)^2}{192}\left[\frac{q(b_1,b_2,b_3,A,B,\lambda)}{(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\right]$$

(3.10)

where

$$q(b_1,b_2,b_3,A,B,\lambda)=16(1+2\lambda)^2b_1b_3-12(1+\lambda)(1+3\lambda)b_2^2+$$
$$\big[12A(1+2\lambda)^2-12A(1+\lambda)(1+3\lambda)-28B(1+2\lambda)^2+24B(1+\lambda)(1+3\lambda)-$$
$$16(1+2\lambda)^2+12(1+\lambda)(1+3\lambda)\big]b_2b_1^2+\big[2A^2(1+2\lambda)^2-3A^2(1+\lambda)(1+3\lambda)+$$
$$12B^2(1+2\lambda)^2-12B^2(1+\lambda)(1+3\lambda)-10AB(1+2\lambda)^2+12AB(1+\lambda)(1+3\lambda)-$$
$$6A(1+2\lambda)^2+6A(1+\lambda)(1+3\lambda)+14B(1+2\lambda)^2-12B(1+\lambda)(1+3\lambda)+$$
$$4(1+2\lambda)^2-3(1+\lambda)(1+3\lambda)\big]b_1^4.$$

If Lemma 2.1 and Lemma 2.2 are applied to (3.10), we have

$$\left|a_2a_4-a_3^2\right|=\frac{(A-B)^2}{192(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\left|r(A,B,\lambda,b_1,x,z)\right|$$

such that

$$r(A,B,\lambda,b_1,x,z)=\big[2A^2(1+2\lambda)^2-3A^2(1+\lambda)(1+3\lambda)+$$
$$12B^2(1+2\lambda)^2-12B^2(1+\lambda)(1+3\lambda)-10AB(1+2\lambda)^2+12AB(1+\lambda)(1+3\lambda)\big]b_1^4+$$
$$\big[6A(1+2\lambda)^2-6A(1+\lambda)(1+3\lambda)-14B(1+2\lambda)^2+12B(1+\lambda)(1+3\lambda)\big]b_1^2\left(4-b_1^2\right)x+$$
$$8(1+2\lambda)^2b_1\left(4-b_1^2\right)\left(1-|x|^2\right)z-\left(4-b_1^2\right)x^2\left[4b_1^2(1+2\lambda)^2+3(1+\lambda)(1+3\lambda)\left(4-b_1^2\right)\right].$$

By assuming that $b_1=b$ and $b\in[0,2]$, with triangular inequality and $|z|\le 1$, the following

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{192(1+\lambda)(1+3\lambda)(1+2\lambda)^2}F(\delta),$$

is also obtainable where $\delta=|x|\le 1$ and

$$F(\delta)=\Big[|A|^2\left|2(1+2\lambda)^2-3(1+\lambda)(1+3\lambda)\right|+12|B|^2\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)+\right.$$
$$\left.2|AB|\left|6(1+\lambda)(1+3\lambda)-5(1+2\lambda)^2\right|\right|\Big]b^4+8(1+2\lambda)^2b\left(4-b^2\right)+$$
$$\Big[6|A|\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|+2|B|\left|7(1+2\lambda)^2-6(1+\lambda)(1+3\lambda)\right|\Big]b^2\left(4-b^2\right)\delta+$$
$$\Big\{\left(4-b^2\right)\left[4b^2(1+2\lambda)^2+3(1+\lambda)(1+3\lambda)\left(4-b^2\right)\right]-8(1+2\lambda)^2b\left(4-b^2\right)\Big\}\delta^2.$$

As $F(\delta)$ is an increasing function, $Max F(\delta)=F(1)$ is also satisfactorily applicable.
Therefore,

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{192(1+\lambda)(1+3\lambda)(1+2\lambda)^2}G(b)$$

(3.11)

where
$G(b)=F(1)$.
So,

$$G(b)=\Big[|A|^2\left|2(1+2\lambda)^2-3(1+\lambda)(1+3\lambda)\right|+12|B|^2\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|+2|AB|\left|6(1+\lambda)(1+3\lambda)-5(1+2\lambda)^2\right|-6|A|\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|-2|B|\left|7(1+2\lambda)^2-6(1+\lambda)(1+3\lambda)\right|-4(1+2\lambda)^2+3(1+\lambda)(1+3\lambda)\Big]b^4+\Big[24|A|\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|+8|B|\left\|7(1+2\lambda)^2-6(1+\lambda)(1+3\lambda)\right|-8\left(3(1+\lambda)(1+3\lambda)-2(1+2\lambda)^2\right)\Big]b^2+48(1+\lambda)(1+3\lambda).$$

Now

$$G'(b)=4\Big[|A|^2\left|2(1+2\lambda)^2-3(1+\lambda)(1+3\lambda)\right|+12|B|^2\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|+2|AB|\left|6(1+\lambda)(1+3\lambda)-5(1+2\lambda)^2\right|-6|A|\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|-2|B|\left|7(1+2\lambda)^2-6(1+\lambda)(1+3\lambda)\right|-4(1+2\lambda)^2+3(1+\lambda)(1+3\lambda)\Big]b^3+2\Big[24|A|\left|(1+2\lambda)^2-(1+\lambda)(1+3\lambda)\right|+8|B|\left\|7(1+2\lambda)^2-6(1+\lambda)(1+3\lambda)\right|-8\left(3(1+\lambda)(1+3\lambda)-2(1+2\lambda)^2\right)\Big]b.$$

Since $G'(b)<0$ for $b\in[0,2]$, the maximum value of $G(b)$ at $b=0$ indicates that $Max\ G(b)=G(0)$.
Thus, (3.1) can be achieved from (3.11), that is

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{192(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\times48(1+\lambda)(1+3\lambda)\le\frac{(A-B)^2}{4(1+2\lambda)^2}$$

Whereby for $b_1=0$, $b_2=2$ and $b_3=0$ the resulting value is sharp.

Based on Theorem 3.1, we shall obtain several corollaries given as follows:

Corollary 3.1. If $F_\lambda(z)\in S^*$, then $\left|a_2a_4-a_3^2\right|\le\frac{1}{(1+2\lambda)^2}$.

This is obtained for $A=1$ and $B=-1$.

By applying $\lambda=0$ in Theorem3.1, the finding obtained is in line with Singh and Singh (3), stated below:

Corollary 3.2. If $f(z)\in S^*(A,B)$, then $\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{4}$.

Similarly, if $A=1$, $B=-1$ and $\lambda=0$ and applied in the Theorem 3.1, we shall have the following:

Corollary 3.3. If $f(z)\in S^*$, then $\left|a_2a_4-a_3^2\right|\le1$.

Theorem 3.2. If $F_\lambda(z)\in K_\lambda(A,B)$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{576(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\left(\frac{r(A,B,\lambda)}{q(A,B,\lambda)}\right)$$

(3.12)

where

$$r(A,B,\lambda)=16(1+\lambda)(1+3\lambda)\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|^2-12(1+2\lambda)^2\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-36(1+2\lambda)^4$$

and

$$q(A,B,\lambda)=\left|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\right|-\left|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\right|-\left[6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\right].$$

Proof. If $F_\lambda(z)\in K_\lambda(A,B)$, then a Schwarz function $w(z)$ is undeniably existing, such that

$$\frac{(zF_\lambda'(z))'}{F_\lambda'(z)}=\phi(w(z))$$
(3.13)

where

$$\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+B^2(A-B)z^3+\ldots$$
$$=1+A_1z+A_2z^2+A_3z^3+\ldots .$$
(3.14)

Since the function $t_1(z)$ can be define as

$$t_1(z)=\frac{1+w(z)}{1-w(z)}=1+b_1z+b_2z^2+b_3z^3+\ldots ,$$
(3.15)

and $w(z)$ is a Schwarz function, we can write $Re(t_1(z))>0$ and $t_1(0)=1$. The function $h(z)$ can subsequently be defined as

$$h(z)=\frac{(zF_\lambda'(z))'}{F_\lambda'(z)}=1+c_1z+c_2z^2+c_3z^3+\ldots .$$
(3.16)

By using the equations (3.13), (3.15) and (3.16), we have

$$h(z)=\phi\left(\frac{t_1(z)-1}{t_1(z)+1}\right)=\phi\left(\frac{b_1z+b_2z^2+b_3z^3+\ldots}{2+b_1z+b_2z^2+b_3z^3+\ldots}\right)$$
$$=\phi\left(\frac{1}{2}b_1z+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)z^2+\frac{1}{2}\left(b_3-b_1b_2-\frac{b_1^3}{4}\right)z^3+\ldots\right)$$
$$=1+\frac{A_1b_1}{2}z+\left[\frac{A_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_2b_1^2}{4}\right]z^2+\left[\frac{A_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{A_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_3b_1^3}{8}\right]z^3+\ldots .$$

Hence,

$$c_1=\frac{A_1b_1}{2};c_2=\frac{A_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_2b_1^2}{4}$$

and

$$c_3=\frac{A_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{A_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{A_3b_1^3}{8}.$$
(3.17)

Similarly, if (3.14) and (3.16) are applied in (3.17) respectively, we have

$$a_2=\frac{(A-B)b_1}{4(1+\lambda)},$$
(3.18)

$$a_3=\frac{A-B}{24(1+2\lambda)}\left[2b_2+(A-2B-1)b_1^2\right],$$
(3.19)

and

$$a_4=\frac{A-B}{576(1+3\lambda)}\left[24b_3+(18A-42B-24)b_1b_2+\left(3A^2+18B^2-15AB-9A+21B+6\right)b_1^3\right].$$
(3.20)

Then, from (3.18), (3.19), (3.20) we get

$$a_2a_4-a_3^2=\frac{(A-B)^2}{2304}\left(\frac{s(b_1,b_2,b_3,A,B,\lambda)}{(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\right)$$
(3.21)

Where

$$s(b_1,b_2,b_3,A,B,\lambda)=24(1+2\lambda)^2b_1b_3+\left[18A(1+2\lambda)^2-16A(1+\lambda)(1+3\lambda)-42B(1+2\lambda)^2+32B(1+\lambda)(1+3\lambda)-24(1+2\lambda)^2+16(1+\lambda)(1+3\lambda)\right]b_1^2b_2+\left[3A^2(1+2\lambda)^2-4A^216(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)-9A(1+2\lambda)^2+8A(1+\lambda)(1+3\lambda)+21B(1+2\lambda)^2-16B(1+\lambda)(1+3\lambda)+6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\right]b_1^4-16(1+\lambda)(1+3\lambda)b_2^2.$$

Furthermore, by utilizing Lemma 2.1 and Lemma 2.2 in (3.21), yields

$$\left|a_2a_4-a_3^2\right|=\frac{(A-B)^2}{2304(1+\lambda)(1+3\lambda)(1+2\lambda)^2}\Big|\Big[3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-$$

$$16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big]b_1^4+\Big[9A(1+2\lambda)^2-$$

$$8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big]b_1^2\left(4-b_1^2\right)x-$$

$$x^2\left(4-b_1^2\right)\Big[16(1+\lambda)(1+3\lambda)+\left(6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\right)b_1^2\Big]+$$

$$12(1+2\lambda)^2b_1\left(4-b_1^2\right)\left(1-|x|^2\right)z\Big|.$$

If $b_1=b$ and $b\in[0,2]$, with triangular inequality and $|z|\le1$, the following inequality

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{2304(1+\lambda)(1+3\lambda)(1+2\lambda)^2}F(\delta),$$

can be observed satisfactorily, where $\delta=|x|\le1$ and

$$F(\delta)=\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-$$

$$15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|b^4+12(1+2\lambda)^2b\left(4-b^2\right)+\Big|9A(1+2\lambda)^2-$$

$$8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|b^2\left(4-b^2\right)\delta+$$

$$\Big[\left(4-b^2\right)\Big|16(1+\lambda)(1+3\lambda)+\left(6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\right)b^2\Big|-$$

$$12(1+2\lambda)^2b\left(4-b^2\right)\Big]\delta^2.$$

Since $F(\delta)$ is an increasing function, we have $Max\ F(\delta)=F(1)$. Therefore,

$$\left|a_2a_4-a_3^2\right|\le\frac{(A-B)^2}{2304(1+\lambda)(1+3\lambda)(1+2\lambda)^2}G(b)$$

(3.22)

where $G(b)=F(1)$.

Thus, $G(b)$ can be expressed according to

$$G(b)=\Big[\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-$$

$$15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-$$

$$21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-\Big[6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\Big]\Big]b^4+$$

$$\Big[4\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-$$

$$32(1+\lambda)(1+3\lambda)+24(1+2\lambda)^2\Big]b^2+64(1+\lambda)(1+3\lambda).$$

Then

$$G'(b)=4\Big[\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-$$

$$15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-$$

$$21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-\Big[6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\Big]\Big]b^3+$$

$$2\Big[4\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-$$

$$32(1+\lambda)(1+3\lambda)+24(1+2\lambda)^2\Big]b.$$

and

$$G''(b)=12\Big[\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-\Big[6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\Big]\Big]b^2+2\Big[4\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-32(1+\lambda)(1+3\lambda)+24(1+2\lambda)^2\Big].$$

Whereby if $G'(b)=0$, then

$$G'(b)=4\Big[\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-\Big[6(1+2\lambda)^2-4(1+\lambda)(1+3\lambda)\Big]\Big]b^3+2\Big[4\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-32(1+\lambda)(1+3\lambda)+24(1+2\lambda)^2\Big]b=0,$$

can be obtained.

Here $G''(b)$ is negative at $b=\sqrt{\frac{p(A,B,\lambda)}{g(A,B,\lambda)}}=b'$ where

$$p(A,B,\lambda)=16(1+\lambda)(1+3\lambda)-12(1+2\lambda)^2-2\Big|9A(1+2\lambda)^2-8(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|$$

and

$$g(A,B,\lambda)=\Big|3A^2(1+2\lambda)^2-4A^2(1+\lambda)(1+3\lambda)+18B^2(1+2\lambda)^2-16B^2(1+\lambda)(1+3\lambda)-15AB(1+2\lambda)^2+16AB(1+\lambda)(1+3\lambda)\Big|-\Big|9A(1+2\lambda)^2-8A(1+\lambda)(1+3\lambda)-21B(1+2\lambda)^2+16B(1+\lambda)(1+3\lambda)\Big|-6(1+2\lambda)^2+4(1+\lambda)(1+3\lambda).$$

Thus *Max G*(*b*)=*G*(*b*') and (3.12) is henceforth obtainable from (3.22).

If $b_1=b'$, $b_2=b_1^2-2$ and $b_3=b_1(b_1^2-3)$, then the resulting value yielded is sharp.

Furthermore, if $A=1$ and $B=-1$ is applied in Theorem 3.2 accordingly, the following corollaries are achieved.

Corollary 3.4. If $F_\lambda(z)\in K$, then

$$|a_2a_4-a_3^2|\le\frac{1}{144(1+\lambda)(1+3\lambda)(1+2\lambda)^2}.\left(\frac{g(\lambda)}{h(\lambda)}\right)$$

where

$$g(\lambda)=16(1+\lambda)(1+3\lambda)\Big|36(1+2\lambda)^2-36(1+\lambda)(1+3\lambda)\Big|-\Big|30(1+2\lambda)^2-24(1+\lambda)(1+3\lambda)\Big|^2-12(1+2\lambda)^2\Big|30(1+2\lambda)^2-24(1+\lambda)(1+3\lambda)\Big|-36(1+2\lambda)^4$$

and

$$h(\lambda)=\Big|36(1+2\lambda)^2-36(1+\lambda)(1+3\lambda)\Big|-\Big|30(1+2\lambda)^2-24(1+\lambda)(1+3\lambda)\Big|^2-6(1+2\lambda)^2+4(1+\lambda)(1+3\lambda).$$

Likewise, if $\lambda=0$ in Theorem 3.2, the resulting value yielded is as per Singh and Singh(3).

Corollary 3.5. If $f(z)\in K(A,B)$, then

$$|a_2a_4-a_3^2|\le\frac{(A-B)^2}{576}.\left(\frac{16|-A^2+2B^2+AB|-|A-5B|^2-12|A-5B|-36}{|-A^2+2B^2+AB|-|A-5B|-2}\right)$$

.

Similarly, applying $A=1$, $B=-1$ and $\lambda=0$ in the Theorem 3.2 will also result in having the following:

Corollary 3.6. If $f(z)\in K$, then $|a_2a_4-a_3^2|\le\frac{1}{8}.$

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دراسة محدد هانكل الثاني لمجموعة من الدوال المعرفة بالمؤثر التفاضلي للعبودي

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الخلاصة:

هذا البحث يهتم في حساب الحد الأعلى من محدد هانكل الثاني لمجموعة من الدوال المعرفة بالمؤثر التفاضلي للعبودي المعرف على القرص الأحادي. لدراسة الحالات الخاصة من نتائج هذا البحث، اعطى الباحثين قيم خاصة للمؤثرات المعرفة ك A، B وλ.

الكلمات المفتاحية: مؤثرالعبودي التفاضلي، الدوال المحدبة، محددةهانكل الثانية، الدوال النجمية، خاصية التابع.