Variational Formulation with Deviating Arguments of Movable boundaries

Nadia K. Marie *

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Abstract

In this paper, we study, in details the derivation of the variational formulation corresponding to functional with deviating arguments corresponding to movable boundaries. Natural or transversility conditions are also derived, as well as, the Eulers equation. Example has been taken to explain how to apply natural boundary conditions to find extremal of this functional.

Introduction

Variational problems with retarded time arguments are of great importance, since we consider such variational problems in order to study and justify the statement of the basic boundary value problems of differential equation with deviated arguments [3]. The variational problem with one deviating argument can be written in the following form [1]:

$$\int_{t_0}^{t_1} F(t, x(t), x'(t), x(t-\tau), x'(t-\tau))\, dt$$

or in more simplest form as:

$$\int_{t_0}^{t_1} F(t, x(t-\tau), x'(t-\tau))\, dt$$

variational problem with lag differs form the classical problems of calculus of variation in that the terms $x(t-\tau)$ and or $x'(t-\tau)$ are present in the functional. Variational problems for functionals depending on functions with deviating arguments have been studied by L .E .El'sgolg's [3], L. D. Sabbach [8], G. A. Kamenskii [4], A.D. Myshkis [5], A. M. Popov [7],N.K. Marie [6], S. B. Norkin, A. L. Skubachevskii, D.K. Hughes and others.

Variational Problems with Movable boundaries:

As a general case for such type of problems, let us consider the problem of the extremal of the functional

$$J(x(t)) = F(t, x(t-\tau), x'(t-\tau))\, dt ....(1)$$

* Instructor- Mathmatics, Dept. – College of Science for Women – Baghdad University

under the assumption that some or all of the points with coordinates $t_0 - \tau$ and $t_1 - \tau$ are unknown: The increment of the functional (1) can be expressed in the form:

$\delta J = J(x + \delta x) - J(x)|_{\text{linear part}}$

where

$$J(x,\delta x) = \int_{t_0}^{t_1} F(t, x(t-\tau) + \delta x(t-\tau), x'(t-\tau) + \delta x'(t-\tau))\,dt$$

Hence

$$\delta J = \int_{t_0}^{t_1} F(t, x(t-\tau) + \delta x(t-\tau), x'(t-\tau) + \delta x'(t-\tau))\,dt$$

$$-\int_{t_0}^{t_1} F(t, x(t-\tau), x'(t-\tau))\,dt$$

$$= \int_{t_0}^{t_1} \left\{ \begin{matrix} F(t, x(t-\tau) + \delta x(t-\tau), x'(t-\tau) + \delta x'(t-\tau)) - \\ F(t, x(t-\tau), x'(t-\tau)) \end{matrix} \right\} dt$$

and it follows upon using Taylor's theorem that [2]:

$$\delta J = \int_{t_0}^{t_1} \left\{ \begin{matrix} F_{x(t-\tau)}\delta x(t-\tau) + \\ F_{x'(t-\tau)}\delta x'(t-\tau) \end{matrix} \right\} dt$$

and since, the first variation of the functional (1) equals zero ($\delta J = 0$), when evaluating the linear part we get:

$$\delta J = \int_{t_0}^{t_1} \left\{ \begin{matrix} F_{x(t-\tau)}\delta x(t-\tau) + \\ F_{x'(t-\tau)}\delta x'(t-\tau) \end{matrix} \right\} dt = 0$$

Suppose:

$\overline{F} = F(\gamma(t), x(t), x'(t))$

where $\gamma(t)$ is the inverse function of

$t - \tau$

Let $z = t - \tau$, then we have:

$$\delta J = \int_{t_0}^{t_1} \left\{ \begin{matrix} F_{x(t-\tau)}\delta x(t-\tau) + \\ F_{x'(t-\tau)}\delta x'(t-\tau) \end{matrix} \right\} dt$$

$$= \int_{t_0-\tau}^{t_1-\tau} \left\{ \begin{matrix} \overline{F}_{x(z)}\delta x(z) + \\ \overline{F}_{x'(z)}\delta x'(z) \end{matrix} \right\} \gamma'(z)\,dz = 0$$

Now, returning to the original independent variable, we obtain:

$$\delta J = \int_{t_0-\tau}^{t_1-\tau} \left\{ \begin{matrix} \overline{F}_{x(t)}\delta x(t) + \\ \overline{F}_{x'(t)}\delta x'(t) \end{matrix} \right\} \gamma'(t)\,dt = 0$$

Integrating the second term by parts, we get:

$$\int_{t_0-\tau}^{t_1-\tau} \overline{F}_{x'(t)}\delta x'(t)\gamma'(t)\,dt =$$

$$\overline{F}_{x'(t)}\gamma'(t)\delta x(t)\Big|_{t_0-\tau}^{t_1-\tau} - \int_{t_0-\tau}^{t_1-\tau} \frac{d}{dt}\left(\overline{F}_{x'(t)}\gamma'(t)\right)\delta x(t)\,dt$$

so, we obtain that:

$$\int_{t_0-\tau}^{t_1-\tau} \left\{ \overline{F}_{x(t)}\gamma'(t) - \frac{d}{dt}\left(\overline{F}_{x'(t)}\gamma'(t)\right) \right\} \delta x(t)\,dt +$$

$$\overline{F}_{x'(t)}\gamma'(t)\delta x(t)\Big|_{t_1-\tau} -$$

$$\overline{F}_{x'(t)}\gamma'(t)\delta x(t)\Big|_{t_0-\tau} = 0$$

since in this case the class of admissible functions is wider than in the clase of non movable boundaries, the function x(t) that realizes the extremum in the present case must satisfy the fundamental necessity condition for the case of non movable boundary points, i.e., it must satisfy the Euler equation

$$\overline{F}_{x(t)}\gamma'(t) - \frac{d}{dt}\left(\overline{F}_{x'(t)}\gamma'(t)\right) = 0$$

and in order to make the fundamental lemma of calculus of variation to be satisfied, we must have the following three cases:

1- If $\delta x(t_1 - \tau) = 0$, we get that $\overline{F}_{x'(t)}\gamma'(t)\delta x(t)\Big|_{t_0-\tau} = 0$ since $\delta x(t)$

is arbitrary, then we have $\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_0-\tau} = 0$.

2- If $\delta x(t_0 - \tau) = 0$ we get $\left.\bar{F}_{x'(t)}\gamma'(t)\delta x(t)\right|_{t_1-\tau} = 0$ since $\delta x(t)$ is arbitrary then we have $\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_1-\tau} = 0$.

3- If all the point with coordinates $t_0 - \tau$, $t_1 - \tau$ are unknown, we get:

$$\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_0-\tau} = 0$$

and

$$\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_1-\tau} = 0.$$

Thus , in order to solve the variable end point problem, we must first find a general solution of Euler's equation, and then use the conditions $\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_0-\tau} = 0$ and $\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_1-\tau} = 0$, which are called the natural boundary conditions in deviating arguments. As an illustration, we consider the following example:

Example

Consider the determination of the extremals of the functional

$$J(x) = \int_{t_0=1}^{t_1=2} \left\{ x'^2(t-1) + \frac{t}{2}x(t-1) \right\} dt$$

With the initial condition $x(t_1 - 1) = 1$.

Therefore, the integrand is given by:

$$F(t, x(t-1), x'(t-1)) = x'^2(t-1) + \frac{t}{2}x(t-1)$$

and since the necessary condition is of the form:

$$\bar{F}_{x(t)}\gamma'(t) - \frac{d}{dt}\left(\bar{F}_{x'(t)}\gamma'(t)\right) = 0$$

where:

$\bar{F} = F(\gamma(t), x(t), x'(t))$

$= F(t + 1, x(t), x'(t))$

$= x'^2(t) + \dfrac{t+1}{2}x(t)$

Hence, Euler's equation takes the form:

$$\frac{t+1}{2} - 2x''(t) = 0$$

The final form of Euler's equation takes the form:

$$\frac{t+1-1}{2} - 2x''(t-1) = 0$$

or equivalently:

$$x''(t-1) = \frac{t}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Equation (2) is the delay differential equation of the second order.

In order to solve equation (2), we integrate the above term, we get:

$$x'(t-1) = \frac{t^2}{8} + c_1$$

where c_1 is an arbitrary constant.

To find the value of c_1, the natural boundary condition is $\left.\bar{F}_{x'(t)}\gamma'(t)\right|_{t_0-\tau} = 0$ is taken, we have:

$$\left.\bar{F}_{x'(t)}\gamma'(t) = 2x'(t)\right|_{t_0-\tau} = 0$$

then

$2x'(t_0 - \tau) = 0$, i.e., $2x'(1 - 1) = 0$

and so

$0 = \dfrac{1}{8} + c_1$ then $c_1 = -\dfrac{1}{8}$, so that

$$x'(t-1) = \frac{t^2}{8} - \frac{1}{8}$$

Integrating this equation again, we get:

$$x(t-1) = \frac{t^3}{24} - \frac{1}{8}t + c_2$$

where c_2 is a constant, and since $x(t_1 - 1) = 1$, we can evaluat the value of the constant c_2 to be equals to $\frac{11}{12}$, so that the solution of equation (2) is of the form:

$$x(t-1) = \frac{t^3}{24} - \frac{1}{8}t + \frac{11}{12}$$

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اشتقاق الصياغة التغايرية المتماثلة لدالي مع تباطؤ في الزمن للحدود الغير معروفة

نادية خزعل مرعي

قسم الرياضيات-كلية العلوم للبنات-جامعة بغداد

المستخلص

درس في هذا البحث تفاصيل اشتقاق الصياغة التغايرية المتماثلة لدالي مع تباطؤ في الزمن للحدود الغير معروفة (المتحركة) (Movable boundaries) . وقد تم اشتقاق الشروط الطبيعية (Natural conditions) , بالإضافة إلى ذلك معادلة اويلر (Eulers Eq.) . كما قدم مثال لتوضيح كيفية تطبيق الشروط الطبيعية لايجاد نقاط التطرف للدالي .