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On Hereditarily Codiskcyclic Operators

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Abstract:

Many codiskcyclic operators on infinite-dimensional separable Hilbert space do not satisfy the criterion of codiskcyclic operators. In this paper, a kind of codiskcyclic operators satisfying the criterion has been characterized, the equivalence between them has been discussed and the class of codiskcyclic operators satisfying their direct summand is codiskcyclic. Finally, this kind of operators is used to prove that every codiskcyclic operator satisfies the criterion if the general kernel is dense in the space.

Keywords: Codiskcyclic operators, Criterion for codiskcyclic operators, Generalize kernel, Hereditarily codiskcyclic operators.

Introduction:

Let H be an infinite – dimensional separable Hilbert space, the unit disk is denoted by $\mathbb{D}$, $\mathbb{B}$ is the unit ball, and $B(H)$ is the set of all linear bounded operators onto H. Hilden and Wallen in 1974 introduced the concept of supercyclic operator $T \in B(H)$, as there exists a non-zero vector y in H such that $orbt\,(T,y) := \{\alpha\, T^n y: \mathrm{n} \geq 0, \alpha \in \mathbb{C}\}$ is dense in H [1]. After that in 2002, Jamil in her thesis divided $\mathbb{C}$ into three areas according to the unit circle:

- The interior of the unit circle: thus, an operator is called diskcyclic, if $orbt\,(T,y) := \{\alpha\, T^n y: \mathrm{n} \geq 0, \alpha \in \mathbb{D}\}$ is dense in H.
- The unit circle: then the operator is called circle cyclic, if
  $$orbt\,(T,y) := \{\alpha\, T^n y: \mathrm{n} \geq 0, |\alpha| = 1\}$$
  is dense in H.
- The exterior of the unit circle: hence the operator is called codiskcyclic, if $orbt\,(T,y) := \{\alpha\, T^n y: \mathrm{n} \geq 0, \alpha \in \mathbb{B}^c\}$ is dense in H.

She studied some of their properties like the range of them, some necessarily and sufficient conditions to be [2].

In 2004, Leon-Saavedra and Muller, proved that every circle cyclic operator is hypercyclic, which mean $orbt\,(T,y) := \{\, T^n y: \mathrm{n} \geq 0\}$ is dense in H [3], while the other kinds have been gaining importance in recent years such as Liang and Zhou [4,5], Wong and Zeng [6] …etc.

In 2002, Jamil [2], introduced a criterion for codiskcyclic operators. She showed that there are codiskcyclic operators that do not satisfy this criterion. Thus, the natural question arises is which kind of codiskcyclic operators can satisfy the criterion?

This paper offered a partial solution to this problem by presenting a new kind of codiskcyclic operator that is called hereditarily codiskcyclic. By using the concept of hereditarily codiskcyclic, codiskcyclic operators have been proved to satisfy the criterion whenever the generalize kernel is dense in the space.

Codiskcyclic vectors:

In this section, the focus of attention is on studying the properties of the set of codiskcyclic vectors of $\{T^k\}$ where $\{k\}$ is a sequence of non- negative integers.

Definition (1):

Let $T \in B(H)$, and $\{i\}$ be a non- negative integer, then $\{T^i\}$ is called codiskcyclic if there is non-zero vector y in H such that $\{\beta T^i y: \beta \in \mathbb{B}^c; 0 \leq i\}$ is dense in H in this case y is called codiskcyclic vector for $\{T^i\}$

Remark (1)

If $< i > = < n >$, then $\{T^i\}$ is codiskcyclic, if and only if T is a codiskcyclic operator.

Starting point is discussing how much the set of codiskcyclic vectors for $\{T^k\}$ is

Proposition (1):

Let $< k >$ be a non-negative integers sequence and $T \in B(H)$, then for all $m > 0$, $T^m z$ is codiskcyclic vector for $\{T^k\}$ whenever z is

Proof:

Let z be a codiskcyclic vector for $\{T^k\}$, then for all $m \geq 0$,

$\overline{\mathbb{D}^c orbt\ (T^k, T^m z)} = \overline{\{\beta T^k(T^m z): k \geq 1, \beta \in \mathbb{B}^c\}}$
$= T^m\{\overline{\beta T^k z: k \geq 1, \beta \in \mathbb{B}^c}\} = \overline{T^m(H)} = H$

In addition to discussing how big the set of codiskcyclic vectors is for $\{T^k\}$, the following proposition studying the relation between $\{T^k\}$ is codiskcyclic and topologically transitive.

Since the proof of $3) \Rightarrow 1)$ is trivial, thus it is omitted.

Proposition (2):

Let H be a separable infinite dimensional complex Hilbert space and $T \in B(H)$, Let $\{k\}$ be a non-negative integer sequence. Then the following statements are equivalent

1. $\{T^k\}$ is codiskcyclic.
2. For all V, U non-empty open subsets of H, there is $\ell \in \{k\}$ large enough and $\lambda \in \mathbb{B}^c$ such that $T^\ell(\lambda V) \cap U \neq \emptyset$.
3. $\{T^k\}$ has a dense G_δ-set of codiskcyclic vectors.

Proof:

1)⟹2): Let V, U be open subsets of H. By (2), there is a codiskcyclic vector z for $\{T^k\}$. Thus, there exist $k_1 \in \{k\}$ and $\beta_1 \in \mathbb{B}^c$ such that $\beta_1 T^{k_1} z \in V$. By proposition (1.3), $z_0 = \beta_1 T^{k_1} z$ is a codiskcyclic vector for $\{T^k\}$, hence there is $k_2 \in \{k\}$ and $\beta_2 \in \mathbb{B}^c$ such that $\beta_2 T^{k_2} z_0 \in U$. Therefore $\beta_2 T^{k_2}\ V \cap U \neq \emptyset$.

2)⟹3): Let $\{B_r\}$ be a countable bases for the topology of H. It is easy to see that

$$\mathbb{D}^c C(\{T^k\}) = \bigcap_r \left(\bigcup_{\beta \in \mathbb{D}} \bigcup_{k \in \mathbb{N}} T^{-k}(\beta B_r) \right)$$

So, by the continuity of T, $\bigcup_{\beta \in \mathbb{D}} \bigcup_{k \in \mathbb{N}} T^{-K}(\beta B_r)$ is open set.

Now, let V be a non-empty open set is H. Thus by (4), there are $\ell \in \{k\}$ and $\lambda \in \mathbb{B}^c$ such that $V \cap T^{-\ell}\left(\frac{l}{\lambda} B_r\right) \neq \emptyset$. Therefore, by Bair theorem, $\mathbb{D}^c C(\{T^k\})$ is dense G_δ- set.

Hereditarily Codiskcyclic Operators:

It is well known that not every codiskcyclic operators satisfies codiskcyclic criterion, so this section introduces the following concept and argue the relation between these operators with codiskcyclic criterion.

Definition (2):

A bounded linear operator T is called here ditarilty codiskcyclic if there is a sequence $\{n_k\}$ such that for all subsequence $\{m_k\}$ of $\{n_k\}$, $\{T^{m_k}\}$ is codiskcyclic.

Every hereditarily codiskcyclic is a codiskcyclic operator. One question raises is the converse true.

The following definition and proposition are needed to answer this question.

Definition (3):

A bounded linear operator T onto H, satisfies codiskcyclic criterion, if there are increasing sequences of positive integer $\{n_k\}$ and $\{\alpha_{n_k}\}$ in $(1, \infty)$, for which there are dense sets Y, X in H and a sequence of mappings $S_{n_k}: Y \to H$ such that for all $y \in Y, x \in X$ and when $k \to \infty$:

1) $\alpha_{n_k} T^{n_k} x \to 0$
2) $\frac{1}{\alpha_{n_k}} S_{n_k} y \to 0$
3) $T^{n_k} S_{n_k}\ y \to y$

Example (1) [2]**:** Let $T \in B(\ell^2(\mathbb{Z}))$ be the forward weighted shift with weight sequence,

$$w_n = \begin{cases} \dfrac{1}{n+2} & if\ n > 0 \\ \dfrac{1}{2} & othewise \end{cases}$$

Then T satisfies codiskcyclic criterion.

Proposition (3)

$T \in B(H)$ is a codiskcyclic operator whenever T satisfies codiskcyclic criterion

Proof:

Let V,U be non-empty open sets in H. By the density of Y and X, there are $x \in X \cap V$ and $y \in Y \cap U$ such that $x + \frac{1}{\alpha_{n_k}} S^{n_k} y \to x$ and

$\alpha_{n_k} T^{n_k}\left(x + \frac{1}{\alpha_{n_k}} S^{n_k}\ y\right) \to y$

Thus, for suitable $k \in \mathbb{N}$, there are $\alpha = \alpha_{n_k} \in (1, \infty)$ and $n = n_k \in \mathbb{N}$ such that $\alpha\ T^n V \cap U \neq \emptyset$. Therefore, by proposition (2) T is a codiskcyclic operator.

The result now discusses the relation between hereditarily codiskcyclic operator and operator which satisfies codiskcyclic criterion.

Jamil in [2] proved that if $\oplus_{i=1}^n T_i \in B(\oplus_{i=1}^n H)$ is a codiskcyclic, then T_i is a codiskcyclic operater for all i.

Proposition (4):

Let $T \in B(H)$. Then the following statements are equivalent:

1) For all $n \in \mathbb{N}$, $\oplus_{i=1}^n T \in \mathbb{D}^c\ C\ (\oplus_{i=1}^n H)$.
2) T satisfies codiskcyclic criterion.
3) T is hereditarily codiskcyclic operator.

Proof:
1) $\Rightarrow$ 2): By Jamil`s proposition [2], it is enough to prove when $n=2$.
Since $T\oplus T\in \mathbb{D}^cC(H\oplus H)$, then there is $(x,y)\in \mathbb{D}^cC(T\oplus T)$.
Let $Y=X=\mathbb{D}^c orbt\,(T,x)$, for all $\alpha\in\mathbb{B}^c$ and $m\in\mathbb{N}$ since $I\oplus\alpha T^m$ has dense range and commute with $T\oplus T$. Thus $(x,\alpha T^m y)\in \mathbb{D}^cC(T\oplus T)$. but $y\in\mathbb{D}^cC(T)$, therefore for all zero-neighbourhood V_k, there exists $v_k\in V_k$ closed enough to zero, and $\alpha_k\in\mathbb{B}^c$, $m\in\mathbb{N}$ such that $v_k=\alpha_k T^m y$. Thus $(x,v_k)\in\mathbb{D}^cC(T\oplus T)$ for all $k\geq 1$. Then there exist sequences: $\{n_k\}$ in $\mathbb{N}$, $\{\lambda_{n_k}\}$ in $\mathbb{B}^c$ such that when $k\to\infty$,
a) $T^{n_k}v_k\to 0$
b) $\lambda_{n_k}T^{n_k}x\to 0$
c) $\lambda_{n_k}T^{n_k}v_k\to x$
Now define a sequence of operator $S_{n_k}:Y\to H$ as $S_{n_k}(\alpha\, T^i x)=\lambda_{n_k}T^i u_k$ for all $\alpha\in\mathbb{B}^c$ and $i\in\mathbb{N}$. It is easy to prove that T and $\{S_{n_k}\}$ satisfy the conditions of codiskcyclic criterion (3) thus the result is done by (3).
2) $\Rightarrow$ 3): Let (U,V) be a pair of non-empty open sets in H, since T satisfies codiskcyclic criterion, then there is a pair of dense sets, say (X,Y), sequence $\{n_k\}$ in $\mathbb{N}$, $\{\alpha_k\}$ in $\mathbb{B}^c$, and $S_{n_k}:Y\to H$ such that for all $x\in X$ and $y\in Y$,
a) $\alpha_{n_k}T^{n_k}x\to 0$
b) $\frac{1}{\alpha_{n_k}}S_{n_k}y\to 0$
c) $T^{n_k}S_{n_k}y\to y$
which is true for all subsequence $< m_k >$ of $< n_k >$. Therefor for all $x\in X\cap V$, $y\in y\cap U$، $x+\frac{1}{\alpha_{n_k}}S_{m_k}y\to x$ and

$$\alpha_{n_k}T^{m_k}\left(x+\frac{1}{\alpha_{n_k}}S_{m_k}y\right)\to y.$$

Hence for m_k large enough, $\alpha_{n_k}T^{m_k}(V)\cap U\neq\emptyset$. Thus, by proposition (2) T is a hereditarily codiskcyclic.
3) $\Rightarrow$ 1) Let V_i,U_i be non-empty set in H where $i=1,\dots,n;\ n\geq 1$. Since T is a hereditarily codiskcyclic operator, then there is a sequence $\{n_k\}$ in $\mathbb{N}$ such that for all subsequence $\{m_k\}$ of $\{n_k\}$, $\{T^{m_k}\}$ is codiskcyclic. By proposition (2), there are $m\in\{m_k\}$ and $\alpha_j\in\mathbb{B}^c$ such that $\alpha_j T^m V_1\cap U_1\neq\emptyset$. But $\{T^m\}$ is codiskcyclic, thus there exist subsequences $\{m_i\}$ and $\{\alpha_{j_i}\}$ such that $\alpha_{j_i}T^{m_i}V_2\cap U_2\neq\emptyset$, and so on there are subsequence $\{m_p\}$ and $\{\alpha_{j_p}\}$ such that $\alpha_{j_p}T^{m_p}V_{n-1}\cap V_{n-1}\neq\emptyset$. Because of $\{T^{n_p}\}$ is codiskcyclic, therefore by proposition (2), there are $\ell\in\{m_p\}$ and $\alpha=\{\alpha_{j_p}\}$ such that $\alpha T^{\ell}V_n\cap U_n\neq\emptyset$. Hence $\alpha T^{\ell}V_i\cap U_i\neq\emptyset$ for all $1\leq i\leq n$. Thus $\oplus_{i=1}^n T$ is a codiskcyclic operator.

Remark (2):
By the same argument one can prove that part (2) of proposition (4) is true for $n=\infty$.
One of the applications of proposition (4) is to prove that codiskcyclic operator is hereditarily codiskcyclic under certain condition. But first the following known proposition is important to correctly interpret the results.

Proposition (5) [2]:
Let $T\in B(H)$, T is codiskcyclic if and only if for each $y,x\in H$, there are sequences $\{x_k\}$ in H, $\{n_k\}$ in $\mathbb{N}$, $\{\lambda_k\}$ in $\mathbb{C}$, $\lambda_k\geq 1$ for all k, such that $x_k\to x$ and $T^{n_k}\lambda_k x_k\to y$

Proposition (6):
Let $T\in B(H)$ be a codiskcyclic operator, then T is hereditarily codiskcyclic whenever the generalized kernel, $\bigcup_{n=1}^{\infty} ker\,T^n$, is dense in H.

Proof:
Let y be a non-zero codiskcyclic vector for T. By proposition (5) there are sequences $\{x_k\}$ in H and $\{n_k\}$ in $\mathbb{N}$, $\{\alpha_k\}$ in $\mathbb{B}^c$ such that $x_k\to 0$ and $T^{n_k}\alpha_k x_k\to y$. To satisfiy codiskcyclic criterion. Put $X=\bigcup_{n=1}^{\infty} ker\,T^n$ and $Y=\mathbb{D}^c orbt(T,y)$. Thus X and Y are dense in H. Define $S_{n_k}:Y\to H$ as $S_{n_k}(\lambda T^n y)=\alpha_{n_k}T^r(x_k)$ $\quad r\in\mathbb{N}$. Then:
1) For all $\alpha_{n_k}T^{n_k}x_k\to 0$ as $k\to\infty$,
2) For all $z\in Y$, there are $\lambda\in\mathbb{B}^c$, $r\in\mathbb{N}$ such that $z=\lambda\,T^r x$. Hence as $k\to\infty$
a. $\frac{1}{\alpha_{n_k}}S_{n_K}(z)=T^r x_k\to 0$
b. $T^{n_k}S_{n_k}(z)=\lambda T^r\left(\alpha_{n_k}T^{n_k}x_k\right)\to\lambda T^r y=z$.

Therefore, by proposition (4), T is a hereditarily codiskcyclic operator.

Conclusions:
The open problem is “Which kind of codiskcyclic operators satisfy codiskcyclic criterion? This paper introduced a hereditarily codiskcyclic operators, studied some characterization, and proved that every codiskcyclic operator satisfies the codiskcyclic criterion if the space contains dense general kernel set.

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- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

References:

1. Hilden HM, Wallen LJ. Some cyclic and non-cyclic vectors of certain operators. Indiana Univ. Math. J. 1974; 23: 557–565.
2. Jamil Z Z. Cyclic phenomena of operators on Hilbert space. Ph.D. Thesis 2002, University of Baghdad, Iraq.
3. Leon-Saavedra F, Muller V. Rotations of Hypercyclic and Supercyclic Operators. Integr. equ. oper. Theory. 2004; 50: 385–391
4. Liang Y, Zhou Z. Disk–cyclicity and codisk–cyclicity of certain shift operators. Operators and Matrices. 2015; 4 (9): 831–846.
5. Liang Y, Zhou Z. Disk-cyclic and Codisk-cyclic tuples of the adjoint weighted composition operators on Hilbert spaces. Bull. Belg. Math. Soc. Simon Stevin. 2016; 2 (23): 203-215.
6. Wang Y, Zeng H. Disk-cyclic and codisk-cyclic weighted pseudo-shifts. Bull. Belg. Math. Soc. Simon Stevin. 2018; 2 (25): 209-224.

خواص المؤثرات القرصية الدوراية المشاركة الوراثية

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قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد ، العراق

الخلاصة:

العديد من المؤثرات القرصية المشاركة المعرفة على فضاء هلبرت منفصل وغير نهائي الابعاد لا تحقق معيارية المؤثرات القرصية الدوارية المشاركة. في هذا البحث شخصنا نوع من أنواع المؤثرات القرصية الدوارية المشاركة التي تحقق معيارية المؤثرات القرصية الدوارية المشاركة وستخدمناها لبرهان انه أي مؤثر قرصي دواري المشارك يحقق معيارية المؤثرات القرصية الدوارية المشاركة اذا كان نواة العامة كثيفة في الفضاء.

الكلمات المفتاحية: المؤثرات القرصية الدوارية المشتركة، معيار المؤثرات القرصية الدوراية المشتركة، المؤثرات القرصية الدوراية المشتركة الوراثية، النواة العمومية.