Schultz and Modified Schultz Polynomials for Edge – Identification Chain and Ring – for Square Graphs

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Abstract:
In a connected graph $G$, the distance function between each pair of two vertices from a set vertex $V(G)$ is the shortest distance between them and the vertex degree $u$ denoted by $deg_u$ is the number of edges which are incident to the vertex $u$. The Schultz and modified Schultz polynomials of $G$ are have defined as: $Sc(G;x) = \sum(deg_u + deg_v)x^{d(u,v)}$ and $Sc^*(G;x) = \sum(deg_u.deg_v)x^{d(u,v)}$, respectively, where the summations are taken over all unordered pairs of distinct vertices in $V(G)$ and $d(u,v)$ is the distance between $u$ and $v$ in $V(G)$. The general forms of Schultz and modified Schultz polynomials shall be found and indices of the edge – identification chain and ring – square graphs in the present work.

Keywords: Edge – Identification Chain and Ring , Schultz index , Modified Schultz index, Polynomials.

Introduction:
All the connected graphs were considered in this paper simple, finite and undirected.$V = V(G)$ and $E = E(G)$ denotes the set of vertices and the set of edges respectively of $G$, the number of vertices of $G$ is called the order of $G$, that is $p = p(G) = |V(G)|$ and the number of edges of $G$ is called the size of $G$, that is $q = q(G) = |E(G)|$. The vertex degree $deg_u$ or $deg_v$ refers to the number of vertices that are incident to the vertex $u$. The distance between any two arbitrary vertices $u$ and $v$ of $G$ is the length of the shortest path joining $u$ to $v$, and it is denoted by $d_G(u,v)$ or $d(u,v)$. The diameter of a connected graph $G$ is the maximum distance between any two vertices in $V(G)$ and denoted by $diam_G$, 1,2, that is,

$$diam_G = max_{u,v \in V(G)} \{d(u,v)\}.$$  

There are many studies on polynomials and indices with respect the Schultz and modified Schultz, 3-6, and also, there are studies on applied of tts, 7-11.

The Schultz index (molecular topological index) was introduced by Schultz in 1989, 12, and the modified Schultz index was defined by Klavžar and Gutman in 1997, 13.

The Schultz and modified Schultz indices are defined respectively as: $Sc(G) = \sum_{(u,v) \in V(G)} (deg_u + deg_v) d(u,v)$. $Sc^*(G) = \sum_{(u,v) \in V(G)} (deg_u.deg_v) d(u,v)$.

The Schultz and Modified Schultz polynomials are important polynomials which can obtain the indices and study some properties of the coefficients. The Schultz polynomial is defined as: $Sc(G;x) = \sum_{(u,v) \in V(G)} (deg_u + deg_v)x^{d(u,v)}$.

In addition, the modified Schultz polynomial is defined as: $Sc^*(G;x) = \sum_{(u,v) \in V(G)} (deg_u.deg_v)x^{d(u,v)}$.

The indices of Schultz and modified Schultz can be obtained by the derivative of Schultz and modified Schultz polynomials with respect to $x$ at $x = 1$, that are:

$$Sc(G) = \frac{d}{dx} (Sc(G;x))|_{x=1}$$ and

$$Sc^*(G) = \frac{d}{dx} (Sc^*(G;x))|_{x=1}.$$ 

The average distance of a connected graph $G$ of order $p(G)$ with respect Schultz and modified Schultz are have defined as:
\[
\overline{Sc}(G) = 2Sc(G)/p(G) \ (p(G) - 1) \quad \text{and} \quad \overline{Sc}^* (G) = 2Sc^*(G)/p(G) \ (p(G) - 1).
\]

The graph \( G \) is said to be \( r \) - regular graph if all vertex of \( G \) has \( r \) degree.

If \( G \) is an \( r \)-regular graph, then

\[
Sc(G; x) = \sum_{(u,v) \in V(G)} 2r \ x^{d(u,v)} \quad \text{and} \quad Sc^*(G; x) = \sum_{(u,v) \in V(G)} r^2 \ x^{d(u,v)}.
\]

Hence \( Sc^*(G, x) = \frac{r}{2} Sc(G, x) \). …(1.1)

With simplified, can be obtained as the following

\[
Sc^*(G) = \frac{r}{2} Sc(G) \quad \text{and} \quad \overline{Sc}^*(G) = \frac{r}{2} \overline{Sc}(G) \quad \text{…(1.2)}
\]

The number of pairs of vertices of \( G \) that are distance \( k \) which denoted by \( D(G, k) \). Let \( D_k(r, h) \) be the set of all unordered pairs of vertices \((u, v) \) of \( G \) with distance \( k \) such \( degu = r \) and \( degv = h \). From clearly that :

\[
\sum_{k=1}^{\text{diam}(G)} |D_k(G)| = p(G)(p(G) - 1)/2,
\]

where \( D(G, k) = |D_r(G)| \).

Finally, the topological indices such as Schultz and modified Schultz indices determine some properties of chemical structures, see, 14–16.

In this work, the general forms can be identified of Schultz and modified Schultz polynomials and indices of the edge – identification chain and ring – square graphs.

**Main Results:**

Binary operations are construct the new graph from any two graphs such as: Cartesian graph product, strong graph product, tensor graph product, … etc. In this paper, the special definitions are given of operation for edge identification chain and ring graphs.

**Definition 1:** Edge – Identification Chain (EIC) – Graphs:

Let \( \{G_1, G_2, \ldots, G_n\} \) be a set of pairwise disjoint graphs with non – adjacent edge \( u_iv_i, x_1y_i \in E(G_i), i = 1, 2, \ldots, n, n \geq 2 \), then the edge-identification chain graph \( C_e(G_1, G_2, \ldots, G_n) \equiv \{G_{ij}\}_{i=1}^{n} \) of \( G_{ij} \) with respect to the edges \( \{u_iv_i, x_1y_i\}_{i=1}^{n} \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_n \) by identifying the edge \( x_1y_i \) with the edge \( u_{i+1}v_{i+1} \) for all \( i = 1, 2, \ldots, n - 1 \). (Fig. 1) in which:

\[
\begin{align*}
C_e &= y_1 \quad u_1 \quad y_2 \quad u_2 \quad y_3 \quad u_3 \quad y_4 \quad u_4 \\
G_1 &= \ldots \quad y_1 \quad x_1 \quad u_2 \quad x_2 \quad u_3 \quad x_3 \quad u_4 \\
G_2 &= \ldots \quad y_{n-1} \quad x_{n-1} \quad u_n \quad x_n \\
G_n &= \ldots
\end{align*}
\]

**Figure 1.** \( C_e(G_1, G_2, \ldots, G_n) \). Edge – Identification Chain (EIC) – Graphs.

The graph is noted as \( C_e(G_1, G_2, \ldots, G_n) \) has:

1. \( p(C_e(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^{n} p(G_i) - 2(n - 1) \).
2. \( q(C_e(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^{n} q(G_i) - (n - 1) \).
3. \( n \leq diam(C_e(G_1, G_2, \ldots, G_n)) \leq \sum_{i=1}^{n} diam(G_i) - (n - 1) \).

The equality of lower bound is satisfied at complete graph but the upper bound is satisfied at path graph.

If \( G_i \equiv H_p \), for all \( 1 \leq i \leq n \), where \( H_p \) is a connected graph of order \( p \), the \( C_e(H_p, H_p, \ldots, H_p) \) by \( C_e(H_p) \) is denoted.

**Definition 2:** Edge – Identification Ring (EIR) – Graph:

Let \( \{G_1, G_2, \ldots, G_n\} \) be a set of pairwise disjoint graphs with non – adjacent edge \( u_iv_i, x_1y_i \in E(G_i), i = 1, 2, \ldots, n, n \geq 2 \), then the edge-identification ring graph \( R_e(G_1, G_2, \ldots, G_n) \equiv \{G_{ij}\}_{i=1}^{n} \) of \( G_{ij} \) with respect to the edges \( \{u_iv_i, x_1y_i\}_{i=1}^{n} \) is the graph obtained from the graphs \( G_1, G_2, \ldots, G_n \) by identifying the edge \( x_1y_i \) with the edge \( u_{i+1}v_{i+1} \) for all \( i = 1, 2, \ldots, n \), where \( u_{n+1}v_{n+1} \equiv u_1v_1 \). (Fig. 2).

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In addition, the graph is denoted $R_e(G_1, G_2, \ldots, G_n)$ has:
1. $p(R_e(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^{n} p(G_i) - 2n$.
2. $q(R_e(G_1, G_2, \ldots, G_n)) = \sum_{i=1}^{n} q(G_i) - n$.

3. $\frac{n-1}{2} \leq \text{diam}(R_e(G_1, G_2, \ldots, G_n)) \leq \frac{\sum_{i=1}^{n} \text{diam}(G_i) - n - 1}{2}$.

Also, the equality of lower bound is satisfied at complete graph but the upper bound is satisfied at path graph.

If $G_i \equiv H_p$, for all $1 \leq i \leq n$, where $H_p$ is a connected graph of order $p$, the $R_e(H_p, H_p, \ldots, H_p)$ by $R_e(H_p)_n$ is denoted.

Finally, there are more chains and rings consisting of special graphs about finding the polynomials and indices of types distance such as: ordinary distance, Detour distance, … etc., (17-19). Therefore, the Schultz and modified Schultz will be continued to be found of them.

Schultz and modified Schultz of $C_e(C_4)_{p-1}$ and $R_e(C_4)_p$:

(a) – Edge – Identification Chain of $C_4$, $C_e(C_4)_{p-1}$.
(b) – Edge – Identification Ring of $C_4$, $R_e(C_4)_p$.

From Fig. 3 (a), can be noted that $(C_e(C_4)_{p-1}) = 2p, q(C_e(C_4)_{p-1}) = 3p - 2$.

diam $(C_e(C_4)_{p-1}) = p$ and $2 \leq i, j \leq p - 1, i \neq j$.

Table 1. The degree vertices for edge – identification chain of $C_4$, $C_e(C_4)_{p-1}$.

<table>
<thead>
<tr>
<th></th>
<th>$deg_{u_1} = 2$</th>
<th>$deg_{v_1} = 2$</th>
<th>$deg_{u_2} = 3$</th>
<th>$deg_{v_2} = 3$</th>
<th>$deg_{u_p} = 2$</th>
<th>$deg_{v_p} = 2$</th>
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<tbody>
<tr>
<td>$deg_{u_1} = 2$</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>4</td>
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<tr>
<td>$deg_{v_1} = 2$</td>
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<td>4</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$deg_{u_2} = 3$</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$deg_{v_2} = 3$</td>
<td>5</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$deg_{u_p} = 2$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>6</td>
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</tr>
<tr>
<td>$deg_{v_p} = 2$</td>
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<td>4</td>
<td>6</td>
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</table>

Theorem 1.1: For $p \geq 4$, then they will be given:

1. $Sc(C_e(C_4)_{p-1}; x) = 2(9p - 10)x + 4\sum_{k=2}^{p-1}(6p - 6k + 1)x^k + 8x^p$.

2. $Sc^*(C_e(C_4)_{p-1}; x) = (27p - 40)x + 6\sum_{k=2}^{p-1}(6p - 6k - 1)x^k + 32x^{p-1} + 8x^p$.

Proof: For all $p \geq 5$ and every two vertices $u, v \in V(C_e(C_4)_{p-1})$, there is a $d(u, v) = k, 1 \leq k \leq p$. There will be five partitions for proof:

P1. If $d(u, v) = 1$, then $|D_1| = 3p - 2 = q(C_e(C_4)_{p-1})$, three subsets will be of it:

P1.1 $|D_1(2,2)| = |\{(u_i, v_i) : i = 1, p\}| = 2$.

P1.2 $|D_1(2,3)| = |\{(u_i, u_2), (u_p, u_{p-1}), (v_1, v_2), (v_p, v_{p-1})\}| = 4$.

P1.3 $|D_1(3,3)| = |\{(u_i, u_{i+1}), (v_i, v_{i+1}) : 2 \leq i \leq p - 2\} \cup \{(u_i, v_i) : 2 \leq i \leq p - 1\}| = 3p - 8$.

P2. If $d(u, v) = k, 2 \leq k \leq p - 3$, then $|D_k| = 4p - 4k + 2$, two subsets will be of it:
P2.1 \(|D_k(2,3)| = |\{(u, u_1+k), (u_p, u_{p-k}), (v_1, v_{1+k})
(v_p, v_{p-k}), (u_1, v_p), (v_1, u_p), (u_p, v_{p+k})\}| = 8.\]

P2.2 \(|D_k(3,3)| = |\{ (u_i, u_{i+k}), (v_i, v_{i+k}) ; 2 \leq i \leq p - k - 1 \} \cup\{(u_1, v_{i+k-1}), (v_1, u_{i+k-1}) ; 2 \leq i \leq p - k\}| = 4p - 4k - 6.\

P3. If \(d(u, v) = p - 2\), then \(|D_p-2| = 10\), two subsets will be of it:
P3.1 \(|D_{p-2}(2,3)| = |\{(u, u_{p-1}), (u_1, v_p), (v_1, u_{p-1}), (u_p, v_1), (u_1, v_1), (u_p, v_p)\}| = 2.\

P4. If \(d(u, v) = p - 1\), then \(|D_{p-1}| = 6\), two subsets will be of it:
P4.1 \(|D_{p-1}(2,2)| = |\{(u, u_{p-1}), (v_1, v_p)\}| = 2.\

P4.2 \(|D_{p-1}(2,3)| = |\{(u, u_{p-1}), (v_1, u_{p-1}), (u_p, v_1), (u_1, v_2)\}| = 4.\

P5. If \(d(u, v) = p\), then \(|D_p| = 2\), one subset will be of it: \(|D_p(2,2)| = |\{(u_1, v_1), (v_1, u_1)\}| = 2.\

From P1 – P5 and Table 1, there will be:
1. \(Sc(C_e(d,4)_{p-1}; x) = 2(9p - 10)x + 4 \sum_{k=2}^{p-1} (6p - 6k + 1)x^k + 8x^p.\)

Now, modified Shultz polynomial can be found:
2. \(Sc^*(C_e(d,4)_{p-1}; x) = (27p - 40)x + 6 \sum_{k=2}^{p-2} (6p - 6k - 1)x^k + 32x^{p-1} + 8x^p.\)

Simply, calculating the following:
\(Sc(C_e(d,4)_{3}; x) = 52x + 52x^2 + 28x^3 + 8x^4.\)
\(Sc^*(C_e(d,4)_{3}; x) = 68x + 66x^2 + 32x^3 + 8x^4.\)

With this, the proof is completed. 

Remark:
1. \(Sc(C_e(d,4)_{2}; x) = 34x + 28x^2 + 8x^3.\)
2. \(Sc^*(C_e(d,4)_{2}; x) = 41x + 32x^2 + 8x^3.\)

Corollary 1.2: For \(p \geq 3\), then they will be given:
1. \(Sc(C_e(d,4)_{p-1}) = 2p(2p^2 + p - 2).\)
2. \(Sc^*(C_e(d,4)_{p-1}) = p(6p^2 - 3p - 2).\)

Corollary 1.3: If \(n\) is a number of rings \(C_e, n \geq 2\), then they will be given:
1. \(Sc(C_e(d,4)_{n}) = 2(2n^3 + 7n^2 + 6n + 1).\)
2. \(Sc^*(C_e(d,4)_{n}) = (6n^3 + 15n^2 + 10n + 1).\)

Corollary 1.4: For \(p \geq 3\), the they will be given:
1. \(Sc(C_e(d,4)_{p-1}) = 2(p + 1) - \frac{2}{2p-1}\).
2. \(Sc^*(C_e(d,4)_{p-1}) = 3p - \frac{2}{2p-1} .\)

From Fig. 3(b), can be noted that \(p(R_e(d,4)_{p}) = 2p, q(R_e(d,4)_{p}) = 3p,\)
diam \(R_e(d,4)_{p} = \frac{p+1}{2}\) and for \(1 \leq i, j \leq p, i \neq j\), there will be:

| Table 2: – Edge – Identification Ring of \(C_e, R_e(d,4)_{p}.\) |
| degv_i = 3 | degv_i = 3 |
| degv_j = 3 | degv_j = 3 |

Theorem 2.1: For \(p \geq 6\), then they will be given:
1. \(Sc(R_e(d,4)_{p}; x) = 18px + 24p \sum_{k=2}^{\frac{p+1}{2}} x^k + 6p \left(4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}\right),\) when \(p\) is an odd,
\(3x^2 + x^{\frac{p+1}{2}},\) when \(p\) is an even.\

2. \(Sc^*(R_e(d,4)_{p}; x) = 27px + 36p \sum_{k=2}^{\frac{p+1}{2}} x^k + 9p \left(4x^{\frac{p-1}{2}} + 2x^{\frac{p+1}{2}}\right),\) when \(p\) is an odd,
\(3x^2 + x^{\frac{p+1}{2}},\) when \(p\) is an even.\

Proof: For every two vertices \(u, v \in V(R_e(d,4)_{p})\) there is \(d(u, v) = k, 1 \leq k \leq \left\lfloor \frac{p+1}{2} \right\rfloor.\)
There will be four partitions for proof:

P1. If \(d(u, v) = 1\), then \(|D_1| = 3p = q(R_e(d,4)_{p}),\) and one subsets will be of it:
\(|D_1(3,3)| = |\{(u_1, u_{i+k}), (v_i, v_{i+k}), (u_1, v_i); 1 \leq i \leq p\}| = 3p,\) where \(u_{p+a} \equiv u_1\) and \(v_{p+a} \equiv v_1.\)

P2. If \(d(u, v) = k, 2 \leq k \leq \left\lfloor \frac{p+1}{2} \right\rfloor - 2,\) then \(|D_k| = 4p,\) and one subsets will be of it:
\(|D_k(3,3)| = |\{(u_i, u_{i+k}), (v_i, v_{i+k}), (u_1, v_{i+k-1}); 1 \leq i \leq p\}| = 4p,\) where \(u_{p+a} \equiv u_1\) and \(v_{p+a} \equiv v_1.\)

P3. If \(d(u, v) = \left\lfloor \frac{p+1}{2} \right\rfloor - 1,\) then
\(|D_\left\lfloor \frac{p+1}{2} \right\rfloor - 1| = 4p,\) when \(p\) is an odd,
\(3p,\) when \(p\) is even.

a. If \(p\) is an odd, one subset will be of it:
\(|D_\left\lfloor \frac{p+1}{2} \right\rfloor - 1(3,3)| = |\{(u_i, u_{i+k}), (v_i, v_{i+k}), (u_1, v_{i+k-1}); 1 \leq i \leq p\}| = 4p,\) where
u_{p+a} \equiv u_a \quad \text{and} \quad v_{p+a} \equiv v_a, \quad a = 1, 2, 3, \ldots, \frac{p-3}{2}.

\left| D_{\frac{p+1}{2}} \begin{pmatrix} 3,3 \end{pmatrix} \right| = \left| \left\{ (u_i, v_{i+\frac{p-1}{2}}), (v_i, v_{i+\frac{p-1}{2}}) : 1 \leq i \leq \frac{p}{2} \right\} \cup \left\{ (u_i, v_{i+\frac{p+1}{2}}), (v_i, u_{i+\frac{p+1}{2}}) : 1 \leq i \leq p \right\} \right| = 3p,

\text{where } u_{p+a} \equiv u_a \text{ and } v_{p+a} \equiv v_a, \quad a = 1, 2, 3, \ldots, \frac{p-1}{2}.

b. If } p \text{ is an even, one subset will be of it:

\left| D_{\frac{p+1}{2}} \begin{pmatrix} 3,3 \end{pmatrix} \right| = \left| \left\{ (u_i, v_{i+\frac{p+1}{2}}), (v_i, v_{i+\frac{p+1}{2}}) : 1 \leq i \leq \frac{p}{2} \right\} \cup \left\{ (u_i, u_{i+\frac{p+1}{2}}), (v_i, u_{i+\frac{p+1}{2}}) : 1 \leq i \leq p \right\} \right| = 2p,

\text{where } u_{p+a} \equiv u_a \text{ and } v_{p+a} \equiv v_a, \quad a = 1, 2, 3, \ldots, \frac{p-1}{2}.

P4. If } d(u, v) = \left[ \frac{p+1}{2} \right], \text{ then:

\left| D_{\left[ \frac{p+1}{2} \right]} \begin{pmatrix} 3,3 \end{pmatrix} \right| = \left\{ (u_i, v_{i+\frac{p-1}{2}}), (v_i, v_{i+\frac{p-1}{2}}) : 1 \leq i \leq \frac{p}{2} \right\}, \quad \text{where } u_{p+a} \equiv u_a \text{ and } v_{p+a} \equiv v_a, \quad a = 1, 2, 3, \ldots, \frac{p-1}{2}.

b. If } p \text{ is an even one subset will be of it:

\left| D_{\left[ \frac{p+1}{2} \right]} \begin{pmatrix} 3,3 \end{pmatrix} \right| = \left| \left\{ (u_i, v_{i+\frac{p+1}{2}}), (v_i, v_{i+\frac{p+1}{2}}) : 1 \leq i \leq \frac{p}{2} \right\} \cup \left\{ (u_i, v_{i+\frac{p+1}{2}}), (v_i, u_{i+\frac{p+1}{2}}) : 1 \leq i \leq p \right\} \right| = 2p,

\text{where } u_{p+a} \equiv u_a \text{ and } v_{p+a} \equiv v_a, \quad a = 1, 2, 3, \ldots, \frac{p-1}{2}.

Corollary (1.1). Since the graph } R_e(C_4) \text{ is 3-regular graph, then:

2. } Sc^e(R_e(C_4); p; x) = \frac{3}{2} Sc(R_e(C_4); x) = 27px + 36p \sum_{k=2}^{\frac{p+1}{2}} x^k + \frac{9}{4} \left( p^2 + 2p - 1 \right), \quad \text{when } p \text{ is an odd,}

\text{and } \frac{9}{4} \left( p^2 + 2p - 1 \right), \quad \text{when } p \text{ is an even.}

With this, the proof is completed.

Remark:
1. } poly(R_e(C_4); x) = \alpha(9x + 6x^2).
2. } poly(R_e(C_4); x) = \alpha(12x + 12x^2 + 4x^3).
3. } poly(R_e(C_4); x) = \alpha(15x + 20x^2 + 10x^3).

\text{Where } \alpha = 6 \text{ when the polynomial poly is Schultz and } \alpha = 9 \text{ when the polynomial poly is modified Schultz.}

Corollary 2.2: For } p \geq 3, \text{ they will be given:

1. } Sc(R_e(C_4); p) = 3p \left( p^2 + 2p - 1 \right), \quad \text{when } p \text{ is an odd,}

2. } Sc^e(R_e(C_4); p) = \frac{3}{2} \left( p^2 + 2p - 1 \right), \quad \text{when } p \text{ is an even.}

Corollary 2.3: For } p \geq 3, \text{ they will be given:

1. } Sc(R_e(C_4); p) = \frac{3}{4} \left( 2p + 5 \right) \left( 1 \right), \quad \text{when } p \text{ is an odd,}

2. } Sc^e(R_e(C_4); p) = \frac{3}{4} \left( 2p + 5 \right) \left( 1 \right), \quad \text{when } p \text{ is an even.}

Schultz and Modified Schultz of } C_e(C_4)_{p-1} \text{ and } R_e(C_4):
From Fig. 4 (a), can be noted that 
\( p(C_e(\mathcal{L}_4)_{p-1}) = 2p, \) \( q(C_e(\mathcal{L}_4)_{p-1}) = 4p - 3 \) and 
\( \text{diam}(C_e(\mathcal{L}_4)_{p-1}) = p. \) For all \( 2 \leq i, j \leq p - 1, \)
i \( \neq j, \) there will be:

| Table 3. Edge – Identification Chain of \( \mathcal{L}_4, C_e(\mathcal{L}_4)_{p-1}. \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| +   | \( \times \) | \( \text{deg}u_1 = 2 \) | \( \text{deg}v_1 = 2 \) | \( \text{deg}u_1 = 3 \) | \( \text{deg}v_1 = 3 \) |
| \( \text{deg}v_1 = 3 \) | 5               | 6               | 6               | 8               | 6               | 8               | 5               | 6               | 4               | 4               |
| \( \text{deg}u_1 = 4 \) | 6               | 6               | 8               | 6               | 8               | 5               | 6               | 4               | 4               | 4               |
| \( \text{deg}v_1 = 4 \) | 8               | 7               | 12              | 8               | 6               | 12              | 7               | 6               | 8               | 4               | 8               |
| \( \text{deg}u_1 = 5 \) | 6               | 8               | 7               | 12              | 8               | 6               | 12              | 7               | 6               | 8               | 4               | 8               |
| \( \text{deg}v_1 = 5 \) | 8               | 7               | 12              | 8               | 6               | 12              | 7               | 6               | 8               | 4               | 8               |
| \( \text{deg}u_1 = 6 \) | 6               | 6               | 9               | 7               | 12              | 7               | 12              | 6               | 8               | 4               | 8               |
| \( \text{deg}v_1 = 6 \) | 6               | 6               | 9               | 7               | 12              | 7               | 12              | 6               | 8               | 4               | 8               |

**Theorem 3.1:** For \( p \geq 4, \) then they will be given:
1. \( Sc(C_e(\mathcal{L}_4)_{p-1}; x) = 2(16p - 19)x \)
   \[ + 4 \sum_{k=2}^{p-1}(8p - 8k - 1)x^k + 4x^p. \]
2. \( Sc^*(C_e(\mathcal{L}_4)_{p-1}; x) = 4(16p - 25)x \)
   \[ + 32 \sum_{k=2}^{p-2}(2p - 2k - 1)x^k + 37x^{p-1} + 4x^p. \]

**Proof:** For all \( p \geq 5 \) and every two vertices \( u, v \in V(C_e(\mathcal{L}_4)_{p-1}), \)
there is \( d(u,v) = k, \) \( 1 \leq k \leq p. \) There will be five partitions for proof:

**P1.** If \( d(u,v) = 1, \) then
\( |D_1| = 4p - 3 = q(C_e(\mathcal{L}_4)_{p-1}) \) and four subsets will be of it:

**P1.1** \( |D_1(2,3)| = |\{(u_1, v_1), (v_1, u_p)\}| = 2. \)

**P1.2** \( |D_1(2,4)| = |\{(u_1, u_2), (v_p, v_{p-1})\}| = 2. \)

**P1.3** \( |D_1(3,4)| = |\{(v_1, v_2), (v_2, u_2), (u_p, u_{p-1}), (u_p, v_{p-1})\}| = 4. \)

**P1.4** \( |D_1(4,4)| = |\{(u_i, u_{i+1}), (v_i, v_{i+1}), (u_i, v_i), (v_i, u_{i+1})\}: 2 \leq i \leq p - 2 \} \cup \{(v_{p-1}, u_{p-1})\}| = 4p - 11. \)

**P2.** If \( d(u,v) = k, \) \( 2 \leq k \leq p - 3, \) then
\( |D_k| = 4p - k + 1 \) and three subsets will be of it:

**P2.1** \( |D_k(2,4)| = |\{(u_1, u_{1+k}), (v_1, v_k), (v_p, v_{p-k}), (u_p, u_{p-k+1})\}| = 4. \)

**P2.2** \( |D_k(3,4)| = |\{(v_1, v_{1+k}), (v_1, u_{1+k}), (u_p, v_{p-k}), (u_p, u_{p-k})\}| = 4. \)

**P2.3** \( |D_k(4,4)| = |\{(u_i, u_{i+k}), (v_i, v_{i+k}), (v_i, u_{i+k}), (u_i, v_{i+k})\}: 2 \leq i \leq p - k - 1 \} \cup \{(u_{p-k}, v_{p-1})\}| = (4p - 4k - 7. \)

**P3.** If \( d(u,v) = p - 2, \) then \( |D_{p-2}| = 9, \) and three subsets will be of it:

**P3.1** \( |D_{p-2}(2,4)| = |\{(u_1, u_{p-2}), (u_1, v_{p-2}), (v_p, u_2), (v_p, v_2)\}| = 4. \)

**P3.2** \( |D_{p-2}(3,4)| = |\{(v_1, v_{p-1}), (v_1, u_{p-1}), (u_p, v_2), (u_p, u_2)\}| = 4. \)

**P3.3** \( |D_{p-2}(4,4)| = |\{(u_2, v_{p-1})\}| = 1. \)

**P4.** If \( d(u,v) = p - 1, \) then \( |D_{p-1}| = 5 \) and three subsets will be of it:

**P4.1** \( |D_{p-1}(2,3)| = |\{(u_1, u_p), (v_p, v_1)\}| = 2. \)

**P4.2** \( |D_{p-1}(2,4)| = |\{(u_1, v_{p-1}), (v_p, u_2)\}| = 2. \)

**P4.3** \( |D_{p-1}(3,3)| = |\{(v_1, u_p)\}| = 1. \)

**P5.** If \( d(u,v) = p, \) then \( |D_p| = 1 \) and one subsets will be of it: \( |D_p(2,2)| = \{|(u_1, v_p)\} = 1. \)

From P1 – P5 and Table 3, there will be:

\( Sc(C_e(\mathcal{L}_4)_{p-1}; x) = 2(16p - 19)x \)
\[ + 4 \sum_{k=2}^{p-1}(8p - 8k - 1)x^k + 4x^p. \]

And
\( Sc^*(C_e(\mathcal{L}_4)_{p-1}; x) = 4(16p - 25)x \)
\[ + 32 \sum_{k=2}^{p-2}(2p - 2k - 1)x^k + 37x^{p-1} + 4x^p. \]

It is easy to calculate that:
\( Sc(C_e(\mathcal{L}_4)_{3}; x) = 90x + 60x^2 + 28x^3 + 4x^4. \)
\( Sc^*(C_e(\mathcal{L}_4)_{3}; x) = 156x + 96x^2 + 37x^3 + 4x^4. \)
With this, the proof is completed.

**Remark:**
1. \( Sc(C_e(\mathcal{L}_4)_{2}; x) = 58x + 28x^2 + 4x^3. \)
2. \( Sc^*(C_e(\mathcal{L}_4)_{2}; x) = 92x + 37x^3 + 4x^4. \)

**Corollary 3.2:** For \( p \geq 3, \) then they will be given:

1. \( Sc(C_e(\mathcal{L}_4)_{p-1}) = \frac{2(8p^3 - 3p^2 + p - 3)}{3}. \)
2. \( Sc^*(C_e(\mathcal{L}_4)_{p-1}) = \frac{32p^3 - 48p^2 + 43p - 27}{3}. \)

**Corollary 3.3:** If \( n \) is a number of rings \( \mathcal{L}_4, n \geq 2, \) then they will be given:

1. \( Sc(C_e(\mathcal{L}_4)_{n}) = \frac{2(8n^3 + 21n^2 + 19n + 3)}{3}. \)
2. \( Sc^*(C_e(\mathcal{L}_4)_{n}) = \frac{n(32n^3 + 48n^2 + 43n + 3)}{3}. \)

**Corollary 3.4:** For \( p \geq 3, \) then they will be given:

1. \( Sc(C_e(\mathcal{L}_4)_{p-1}) = \frac{8p^3 + 11p^2 + 6p - 2}{3} + \frac{p - 2}{p(2p-1)}. \)
2. \( \overline{Sc}^c(C_e(\mathcal{L}_4)p^{-1}) = \frac{16(p-1)}{3} + \frac{9(p-1)}{p(2p-1)}. \)

From Fig. 4 (b), can be noted that
\( p(R_e(\mathcal{L}_4)p) = 2p, \quad q(R_e(\mathcal{L}_4)p) = 4p \)
and\( \text{diam}(R_e(\mathcal{L}_4)p) = \frac{p}{2}. \)
For \( 1 \leq i, j \leq p, \ i \neq j, \) there will be:

<p>| Table 4. Edge – Identification Ring of ( \mathcal{L}_4, R_e(\mathcal{L}_4)p. ) |</p>
<table>
<thead>
<tr>
<th>+</th>
<th>x</th>
<th>deguv_1 = 4</th>
<th>deguv_2 = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>deguj_1 = 4</td>
<td>8</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>deguj_2 = 4</td>
<td>8</td>
<td>16</td>
<td>8</td>
</tr>
</tbody>
</table>

\textbf{Theorem 4.1:} For \( p \geq 5, \) then they will be given:
1. \( Sc(R_e(\mathcal{L}_4)p; x) = 32p \sum_{k=1}^{\left[ \frac{p}{2} \right]-1} x^k + 8p \begin{cases} x^{\frac{p+1}{2}}, & \text{when } p \text{ is an odd,} \\ 3x^{\frac{p}{2}}, & \text{when } p \text{ is an even.} \end{cases} \)
2. \( Sc^c(R_e(\mathcal{L}_4)p; x) = 64p \sum_{k=1}^{\left[ \frac{p}{2} \right]-1} x^k + 16p \begin{cases} x^{\frac{p+1}{2}}, & \text{when } p \text{ is an odd,} \\ 3x^{\frac{p}{2}}, & \text{when } p \text{ is an even.} \end{cases} \)

\textbf{Proof:} For every two vertices \( u, v \in V(R_e(\mathcal{L}_4)p), \) there is \( d(u, v) = k, \ 1 \leq k \leq \left[ \frac{p}{2} \right]. \)
There will be two partitions for proof:

\textbf{P1.} If \( d(u, v) = k, \ 1 \leq k \leq \left[ \frac{p}{2} \right]-1, \) then
\( |D_k| = 4p \) and one subsets will be of it:
\( |D_k(4, A)| = \left\{ (u_i, u_{i+k}), (v_i, v_{i+k}), (u_i, v_{i+k-1}), (v_i, u_{i+k}) \mid 1 \leq i \leq p \right\} = 4p, \) where \( u_{p+a} \equiv u_a \) and \( v_{p+a} \equiv v_a, a = 1, 2, 3, \ldots, k. \)

\textbf{P2.} If \( d(u, v) = \left[ \frac{p}{2} \right], \) then
\( |D_{\left[ \frac{p}{2} \right]}| = \begin{cases} p, & \text{when } p \text{ is an odd,} \\ 3p, & \text{when } p \text{ is an even.} \end{cases} \)

\textbf{a.} If \( p \) is an odd, and one subset will be of it:
\( |D_{\left[ \frac{p}{2} \right]}(4, A)| = \left\{ (u_i, v_{i+\frac{p-1}{2}}) \mid 1 \leq i \leq l \right\}, \) where \( v_{p+a} \equiv v_a, a = 1, 2, 3, \ldots, \frac{p-1}{2}. \)

\textbf{b.} If \( p \) is an even, and one subset will be of it:
\( |D_{\left[ \frac{p}{2} \right]}(4, A)| = \left\{ (u_i, u_{i+\frac{p}{2}}), (v_i, v_{i+\frac{p}{2}}) \mid 1 \leq i \leq \left\lfloor \frac{p}{2} \right\rfloor \\ \cup \left\{ (u_i, v_{i+\frac{p-1}{2}}), (v_i, u_{i+\frac{p-1}{2}}) \mid 1 \leq i \leq \lfloor \frac{p}{2} \rfloor \right\} \right\} = 3p, \) where \( u_{p+a} \equiv u_a, \) and \( v_{p+a} \equiv v_a, a = 1, 2, 3, \ldots, \frac{p}{2}. \)

From P1 and P2 and Table 4, there will be:
1. \( Sc(R_e(\mathcal{L}_4)p; x) = 32p \sum_{k=1}^{\left[ \frac{p}{2} \right]-1} x^k + 8p \begin{cases} x^{\frac{p+1}{2}}, & \text{when } p \text{ is an odd,} \\ 3x^{\frac{p}{2}}, & \text{when } p \text{ is an even.} \end{cases} \)

\textbf{Remark:}
1. \( poly(R_e(\mathcal{L}_4), x) = \alpha \{ 12x + 3x^2 \}. \)
2. \( poly(R_e(\mathcal{L}_4), x) = \alpha \{ 16x + 12x^2 \}. \)

Where \( \alpha = 8 \) when \( poly \) is polynomial Schultz and \( \alpha = 16 \) when \( poly \) is polynomial modified Schultz.

\textbf{Corollary 4.2:} For all \( p \geq 3, \) then they will be given:
1. \( Sc(R_e(\mathcal{L}_4)p) = 4p^2(p + 1). \)
2. \( Sc^c(R_e(\mathcal{L}_4)p) = 8p^2(p + 1). \)

\textbf{Corollary 4.3:} For all \( p \geq 3, \) they will be given:
1. \( \overline{Sc}(R_e(\mathcal{L}_4)p) = 2p + 3 + \frac{3}{2p-1}. \)
2. \( \overline{Sc}^c(R_e(\mathcal{L}_4)p) = 2 \left( 2p + 3 + \frac{3}{2p-1} \right). \)

\textbf{Some Properties of the Coefficients of Schultz and Modified Schultz Polynomials:}
A finite sequence \( (a_1, a_2, \ldots, a_h) \) of \( h \) positive integers is coefficients of polynomial \( P(x) = \sum_{i=1}^{h} a_i x^i. \) Then (Table 5):
1. The polynomial \( P(x) \) is called j-unimodal if, for some index \( j, \ a_1 \leq a_2 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_h \) and it is strictly j-unimodal if the inequality holds without equalities.
2. The polynomial \( P(x) \) is called monotonically increasing (or monotonically decreasing) if, \( a_i \leq a_{i+1} \) or \( a_i \geq a_{i+1} \), respectively, for all \( 1 \leq i \leq h \) and it is strictly-increasing or strictly-decreasing respectively if the inequalities holds without equalities.
3. The polynomial \( P(x) \) is called palindromic if \( a_i = a_{h-i} \), for all \( 1 \leq i \leq h \) and is called semi-palindromic if \( a_j = a_{h-j+1}, \ 1 + i \leq j \leq h \) and for all \( 1 \leq i \leq h-2 \).
4. The polynomial \( P(x) \) is called troubled if \( a_i \neq a_{i+1} \), for all \( 1 \leq i \leq h \).
5. The polynomial $P(x)$ is called equality if $a_i = a_{i+1}$, for all $1 \leq i \leq h$ and is called semi-

Table 5. Some Properties of the Coefficients of Schultz and Modified Schultz Polynomials

<table>
<thead>
<tr>
<th>Polynomials of Types graphs</th>
<th>Property 1</th>
<th>Property 2</th>
<th>Property 3</th>
<th>Property 4</th>
<th>Property 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Sc(G_{c}(G_{p-1}; x))$</td>
<td>Satisfy at $j = 2$</td>
<td>Not Satisfy</td>
<td>Satisfy</td>
<td>Not Satisfy</td>
<td>Not Satisfy</td>
</tr>
<tr>
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<td>Satisfy semi at $j = 2$</td>
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<tr>
<td>$Sc^{*}(R_{c}(G_{p}; x))$</td>
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<td>Satisfy</td>
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</tr>
</tbody>
</table>

Conclusion:
From this paper, the general formulas can be obtained for square graphs for the modified Schultz and Schultz polynomials and some of their properties have been discussed.

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Authors’ declaration:
- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Mosul.

Authors’ contributions statement:
Mahmood M. and Ahmed M. are find the new polynomials which are called Schultz and modified Schultz using a new technique to finding general formulas, indices, average and then studied some properties of coefficients polynomials of Schultz and modified Schultz polynomials.

References:
متعددات حدود شوتز وشوتز المعدلة لتطابق حافة لسلسلة وحلقة للبيانات المربعة

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الخلاصة:
في البيان المتصل G، دالة المسافة بين أي رأسين من رؤوس البيان V(G) هي أقصر مسافة بينهما، كما تعترف درجة الرأس u والتي يرمز لها الدالة 
d(e(u,v)) = deg(u) + deg(v) x d(u,v)
والتي يرمز لها بالعربية: متعددة حدود شوتز وشوتز المعدلة تعرف كالأتالي:

Sc(G;x) = ∑(deg(u) + deg(v)) x d(u,v) and Sc ∗(G;x) = ∑(deg(u).deg(v)) x d(u,v)

على التوالي، حيث أن المجموع يؤخذ لكل الازوج غير المرتبة من الرؤوس المختلفة في V(G) و أن V(G) هو المجموع بين الرؤوس d(u,v)
والتي يرمز لها بالعربية: متعددة حدود شوتز وشوتز المعدلة تعرف كالأتالي:

Sc(G;x) = ∑(deg(u) + deg(v)) x d(u,v) and Sc ∗(G;x) = ∑(deg(u).deg(v)) x d(u,v)

في هذا البحث استطعنا الحصول على صيغ عامة لكل من متعددة حدود شوتز وشوتز المعدلة ودليليهما ومعادلاتها لتطبيق الحافة لسلسلة وحلقة للبيانات المربعة.

الكلمات المفتاحية: تطابق الحافة لسلسلة وحلقة للبيانات المربعة، متعددات الحدود، الأدلة.