DOI: http://dx.doi.org/10.21123/bsj.2021.18.2.0430

Reliability and Failure Probability Functions of the *m*-Consecutive-*k*-out-of-*n*: F Linear and Circular Systems

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Received 19/3/2020, Accepted 3/11/2020, Published Online First 11/1/2021, Published 1/6/2021

Abstract:

The m-consecutive-k-out-of-n: F linear and circular system consists of n sequentially connected components; the components are ordered on a line or a circle; it fails if there are at least m non-overlapping runs of consecutive-k failed components. This paper proposes the reliability and failure probability functions for both linearly and circularly m-consecutive-k-out-of-n: F systems. More precisely, the failure states of the system components are separated into two collections (the working and the failure collections); where each one is defined as a collection of finite mutual disjoint classes of the system states. Illustrative example is provided.

Key words: Consecutive-k-out-of-n: F system, Equivalence relation, Failure probability function, Reliability function.

Introduction:

The reliability of engineering systems has become a pivotal issue through the design phase; the daily life of people depends on the satisfactory functioning of these systems. The engineers started with few components to design these systems, and by the time, the systems have become highly complex and sophisticated, and may consist of hundreds or perhaps thousands of components. In this context, the engineers developed theories for such systems, and applied the available results for all types of systems, including system reliability, optimal system design, component reliability importance, and reliability bounds.

The family of the consecutive systems had a wide area of interest for many engineers through the last few decades, the members of this family have been used to model and create optimal designs for many engineering systems, such as, the telecommunication networks, spacecraft relay stations, vacuum system of the electron accelerator, oil pipeline systems, microwave stations, photographing of a nuclear accelerator etc. where the “consecutive failures” is the basic condition for all these systems to shut down, which makes the system more reliable than the series system and less expense than that in the parallel system.

The first member in this family which appeared in literature is the consecutive-k-out-of-n: F system, Kontoleon (1) mentioned it firstly, the system consists of n connected components (arranged in a line (a circle)), and it fails if at least k consecutive components fail. Nashwan (2) proposed an algorithm to compute the sample space of all failure states of the components for the consecutive k-out-of-n: F linear (circular) system, and classified them into a working and failure spaces, also Gökdere et al. (3) proposed a new method to compute the reliability of consecutive k-out-of-n: F linear and circular systems using combinatorial approach. Shingyochit and Yamarnoto (4), and Cai et al. (5) found an optimal component arrangement where n components are assigned to n positions to maximize the system reliability. The maintenance problem was discussed by Endharta et al. (6); they evaluated the maintenance policy by comparing the expected cost rate of the proposed policy with those of corrective maintenance and age policy maintenance. Cai et al. (7) discussed the maintenance problem through exchanging the position of components and improving the reliability of components.

Some researchers studied the combination with the consecutive k-out-of-n: F system models, Mohammadi et al. (8) determined multiple simultaneous failure states for k-out-of-n: F and linear consecutive *k*-out-of-*n*: F system in which each component has a constant failure probability. Dui et al. (9) computed the marginal reliability importance and joint reliability importance in *k*-out-of-*n*: F systems and consecutive *k*-out-of-*n*: F systems for some situations. The latest researches that discussed the various generalizations and applications of the system were conducted in (10-16).

Another important member of the consecutive systems family is the *m*-consecutive-*k*-out-of-*n*: F linear (circular) system (L(C) (*m*,*k*,*n*) system), it is more complicated. Griffith (17) was firstly introduced the system, the system that consists of *n* connected components linearly (circularly), and fails if there are at least *m* non-overlapping runs of consecutive *k* failed components. Actually, it is a generalization of the consecutive-*k*-out-of-*n*: F system when *m*=1, and for *k*=1 it will be *m*-out-of-*n*: system. Such system model is applied in the bank automatic payment system, infrared detecting system, inspection in production line and quality control, as well as the above-mentioned applications.

Papastavridis (18) and Ghoraf (19) studied the L(*m*,*k*,*n*) system and provided a recursive algorithm for non-equal components reliabilities, Papastavridis (18) proposed the exact failure probability function, while Makri and Philippou (20) used the multinomial coefficients to obtain the exact reliability of the L(C) (*m*,*k*,*n*) system. Godbole (21) established Poisson approximations for the reliability of the system, Eryilmaz et al. (22) studied the reliability of the system with exchangeable components, Eryilmaz in (23) obtained closed expression for the system signature, and Gharof (24) gave a recursive algorithm to compute the reliability of the system.

The repair problem was discussed by Tang (25), he studied the repairable L(*m*,*k*,*n*) system with *l* repairs, and when the exponential distribution represents the working and the repair time of each component, where the repair is perfect, Sheng and Gen (26) computed the reliability of the repairable L (*m*,*k*,*n*) system, they assumed that both working lifetime and repair lifetime of each component were also an exponentially distributed.

Agarwal and Mohan (27) used the Graphical Evaluation and Review Technique (GERT) analysis to compute the reliability of the L(C) (*m*,*k*,*n*) system for both independent and identically distributed components, and (*k*-1)-step Markov dependent components. Ghoraf (28) used Markov-dependent components to compute the reliability of the L(C) (*m*,*k*,*n*) system.

The combination with the L(C) (*m*,*k*,*n*) system models attracted the attention of Mohan et al. (29), they studied the combination with the consecutive *k*-out-of-*n*: F system, and used the GERT Analysis and applied the model on a various complex systems such as infrared detecting and signal processing, and bank automatic payment systems, after that, Eryilmaz (30) derived the reliability of the system using a combinatorial equation for the number working states, Boushaba and Benyahia (31) computed the reliability and importance measures for the combination with consecutive *k*-out-of-*n*: F system with non-homogeneous Markov-dependent components, while Gera (32) evaluated the reliability of the combination of L(C) (m_1,k_1,*n*) and L(C) (m_2,k_2,*n*) system. Further investigations on reliability, optimal system design, component reliability importance, applications, and generalization models of the L(C) (*m*,*k*,*n*) system are in (33-36).

The main contribution of this paper is to compute the exact reliability function and the exact failure probability function of the L(C) (*m*,*k*,*n*) system. In fact, it developed the classification technique of Nashwan (2) to determine the working and failure states of the L(C) (*m*,*k*,*n*) system, which is the main stone to find the reliability function and failure probability function of the system, which is arranged in the paper as follows:

The second section derives the working and the failure states of the C(*m*,*k*,*n*) system only, and determines the necessary conditions of its failure states. Thereafter, Section 3 withdraws these conditions on the linear type (L(*m*,*k*,*n*) system). Finally, the last section introduced a mathematical algorithm to find the reliability and the failure probability functions of the L(C) (*m*,*k*,*n*) system. Through all, this paper is assuming that (All components of the system are mutually statistically independent, and they are either “failed” or “working” states, as well as the system). Table 1 presents the signs and notations frequently use in this paper.

Table 1. Notations

L(C)	: Linear (circular).
L(C)(*m*,*k*,*n*)	: The *m*-consecutive-*k*-out-of-*n*: F linear (circular) system.
I_s^r	$: =\{r, r+1, ..., s\} \quad 1 \le r < s \le n$.
$P(\mathrm{I}_n^1)$	: The power set of I_n^1.
f_n^t	: Composite function *t* times, such that $f_n(x) = x \bmod_n + 1 : x \in \mathrm{I}_n^1$.
$d_X = (d_1^X, d_2^X, ..., d_r^X)$: The rotations of the set	
	$X = \{x_1, x_2, ..., x_r\}$, such that $d_s^X \ge 1$ is

	the minimum integer number where $f_n^{d_s^X}(x_s)=x_{s+1}$, for $s=1,2,\ldots, r-1$, and $f_n^{d_r^X}(x_r)=x_1$, where $n=\sum_{s=1}^{r} d_s^X$, and for $d_X^t=\left(d_{r-t+1}^X,\ldots,d_r^X,d_1^X,\ldots,d_{r-t}^X\right)$ where $t\in\mathbf{Z}$ and $d_X^0=d_X^r, d_X^{r+s}=d_X^s$.
$\mathrm{Card}(X)$	: Number of elements in the set *X*.
$\equiv,\sim$	: Equivalence relations.
$s\underset{j}{\oplus}r$	$:=(s+r)\bmod j$, if $s+r=tj : t\in\square$, then $s\underset{j}{\oplus}r=j$.
$p_i(q_i)$	: Reliability (failure probability) of the i^{th} component.
$R(X)$	: Reliability of the set *X*. which equal $=p_X=\prod_{i\in X}p_i\prod_{j\in\overline{X}}q_j$ and hence define $\mathrm{R}[X]=\sum_{Y\in[X]}p_Y$, where $\overline{X}$ is the complement of *X*.
$F(X)$	: The same as $R(X)$, but the symbol used to distinguish that the system is in the failure state.

The circular *m*-consecutive-*k*-out-of-*n*: system

The C(*m*,*k*,*n*) system consists of *n* components (connected sequentially in a circle), it fails when the event “at least *k* consecutive failed components” occurs disjointedly at least *m* times, and denotes the components indices by I_n^1, $P(\mathrm{I}_n^1)$ is the failure space of these components indices. The system is represented by the set of all indices of the failed components $X=\{x_1,\ldots,x_j\}\in P(\mathrm{I}_n^1)$, such that $x_i<x_h$ for all $1\le i<h\le j$. *X* is a failed set (state), if *X* includes *m* disjointed subsets, and each subset consists of at least *k* consecutive failed components, where $j\ge mk$, otherwise it is a working set (state). In this context, divide the whole $P(\mathrm{I}_n^1)$ into two sub collections, $\Psi_{m,k,n}^C\left(\Theta_{m,k,n}^C\right)$ the collection of all failed (working) sets. For example in the C(2,2,8) system, the set $X=\{1,3,4,8\}$ (for simply *X*=1348) means that, all components are in the working state except the components with indices the first, third , fourth, and the eighth components, more precisely *X* is a failed set, it has 2 disjointed subsets, each one consists of 2 consecutive indices (failed components), the first subset is the components with indices 1and 8, while the second subset is the components with indices 3 and 4). Nashwan (2) defined an equivalence relations for any circular system using the bijection function $f_n:\mathrm{I}_n^1\to\mathrm{I}_n^1$, where $f_n(x)=(x \bmod n)+1$ for all $x\in\mathrm{I}_n^1$ in order to partition $P(\mathrm{I}_n^1)$ into finite mutual pairwise disjoint classes as in the form $[X]=\{f_n^t(X):t\in\mathbf{Z}\}$ and as follows:

1. For any $X,Y\in P(\mathrm{I}_n^1)$, $X\equiv Y$ if there exists $t\in\mathbf{Z}$ such that $f_n^t(X)=Y$.
2. For any $X,Y\in P(\mathrm{I}_n^1)$, $X\sim Y$ if there exists $r\in\mathbf{Z}$ such that $d_Y=d_X^r$, (*Y* is a rotation of *X*).
3. $X\equiv Y \Leftrightarrow X\sim Y$.

Now, if $X=\{x_1,\ldots,x_j\}$ represents the C(*m*,*k*,*n*) system, such that $mk\le j\le n$, takes the rotation of the set *X*, $d_X=\left(d_1^X,d_2^X,\ldots,d_j^X\right)$, assumes d_i^X is the number of steps to walk on the circle from the failed component x_i to the next failed component $x_{i\underset{j}{\oplus}1}$, define $S^X=\left(S_1^X,S_2^X,\ldots,S_j^X\right)$ such that $S_i^X=\sum_{r=0}^{k-2}d_{i\underset{j}{\oplus}r}^X$. Actually S_i^X counts the total number of steps to walk on the circle through the failed components starting from the failed component x_i to $x_{i\underset{j}{\oplus}k-1}$, i.e. the subset of *X* $\left\{x_i,x_{i\underset{j}{\oplus}1},\ldots,x_{i\underset{j}{\oplus}k-1}\right\}$. If $S_i^X\le k-1$, then these *k* failed components are consecutive, and if S^X includes *m* elements less than or equal *k*-1, and the corresponding subsets of *X* are disjointed, then *X* is a failed set. See the following figure.

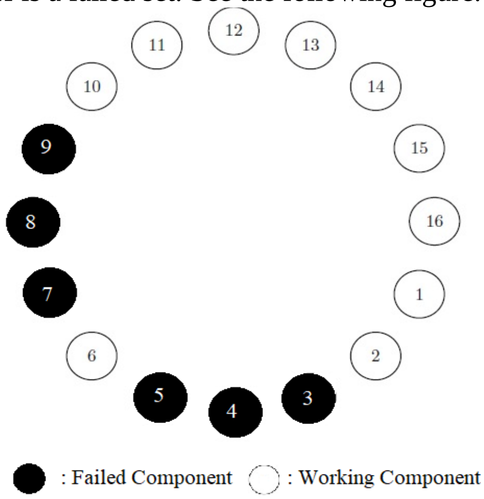

Figure 1. A failure state in the 2-consecutive-3-out-of-16: F circular system

In Fig. 1, the failure components are represented by $X=\{3,4,5,7,8,9\}$, $d_X=(1,1,2,1,1,10)$ is the number of steps between any failed component and the next one. For example, there are 2 steps between the failed components (the 5th and the 7th), i.e. $d_3^X=2$, consequently $S^X = (2,3,3,2,11,11)$, where $S_i^X = \sum_{r=0}^{1} d^X_{i \underset{6}{\oplus} r}$, i.e. $S_1^X = 1+1 = 2$, $S_2^X = 1+2 = 3$, $S_3^X = 2+1 = 3$, $S_4^X = 1+1 = 2$, $S_5^X = 1+10 = 11$, $S_5^X = 10+1 = 11$.

Note that $S_1^X = 2$, $S_4^X = 2 \le 3-1$, the corresponding subsets are $\{3,4,5\}$ and $\{7,8,9\}$, which are disjoint, each one consists of 3 consecutive failed components, hence the C(2,3,16) is in the failure state.

Lemma 1: Let $X = \{x_1,....,x_j\}$ represents the C(*m*,*k*,*n*) system, then *X* is a failed set, if there exists $1 \le \alpha_1 < \alpha_2 < .... < \alpha_m \le j$ such that $S^X_{\alpha_i} \le k-1$ and $\alpha_i \underset{j}{\oplus} k - 1 < \alpha_{i+1}$, where, $i = 1,2,...,m$, and $mk \le j \le n$.

Proof: Since $S^X_{\alpha_i}, S^X_{\alpha_{i+1}} \le k-1$, then the corresponding subsets $X_{\alpha_i} = \left\{x_{\alpha_i}, x_{\alpha_i \underset{j}{\oplus} 1}, ..., x_{\alpha_i \underset{j}{\oplus} k-1}\right\}$ and $X_{\alpha_{i+1}} = \left\{x_{\alpha_{i+1}}, x_{\alpha_{i+1} \underset{j}{\oplus} 1}, ..., x_{\alpha_{i+1} \underset{j}{\oplus} k-1}\right\}$ of X have k consecutive failed components, and since $\alpha_i \underset{j}{\oplus} k - 1 < \alpha_{i+1}$, (i.e. $x_{\alpha_i \underset{j}{\oplus} k-1} < x_{\alpha_{i+1}}$ by definition), then $X_{\alpha_i} \cap X_{\alpha_{i+1}} = \varnothing$. If there exists $S^X_{\alpha_1}, S^X_{\alpha_2}, ..., S^X_{\alpha_m} \le k-1$, then there is *m* pairwise disjointed subsets of *X* ($X_{\alpha_1}, X_{\alpha_2}, ..., X_{\alpha_m} \subseteq X$), each one consists of *k* consecutive failed components, i.e. *X* is a failed set.

Lemma 2: Consider the C(*m*,*k*,*n*) system, and $Y \in [X]$

1. If $X \in \Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$, then $Y \in \Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$.
2. $\Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$ is a union of finite mutual pairwise disjointed classes.

Proof:

1. Consider $X \in \Psi^C_{m,k,n}$ and $Y \in [X]$, then there exist $t \in \mathbf{Z}$ such that $f_n^t(X) = Y$, WLOG let $f_n^t(x_i) = y_i$, then for any $i \in \mathrm{I}^1_{j-1}$, $y_{i+1} = f_n^t(x_{i+1}) = f_n^t\left(f_n^{d_i^X}(x_i)\right) = f_n^{d_i^X}\left(f_n^t(x_i)\right) = f_n^{d_i^X}(y_i)$, which implies that $d_X = d_Y$, consequently $S^X = S^Y$, hence $Y \in \Psi^C_{m,k,n}$.
2. According Nashwan (2), $P\left(\mathrm{I}^1_n\right) = \bigcup_{i=1}^{s} [X_i]$, use 1 to check the sets X_i weather failed or working state, if $X_i \in \Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$, then by 1 $[X_i] \in \Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$, then rearrange these classes from 1 to *q* as a failed sets and from *q*+1 to *s* as a working sets, hence $\Psi^C_{m,k,n} = \bigcup_{i=1}^{q} [X_i]$, and $\Theta^C_{m,k,n} = \bigcup_{i=q+1}^{s} [X_i]$.

Remarks:

1. If the C(*m*,*k*,*n*) system has the state *Y*, and *X* is another state of the system such that $Y \in [X]$, then the reliability of the set (state) *Y* is a function of the reliability of the set (state) *X*, where $R(Y) = p_Y = p_{f_n^t(X)}$ for some $t \in \mathbf{Z}$. On the opposite, the failure probability of the set (state) *Y*, is also is a functioning of the set (state) *X* i.e. $R(Y) = F(Y) = p_{f_n^t(X)}$, for some $t \in \mathbf{Z}$.
2. If the C(*m*,*k*,*n*) system has the state *Y*, where the components are independent and identically distributed (*i.i.d.*), and *X* is another state of the system such that $Y \in [X]$, then $\mathrm{Card}(X) = \mathrm{Card}(Y)$, which implies that $R(X) = p_n^{\mathrm{Card}(X)} = p_n^{\mathrm{Card}(Y)} = R(Y)$.
3. According 1 in lemma 2, if *X* is a failure (working) set, then for all $Y \in [X]$ is also a failure (working) set, hence we may call $[X]$ is a failure (working) class, in this context $\Psi^C_{m,k,n}\left(\Theta^C_{m,k,n}\right)$ is a finite union of mutual pairwise disjoint failure (working) classes.

The linear *m*-consecutive-*k*-out-of-*n*: system

Consider the L(*m*,*k*,*n*) system, and connect the first and last components, treat the system as a C(*m*,*k*,*n*) system. Actually, this connection generates additional failure states, i.e. the failure collection (sets) of the L(*m*,*k*,*n*) system $\Psi^L_{m,k,n}$ is included in $\Psi^C_{m,k,n}$, conversely, $\Theta^C_{m,k,n}$ is included in the collection of the working sets of the L(*m*,*k*,*n*) system $\Theta^L_{m,k,n}$. Hence, our mission is to determine all failure state from the circular type i.e. from $\Psi^C_{m,k,n}$, which is not in the linear one $\Psi^L_{m,k,n}$, which means that, it is in $\Theta^L_{m,k,n}$.

Lemma 3: Let $X = \{x_1,....,x_j\}$ represents the L(*m*,*k*,*n*) system, then *X* is a failed set, if there exists $1 \le \alpha_1 < \alpha_2 < .... < \alpha_m \le j - (k-1)$ such that $S^X_{\alpha_i} \le k-1 : i = 1,2,...,m$, where $\alpha_{i+1} - \alpha_i \ge k$ and $j \ge mk$.

Proof: The same as in lemma 1, but

Baghdad Science Journal
P-ISSN: 2078-8665
E-ISSN: 2411-7986

$1 \le \alpha_1 < \alpha_2 < .... < \alpha_m \le j-(k-1)$.

It is worth mentioning that, the set X may be a failure set in the circular type, but it is a working set in the linear type, for example, in the C(2,3,9) , the set $156789 \in [123456] \in \Psi^C_{2,3,9}$, since $S^{\{156789\}} = (5,2,2,2,2,5) \Rightarrow S_2^{\{123456\}}, S_5^{\{123456\}} \le 2$, but in the L(2,3,9), $156789 \notin \Psi^L_{2,3,9}$ while $123456 \in \Psi^L_{2,3,9}$.

The proposed algorithm.

Consider the L(C)(m,k,n) system, j is the total number of failed components, $\mathrm{F}_j^{L(C)}$ is the failure probability function and $\mathrm{R}_j^{L(C)}$ is the reliability function, then:

1. For j=0, 1, …, mk-1, all states of the system are in a working state, and then $\mathrm{R}_j^{L(C)} = \binom{n}{j} p_n^j$, $\mathrm{F}_j^{L(C)} = 0$.
2. For j=mk, mk+1,…., n, apply Nashwan (2) in the light of lemma 2 and 3.

2.1. Find $d_X = (d_1^X, d_2^X, ..., d_j^X)$ such that $n = \sum_{i=1}^{j} d_i^X$.

2.2. Find the corresponding $X = \{x_1, ..., x_j\} \subseteq \mathrm{I}_n^1$ starting with $x_1 = 1$.

2.3. Find the corresponding class $[X] = \{ f_n^t(X) : t = 1,2,...,n \}$, and determine $S^X = (S_1^X, S_2^X, ..., S_j^X)$.

2.4. Use lemma 2 to classify $X \in \Psi^C_{m,k,n}$ or $(X \in \Theta^C_{m,k,n})$, consequently $[X] \in \Psi^C_{m,k,n}$ or $([X] \in \Theta^C_{m,k,n})$.

2.5. For the linear system, if $[X] \in \Psi^C_{m,k,n}$, apply lemma 3 for their elements,

2.6. Where $\Theta^L_{m,k,n}$ consists of $\Theta^C_{m,k,n}$, and all states $Y \in [X] \in \Psi^C_{m,k,n}$ that is not holding the conditions of lemma 3.

3. Find $\mathrm{F}_j^{C(L)} \left(\mathrm{R}_j^{C(L)} \right)$, which is the summation of the failure function of the failed (working) classes $[X] \in \Psi^{C(L)}_{m,k,n} \left(\Theta^{C(L)}_{m,k,n} \right)$, where $|X| = j$.
4. The reliability function is $\mathrm{R}_{C(L)} = \sum_{j=0}^{n} \mathrm{R}_j^{C(L)}$, while the failure function is $\mathrm{F}_{C(L)} = \sum_{j=mk}^{n} \mathrm{F}_j^{C(L)}$.

Example 1: Consider a 2-consecutive -3-out-of-9: F linear or circular system with independent and identically distributed components, i.e. L(C) (2,3,9) system.

For j = 0, 1, 2,…,5 all states are in a working states

$$\mathrm{R}_j^{L(C)} = \binom{9}{j} p_9^j \qquad \mathrm{F}_j^{L(C)} = 0$$

For j = 6

$(1,1,1,1,1,4) \in \Psi^C_{2,3,9} \Rightarrow S = (2,2,2,2,5,5) \Rightarrow S_1, S_4 \le 2$

$$[123456] = \left\{ \begin{matrix} 123456, 234567, 345678, 456789, \underline{156789}, \\ \underline{126789}, 123789, \underline{123489}, \underline{123459} \end{matrix} \right\}$$

$\{156789, 126789, 123489, 123459\} \in \Theta^L_{2,3,9}$

$(1,1,1,1,2,3) \in \Theta^C_{2,3,9} \Rightarrow S = (2,2,2,3,5,4)$

$$[123457] = \left\{ \begin{matrix} 123457, 234568, 345679, 145678, 256789, \\ 136789, 124789, 123589, 123469 \end{matrix} \right\}$$

$(1,1,1,1,3,2) \in \Theta^C_{2,3,9} \Rightarrow S = (2,2,2,4,5,3)$

$$[123458] = \left\{ \begin{matrix} 123458, 234569, 134567, 245678, 356789, \\ 146789, 125789, 123689, 123479 \end{matrix} \right\}$$

$(1,1,1,2,1,3) \in \Theta^C_{2,3,9} \Rightarrow S = (2,2,3,3,4,4)$

$$[123467] = \left\{ \begin{matrix} 123467, 234578, 345689, 145679, 125678, \\ 236789, 134789, 124589, 123569 \end{matrix} \right\}$$

$(1,1,1,2,2,2) \in \Theta^C_{2,3,9} \Rightarrow S = (2,2,3,4,4,3)$

$$[123468] = \left\{ \begin{matrix} 123468, 234579, 134568, 245679, 135678, \\ 246789, 135789, 124689, 123579 \end{matrix} \right\}$$

$(1,1,1,3,1,2) \in \Theta^C_{2,3,9} \Rightarrow S = (2,2,4,4,3,3)$

$$[123478] = \left\{ \begin{matrix} 123478, 234589, 134569, 124567, 235678, \\ 346789, 145789, 125689, 123679 \end{matrix} \right\}$$

$(1,1,2,1,1,3) \in \Psi^C_{2,3,9} \Rightarrow$

$S = (2,3,3,2,4,4) \Rightarrow S_1^X, S_4^X \le 2$

$$[123567] = \left\{ \begin{matrix} 123567, 234678, 345789, \underline{145689}, \underline{125679}, \\ 123678, 234789, \underline{1345689}, \underline{124569} \end{matrix} \right\}$$

$\{145689, 125679, 134589, 124569\} \in \Theta^L_{2,3,9}$

$(1,1,2,1,2,2) \in \Theta^C_{2,3,9} \Rightarrow S = (2,3,3,3,4,3)$

$$[123568] = \left\{ \begin{matrix} 123568, 234679, 134578, 245689, 135679, \\ 124678, 235789, 134689, 124579 \end{matrix} \right\}$$

$(1,1,2,2,1,2) \in \Theta^C_{2,3,9} \Rightarrow S = (2,3,4,3,3,3)$

$$[123578] = \left\{ \begin{matrix} 123578, 234689, 134579, 124568, 235679, \\ 134678, 245789, 135689, 124679 \end{matrix} \right\}$$

$(1,2,1,2,1,2) \in \Theta^C_{2,3,9} \Rightarrow S = (3,3,3,3,3,3)$

$[124578] = \{124578, 235689, 134679\}$

To explain the algorithm, when j=6, take any 6 arbitrary numbers $d_i^X \ge 1$ such that $9 = \sum_{i=1}^{j} d_i^X$, e.g. $d_X = (1,1,2,1,1,3)$. Assume that the first component is in the failure state, according to the definition of d_X, the steps between the failed component respectively, the failed components have the indices

$X=\{1,2,3,5,6,7\}=123567$. Use the definition of $[X]=\{f_n^t(X):t\in\mathbf{Z}\}$ to find the class of *X*.

$f^1(\{1,2,3,5,6,7\})=\{2,3,4,6,7,8\}$,

$f^2(\{1,2,3,5,6,7\})=\{3,4,5,7,8,9\}$,....,

$f^9(\{1,2,3,5,6,7\})=\{1,2,3,5,6,7\}$, then

$$[X]=\left\{\begin{matrix}123567,234678,345789,\underline{145689},\underline{125679},123678,\\ 234789,\underline{1345689},\underline{124569}\end{matrix}\right\}$$

By definition of S^X, $S^{\{123567\}}=(2,3,3,2,4,4)$ where $S_i^X=\sum_{r=0}^{1}d^X_{i\oplus_6 r}$. According lemma 2, $[123567]$is a failure class, since $S_1^X,S_4^X\le 3-1=2$. i.e. $[123567]\in\Psi^C_{2,3,9}$.

Actually, the failure states of the C(2,3,9) when *j*=6 are those in the failure classes $[123456],[123567]$, and all other states are a working states, we have 18 failure states, so $\mathrm{F}_6^C=18p^3q^6$, and

$$\mathrm{R}_6^C=66p^3q^6=\left(\binom{9}{6}-18\right)p^3q^6.$$

For the linear type, i.e. L(2,3,9), apply lemma 3, and check only the elements of the classes $[123456],[123567]$. Note that the underlined elements (above mentioned) $\{156789,126789,123489,123459\}$ as well as $\{145689,125679,1345689,124569\}$ are not in $\Psi^L_{2,3,9}$, consequently, $\mathrm{F}_6^L=10p^3q^6$ and $\mathrm{R}_6^L=74p^3q^6$.

For *J*=7

$(1,1,1,1,1,1,3)\in\Psi^C_{2,3,9}\Rightarrow S=(2,2,2,2,2,4,4)$

$$[1234567]=\left\{\begin{matrix}1234567,2345678,3456789,1456789,\\ \underline{1256789},1236789,1234789,\underline{1234589},\\ 1234569\end{matrix}\right\}$$

$\{1256789.1234589\}\in\Theta^L_{2,3,9}$

$(1,1,1,1,1,2,2)\in\Psi^C_{2,3,9}\Rightarrow S=(2,2,2,2,3,4,3)$

$$[1234568]=\left\{\begin{matrix}1234568,2345679,1345678,2456789,\\ \underline{1356789},\underline{1246789},1235789,\underline{1234689},\\ \underline{1234579}\end{matrix}\right\}$$

$\{1356789,1246789,1234689,1234579\}\in\Theta^L_{2,3,9}$

$(1,1,1,1,2,1,2)\in\Theta^C_{2,3,9}\Rightarrow S=(2,2,2,3,3,3,3)$

$$[1234578]=\left\{\begin{matrix}1234578,2345689,1345679,1245678,\\ 2356789,1346789,1245789,1235689,\\ 1234679\end{matrix}\right\}$$

$(1,1,1,2,1,1,2)\in\Psi^C_{2,3,9}\Rightarrow S=(2,2,3,3,2,3,3)$

$$[1234678]=\left\{\begin{matrix}1234678,2345789,\underline{1345689},\underline{1245679},\\ 1235678,2346789,1345789,\underline{1245689},\\ \underline{123569}\end{matrix}\right\}$$

$\{1345689,1245679,1245689,123569\}\in\Theta^L_{2,3,9}$

$\mathrm{F}_7^C=27p^2q^7$ $\quad$ $\mathrm{R}_7^C=9p^2q^7$

$\mathrm{F}_7^L=17p^2q^7$ $\quad$ $\mathrm{R}_7^L=19p^2q^7$

For *j*=8

$(1,1,1,1,1,1,1,2)\in\Psi^C_{2,3,9}\Rightarrow S=(2,2,2,2,2,2,3,3)$

$$[12345678]=\left\{\begin{matrix}12345678,23456789,13456789,\\ 12456789,12356789,12346789,\\ 12345789,12345689,12345679\end{matrix}\right\}$$

$\mathrm{F}_8^C=9pq^8$ $\quad$ $\mathrm{R}_8^C=0$

$\mathrm{F}_8^L=9pq^8$ $\quad$ $\mathrm{R}_8^L=0$

For *j*=9

$(1,1,1,1,1,1,1,1)\in\Psi^C_{2,3,9}\Rightarrow S=(2,2,2,2,2,2,2,2,2)$

$\Rightarrow[123456789]=\{123456789\}$

$\mathrm{F}_9^C=q^9$ $\quad$ $\mathrm{R}_9^C=0$

$\mathrm{F}_9^L=q^9$ $\quad$ $\mathrm{R}_9^L=0$

For the C(2,3,9), the reliability function is $\mathrm{R}_C=\sum_{j=0}^{9}\mathrm{R}_j^C$, while the failure function is $\mathrm{F}_C=\sum_{j=6}^{n}\mathrm{F}_j^C$

$$\mathrm{F}_C=\sum_{j=6}^{9}\mathrm{F}_j^C=18p^3q^6+27p^2q^7+9pq^8+9q^9$$

$$\mathrm{R}_C=\sum_{j=0}^{9}\mathrm{R}_j^C=p^9+9p^8q+36p^7q^2+84p^6q^3$$
$$+126p^5q^4+126p^4q^5+66p^3q^6+9p^2q^7$$

Where the failure and the working collection of the C (2,3,9) are respectively

$$\Psi^C_{2,3,9}=\left\{\begin{matrix}[123456],[123567],[1234567],[1234568],\\ [1234678],[12345678],[123456789]\end{matrix}\right\}$$

$$\Theta^C_{2,3,9}=P(\mathrm{I}_n^1)-\Psi^C_{2,3,9}$$

For the L (2,3,9), the reliability function is $\mathrm{R}_L=\sum_{j=0}^{9}\mathrm{R}_j^L$, while the failure function is $\mathrm{F}_L=\sum_{j=6}^{n}\mathrm{F}_j^L$

$$\mathrm{F}_L=\sum_{j=6}^{9}\mathrm{F}_j^L=10p^3q^6+17p^2q^7+9pq^8+q^9$$

$$\mathrm{R}_L=\sum_{j=0}^{9}\mathrm{R}_j^L=p^9+9p^8q+36p^7q^2+84p^6q^3+126p^5q^4$$
$$+126p^4q^5+74p^3q^6+19p^2q^7$$

Where $\Psi^L_{2,3,9}$ the failure collection of the L (2,3,9) are the $\Psi^C_{2,3,9}$ without all underlined elements.

Conclusion:

This paper aimed to compute the exact reliability and failure probability functions for the *m*-consecutive-*k*-out-of-*n*: F linear and circular system. The linear system is treated as a special case of the circular one. An algorithm is suggested to classify the states of the *m*-consecutive-*k*-out-of-*n*: F linear and circular system into a working and a failure collections, moreover these collections are consisting of a finite pairwise disjointed classes, where the reliability and the failure probability functions of the *m*-consecutive-*k*-out-of-*n*: F linear and circular system are the summations of reliabilities of these working and failure collections respectively.

Acknowledgment

I would like to introduce my grateful thanks to the anonymous referees for their important comments.

Author's declaration:

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Besides, the Figures and images, which are not mine, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Al Quds Open University.

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دوال كثافة احتمال موثوقية وفشل الانظمة الخطية والدائرية *m* التتابعية *k* من *n*

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الخلاصة:

في هذه الورقة ، تم دراسة الانواع الخطية والدائرية للنظام *m* لعدد *k* التتابعي من *n* من المكونات، حيث تم تصنيف عناصر فضاء عينة فشل مكونات النظام الى تجمعين من التصنيفات، الأول للحالات التي يكون فيها النظام في حالة العمل والآخر للحالات التي يكون فيها النظام وصل لمرحلة الفشل، ومن ثم تم استخدام هذه التصنيفات لحساب دالة كثافة احتمال موثوقية النظام ودالة كثافة احتمال فشل النظام، وفي النهاية تم اقتراح خوارزمية رياضية لحساب ذلك، تتضمن عمليات التصنيفات هذه وكيفية حساب الدالتين المذكورين من خلال عملية التصنيف.

الكلمات المفتاحية: النظام التتابعي *k* من *n* ، علاقة التكافؤ، دالة كثافة احتمال فشل النظام، دالة كثافة احتمال موثوقية النظام.