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A complete (48, 4)-arc in the Projective Plane Over the Field of Order Seventeen

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Abstract:

The article describes a certain computation method of (k, n)-arcs to construct the number of distinct $(k, 4)$-arcs in PG$(2,17)$ for $k = 7, \ldots, 48$. In this method, a new approach employed to compute the number of (k, n)-arcs and the number of distinct (k, n)–arcs respectively. This approach is based on choosing the number of inequivalent classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ of i-secant distributions that is the number of 4-secant, 3-secant, 2-secant, 1-secant and 0-secant in each process. The maximum size of $(k, 4)$-arc that has been constructed by this method is $k = 48$. The new method is a new tool to deal with the programming difficulties that sometimes may lead to programming problems represented by the increasing number of arcs. It is essential to reduce the established number of (k, n)-arcs in each construction especially for large value of k and then reduce the running time of the calculation. Therefore, it allows to decrease the memory storage for the calculation processes. This method's effectiveness evaluation is confirmed by the results of the calculation where a largest size of complete $(k, 4)$-arc is constructed. This research's calculation results develop the strategy of the computational approaches to investigate big sizes of (k, n)–arcs in $PG(2, q)$ where it put more attention to the study of the number of the inequivalent classes of i-secants of (k, n)-arcs K_i in $PG(2, q)$ which is an interesting aspect. Consequently, it can be used to establish a large value of k.

Keywords: Complete arc, Distinct secant distribution, Group, Projective space, (k, n)-arc.

Introduction:

In 1947, the notions of arcs and also complete arcs were presented. In the projective plane $PG(2, q)$ over the finite field $\mathbf{F}_q$, a (k,n)-arc is a set of k points such that some n are collinear but no $n+1$ are collinear. Also, a (k, n)-arc is complete if it is not subset of $(k + 1, n)$-arc. Moreover, a line contains two points of an arc is called a secant. The study of finite projective planes become one of the most essential problems in combinatorics and several associated areas. For instance, coding theory. In addition, many attempts have been made to determine the size of the maximum (k, n)-arc in $PG(2, q)$ to establish a good upper bound. For a more detailed to the sizes of (k, n)-arcs (1-7).

The main gaols in this research are to construct the number of $(k, 4)$-arcs, to classify the number of distinct $(k, 4)$-arcs in $PG(2, 17)$, to compute the stabiliser group for each distinct arc, and to discuss the group action of the most interesting stabiliser group on an associated arc. In this area of study (k, n)-arcs and for n>2, many researchers have followed the method of constructing a particular (k, n)-arc that contains many points. However, the technique used in this research is a classification method that classifies numbers of (k, n)-arcs in $PG(2, q)$. The spectrum of $(k,4)$-arcs has been calculated using a new technique which is based on the number of inequivalent classes of secant distribution in each construction process.

The start of this technique is fixing a set of six points $S = \{U_0, U_1, U_2, U, P_1, P_2\}$. Here, the points U_0, U_1, U_2, U are the frame points and P_1, P_2 are any two collinear points from the plane. $PG(2, 17)$, implies that the set S containing four collinear points. For example, the first step begins from the set S. Then, the second step is to construct the number of (7,4)-arcs. This construction is made by adding separately to S all the points from the lines l_j of $PG(2, 17)$, $j = 1,\ldots,307$ that satisfy $l_j \cap S = 4$. Then the i-secant distribution for each arc is calculated and then among these classes, the distinct classes of i-secant distribution with their

associated arcs are chosen only. So, the number of distinct (7,4)-arcs is based on the number of distinct classes of secant distribution. All the details of this method are illustrated in section of research method.

This technique is a systematic method. It is used due to the increasing number of arcs especially for large value of k. The maximum size of a $(k, 4)$-arc in $PG(2,17)$ established by this method is $k = 48$. Also, a $(k, 4)$-arc of size 48 was also found in (8). This arc is found by prescribing the group generated and it has the same secant distribution of our arc. However, it has different stabiliser group. In conclusion, a new method established a large k systematically based on the concept of the i-secant distribution of arc. Also, it reduced the programming problems in terms of the constructed number of arcs, the executed time, and the memory storage. The programming language that used in this research is Gap (9). So, the problems of finding the number of (k, n)-arcs and the number of distinct (k, n)-arcs in $PG(2, q)$ are still interesting problems (10). Thus, in the study of projective planes and for a certain value of q, some questions arise .

(i) For given k and n, what is the number of projectively distinct (k, n)-arcs in $PG(2, q)$?

(ii)What is the maximum size $m_n(2, q)$ of a complete (k, n)-arc in $PG(2, q)$ for given values of n and q?

The projective plane $PG(2, 17)$

The projective plane $PG(2, q)$ is an incident structure of points and lines over the Galois field $\mathbf{F}_q$. It contains $q^2 + q + 1$ points and lines. Therefore, the line contains $q + 1$ points and there are $q + 1$ lines concurrent with a unique point. So, the number of points and lines in the plane $PG(2, 17)$ is 307, with 18 points on each line and 18 lines concurrent with a point as given in Tables 1 and 2. (11-13).

Table 1. The lines of $PG(2, 17)$

L1	{1,2,4,46,59,63,74,97,111,123,143,150,179,197,268,278,287,303}
L2	{2,3,5,47,60,64,75,98,112,124,144,151,180,198,269,279,288,304}
L3	{3,4,6,48,61,65,76,99,113,125,145,152,181,199,270,280,289,305}
L4	{4,5,7,49,62,66,77,100,114,126,146,153,182,200,271,281,290,306}
L5	{5,6,8,50,63,67,78,101,115,127,147,154,183,201,272,282,291,307}
L6	{6,7,9,51,64,68,79,102,116,128,148,155,184,202,273,283,292 ,1}
L7	{7,8,10,52,65,69,80,103,117,129,149,156,185,203,274,284,293,2}
L8	{8,9,11,53,66,70,81,104,118,130,150,157,186,204,275,285,294,3}
L9	{9,10,12,54,67,71,82,105,119,131,151,158,187,205,276,286,295,4}
L10	{10,11,13,55,68,72,83,106,120,132,152,159,188,206,277,287,296,5}
L11	{11,12,14,56,69,73,84,107,121,133,153,160,189,207,278,288,297,6}
L12	{12,13,15,57,70,74,85,108,122,134,154,161,190,208,279,289,298,7}
L13	{13,14,16,58,71,75,86,109,123,135,155,162,191,209,280,290,299,8}
L14	{14,15,17,59,72,76,87,110,124,136,156,163,192,210,281,291,300,9}
L15	{15,16,18,60,73,77,88,111,125,137,157,164,193,211,282,292,301,10}
L16	{16,17,19,61,74,78,89,112,126,138,158,165,194,212,283,293,302,11}
L17	{17,18,20,62,75,79,90,113,127,139,159,166,195,213,284,294,303,12}
L18	{18,19,21,63,76,80,91,114,128,140,160,167,196,214,285,295,304,13}
L19	{19,20,22,64,77,81,92,115,129,141,161,168,197,215,286,296,305,14}
L20	{20,21,23,65,78,82,93,116,130,142,162,169,198,216,287,297,306,15}
L21	{21,22,24,66,79,83,94,117,131,143,163,170,199,217,288,298,307,16}
L22	{22,23,25,67,80,84,95,118,132,144,164,171,200,218,289,299, 1,17}
L23	{23,24,26,68,81,85,96,119,133,145,165,172,201,219,290,300, 2,18}
.	.
.	.
.	.
.	.
L307	{307, 1, 3,45,58,62,73,96,110,122,142,149,178,196,267,277,286,302}

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Table 2. Points of $PG(2, 17)$

1	{ 1, 0, 0 }	34	{ 3, 12, 1 }	67	{ 6, 13, 1 }
2	{ 0, 1, 0 }	35	{ 4, 6, 1 }	68	{ 5, 11, 1 }
3	{ 0, 0, 1 }	36	{ 8, 15, 1 }	69	{ 9, 16, 1 }
4	{ 14, 1, 0 }	37	{ 10, 4, 1 }	70	{ 3, 7, 1 }
5	{ 0, 14, 1 }	38	{ 12, 7, 1 }	71	{ 2, 3, 1 }
6	{ 1, 11, 1 }	39	{ 2, 14, 1 }	72	{ 16, 1, 1 }
7	{ 9, 11, 1 }	40	{ 1, 16, 1 }	73	{ 14, 0, 1 }
8	{ 9, 4, 1 }	41	{ 3, 15, 1 }	74	{ 10, 1, 0 }
9	{ 12, 11, 1 }	42	{ 10, 15, 1 }	75	{ 0, 10, 1 }
10	{ 9, 12, 1 }	43	{ 10, 3, 1 }	76	{ 15, 12, 1 }
11	{ 4, 15, 1 }	44	{ 16, 15, 1 }	77	{ 4, 7, 1 }
12	{ 10, 6, 1 }	45	{ 10, 0, 1 }	78	{ 2, 8, 1 }
13	{ 8, 16, 1 }	46	{ 9, 1, 0 }	79	{ 6, 11, 1 }
14	{ 3, 8, 1 }	47	{ 0, 9, 1 }	80	{ 9, 13, 1 }
15	{ 6, 9, 1 }	48	{ 11, 2, 1 }	81	{ 5, 6, 1 }
16	{ 11, 14, 1 }	49	{ 7, 6, 1 }	82	{ 8, 1, 1 }
17	{ 1, 13, 1 }	50	{ 8, 7, 1 }	83	{ 14, 9, 1 }
18	{ 5, 8, 1 }	51	{ 2, 11, 1 }	84	{ 11, 13, 1 }
19	{ 6, 5, 1 }	52	{ 9, 8, 1 }	85	{ 5, 14, 1 }
20	{ 13, 15, 1 }	53	{ 6, 14, 1 }	86	{ 1, 15, 1 }
21	{ 10, 10, 1 }	54	{ 1, 9, 1 }	87	{ 10, 16, 1 }
22	{ 15, 13, 1 }	55	{ 11, 4, 1 }	88	{ 3, 6, 1 }
23	{ 5, 13, 1 }	56	{ 12, 3, 1 }	89	{ 8, 12, 1 }
24	{ 5, 7, 1 }	57	{ 16, 10, 1 }	90	{ 4, 5, 1 }
25	{ 2, 13, 1 }	58	{ 15, 0, 1 }	91	{ 13, 1, 1 }
26	{ 5, 12, 1 }	59	{ 3, 1, 0 }	92	{ 14, 14, 1 }
27	{ 4, 9, 1 }	60	{ 0, 3, 1 }	93	{ 1, 12, 1 }
28	{ 11, 10, 1 }	61	{ 16, 6, 1 }	94	{ 4, 3, 1 }
29	{ 15, 8, 1 }	62	{ 8, 0, 1 }	95	{ 16, 13, 1 }
30	{ 6, 2, 1 }	63	{ 11, 1, 0 }	96	{ 5, 0, 1 }
31	{ 7, 12, 1 }	64	{ 0, 11, 1 }	97	{ 8, 1, 0 }
32	{ 4, 12, 1 }	65	{ 9, 14, 1 }	98	{ 0, 8, 1 }
33	{ 4, 16, 1 }	66	{ 1, 8, 1 }	99	{ 6, 15, 1 }

100	{ 10, 5, 1 }	116	{ 14, 11, 1 }	132	{ 13, 13, 1 }
101	{ 13, 9, 1 }	117	{ 9, 6, 1 }	133	{ 5, 5, 1 }
102	{ 11, 11, 1 }	118	{ 8, 13, 1 }	134	{ 13, 8, 1 }
103	{ 9, 15, 1 }	119	{ 5, 2, 1 }	135	{ 6, 6, 1 }
104	{ 10, 12, 1 }	120	{ 7, 3, 1 }	136	{ 8, 4, 1 }
105	{ 4, 8, 1 }	121	{ 16, 14, 1 }	137	{ 12, 15, 1 }
106	{ 6, 7, 1 }	122	{ 1, 0, 1 }	138	{ 10, 2, 1 }
107	{ 2, 1, 1 }	123	{ 7, 1, 0 }	139	{ 7, 14, 1 }
108	{ 14, 3, 1 }	124	{ 0, 7, 1 }	140	{ 1, 3, 1 }
109	{ 16, 5, 1 }	125	{ 2, 5, 1 }	141	{ 16, 12, 1 }
110	{ 13, 0, 1 }	126	{ 13, 4, 1 }	142	{ 4, 0, 1 }
111	{ 1, 1, 0 }	127	{ 12, 12, 1 }	143	{ 13, 1, 0 }
112	{ 0, 1, 1 }	128	{ 4, 11, 1 }	144	{ 0, 13, 1 }
113	{ 14, 1, 1 }	129	{ 9, 2, 1 }	145	{ 5, 4, 1 }
114	{ 14, 15, 1 }	130	{ 7, 5, 1 }	146	{ 12, 10, 1 }
115	{ 10, 1, 1 }	131	{ 13, 5, 1 }	147	{ 15, 3, 1 }
148	{ 16, 11, 1 }	165	{ 5, 10, 1 }	182	{ 15, 9, 1 }
149	{ 9, 0, 1 }	166	{ 15, 4, 1 }	183	{ 11, 15, 1 }
150	{ 15, 1, 0 }	167	{ 12, 4, 1 }	184	{ 10, 11, 1 }
151	{ 0, 15, 1 }	168	{ 12, 16, 1 }	185	{ 9, 1, 1 }
152	{ 10, 8, 1 }	169	{ 3, 4, 1 }	186	{ 14, 10, 1 }
153	{ 6, 12, 1 }	170	{ 12, 1, 1 }	187	{ 15, 10, 1 }
154	{ 4, 2, 1 }	171	{ 14, 13, 1 }	188	{ 15, 5, 1 }
155	{ 7, 11, 1 }	172	{ 5, 9, 1 }	189	{ 13, 10, 1 }

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156	{ 9, 10, 1 }	173	{ 11, 12, 1 }	190	{ 15, 15, 1 }
157	{ 15, 1, 1 }	174	{ 4, 1, 1 }	191	{ 10, 9, 1 }
158	{ 14, 16, 1 }	175	{ 14, 5, 1 }	192	{ 11, 5, 1 }
159	{ 3, 2, 1 }	176	{ 13, 3, 1 }	193	{ 13, 16, 1 }
160	{ 7, 2, 1 }	177	{ 16, 16, 1 }	194	{ 3, 3, 1 }
161	{ 7, 4, 1 }	178	{ 3, 0, 1 }	195	{ 16, 7, 1 }
162	{ 12, 2, 1 }	179	{ 12, 1, 0 }	196	{ 2, 0, 1 }
163	{ 7, 15, 1 }	180	{ 0, 12, 1 }	197	{ 16, 1, 0 }
164	{ 10, 13, 1 }	181	{ 4, 10, 1 }	198	{ 0, 16, 1}

199	{ 3, 16, 1 }	232	{ 15, 2, 1 }	265	{ 6, 3, 1 }
200	{ 3, 13, 1 }	233	{ 7, 8, 1 }	266	{ 16, 8, 1 }
201	{ 5, 16, 1 }	234	{ 6, 1, 1 }	267	{ 6, 0, 1 }
202	{ 3, 11, 1 }	235	{ 14, 7, 1 }	268	{ 2, 1, 0 }
203	{ 9, 5, 1 }	236	{ 2, 7, 1 }	269	{ 0, 2, 1 }
204	{ 13, 2, 1 }	237	{ 2, 15, 1 }	270	{ 7, 9, 1 }
205	{ 7, 7, 1 }	238	{ 10, 7, 1 }	271	{ 11, 16, 1 }
206	{ 2, 6, 1 }	239	{ 2, 4, 1 }	272	{ 3, 5, 1 }
207	{ 8, 9, 1 }	240	{ 12, 5, 1 }	273	{ 13, 11, 1 }
208	{ 11, 1, 1 }	241	{ 13, 6, 1 }	274	{ 9, 9, 1 }
209	{ 14, 12, 1 }	242	{ 8, 8, 1 }	275	{ 11, 3, 1 }
210	{ 4, 14, 1 }	243	{ 6, 16, 1 }	276	{ 16, 4, 1 }
211	{ 1, 4, 1 }	244	{ 3, 10, 1 }	277	{ 12, 0, 1 }
212	{ 12, 9, 1 }	245	{ 15, 14, 1 }	278	{ 5, 1, 0 }
213	{ 11, 9, 1 }	246	{ 1, 6, 1 }	279	{ 0, 5, 1 }
214	{ 11, 7, 1 }	247	{ 8, 6, 1 }	280	{ 13, 7, 1 }
215	{ 2, 9, 1 }	248	{ 8, 10, 1 }	281	{ 2, 2, 1 }
216	{ 11, 6, 1 }	249	{ 15, 6, 1 }	282	{ 7, 10, 1 }
217	{ 8, 2, 1 }	250	{ 8, 14, 1 }	283	{ 15, 11, 1 }
218	{ 7, 13, 1 }	251	{ 1, 14, 1 }	284	{ 9, 3, 1 }
219	{ 5, 15, 1 }	252	{ 1, 5, 1 }	285	{ 16, 9, 1 }
220	{ 10, 14, 1 }	253	{ 13, 14, 1 }	286	{ 11, 0, 1 }
221	{ 1, 2, 1 }	254	{ 1, 1, 1 }	287	{ 4, 1, 0 }
222	{ 7, 1, 1 }	255	{ 14, 2, 1 }	288	{ 0, 4, 1 }
223	{ 14, 8, 1 }	256	{ 7, 16, 1 }	289	{ 12, 13, 1 }
224	{ 6, 4, 1 }	257	{ 3, 9, 1 }	290	{ 5, 1, 1 }
225	{ 12, 6, 1 }	258	{ 11, 8, 1 }	291	{ 14, 6, 1 }
226	{ 8, 5, 1 }	259	{ 6, 10, 1 }	292	{ 8, 11, 1 }
227	{ 13, 12, 1 }	260	{ 15, 16, 1 }	293	{ 9, 7, 1 }
228	{ 4, 4, 1 }	261	{ 3, 1, 1 }	294	{ 2, 16, 1 }
229	{ 12, 14, 1 }	262	{ 14, 4, 1 }	295	{ 3, 14, 1 }
230	{ 1, 7, 1 }	263	{ 12, 8, 1 }	296	{ 1, 10, 1 }
231	{ 2, 10, 1 }	264	{ 6, 8, 1 }	297	{ 15, 7, 1 }

298	{ 2, 12, 1 }
299	{ 4, 13, 1 }
300	{ 5, 3, 1 }
301	{ 16, 2, 1 }
302	{ 7, 0, 1 }
303	{ 6, 1, 0 }
304	{ 0, 6, 1 }
305	{ 8, 3, 1 }
306	{ 16, 3, 1 }
307	{ 16, 0, 1 }

Constants for an arc

A line L is an *i-secant* of K if $|L \cap K| = i$ (14).

Notation

(i) Let ρ_i be the number of i-secants of a (k, n)-arc K in $PG(2, q)$

(ii) Let $\eta_i(P)$ be the number of i-secant through the point $P \epsilon K$.

(iii) Let $\theta_i(Q)$ be the number of i-secants through the point $Q \epsilon PG(2, q) \setminus K$.

Definitions

1. The secant distribution of a (k, n)-arc K is the ordered (n+1)-tuple $(\rho_1, \rho_2, \ldots, \rho_n)$.

2. Two (k, n)-arcs are equivalent if their i-secant distributions are the same (14,15).

Lemma (16)

The following equations hold for a (k, n)-arc:

1. $\sum_{i=0}^{n} \rho_i = q^2 + q + 1$;
2. $\sum_{i=1}^{n} i\rho_i = k(q + 1)$;
3. $\sum_{i=2}^{n} \frac{1}{2} i(i - 1)\rho_i = \frac{1}{2} k(k - 1)$;
4. $\sum_{i=1}^{n} \eta_i(P) = (q + 1)$;
5. $\sum_{i=2}^{n} (i - 1)\eta_i(P) = (k - 1)$;
6. $\sum_{i=0}^{n} \theta_i(Q) = (q + 1)$;
7. $\sum_{i=1}^{n} i\theta_i(Q) = k$;
8. $i\rho_i \sum_{P \epsilon k} \eta_i(P)$;
9. $(q + 1 - i)\, \rho_i = \sum_{Q \epsilon PG(2,q) \setminus K} \theta_i(Q)$.

Theorem

For q odd, $m_4(2, q) \leq 3q + 1$.

Finite group:

A group G is a finite group if it consists a finite number of elements (17).

Research aim:

The aim of this research is as follows:

(1) What is the number of $(k$,4)-arcs in $PG(2, 17)$?.

(2) What is the number of distinct $(k$,4)-arcs in $PG(2, 17)$?.

(3) What is the largest size of complete $(k$,4)-arc in $PG(2, 17)$ can be obtained by the suggested method?.

The answers to these questions are given in Tables 3-23.

Literature review:

In 1947, Bose (18) proved that the largest (k, n)-arc is $q + 1$ *and* $q + 2$ for q odd and q even respectively. Then Barlotti (19) and Ball (20) showed that the maximum bound of (k, n)-arc is $(r - 1)q + 1$ and for r = $(q + 1)/2$ and r = $(q + 3)/2$ and for q odd prime. Later, a historical survey on (k, n)-arcs was written in (21). Subsequently, many studies and improvements have been given to get a large (k, n)-arc of size two, three or four, etc. In $PG(2, q)$ for a certain value of q. For n=2 see (22-25), for n=3 see (26-28), and for n $\geq$ 4 see (29-31).

Research Method:

The approach that has been used to classify the sets of $(k$,4)-arcs in $PG(2, 17)$ is based on fixing a set of six points, that is $S = \{U_0, U_1, U_2, U, P_1, P_2\}$, where $\{U_0, U_1, U_2, U\}$ is the standard frame; here, U_0=[1,0,0], U_1=[0,1,0], U_2= [0,0,1], U = [1,1,1] and P_1, P_2 lie on the line $U_0\,U_1$. So, S contains four collinear points. Since S meets each line l_j, j = 1,…,307, in at most four points, so by eliminating the points that lie on any 4-secant to S, then there remains the set R. A point Z of R is added separately to S to construct the (7,4)-arc $S_1 = S \cup \{Z\}$. The secant distribution $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ for each (7,4)-arc is calculated, and then the arcs are separated into a number of sets each of which has an inequivalent secant distribution:

$$C_1 = \{J_1^{(1)}, \ldots, J_{m_1}^{(1)}\},\ C_2 = \{J_1^{(2)}, \ldots, J_{m_2}^{(2)}\},\ \ldots,\ C_r = \{J_1^{(r)}, \ldots, J_{m_r}^{(r)}\}.$$

Then, by choosing a (7,4)-arc from each set of J_1, …, J_n, the set X_0 is constructed. This set is the following:

$$X_0 = \{\ J \mid J \text{ is a (7, 4)-arc} : J_1, J_2 \in X_0 \longrightarrow J_1, J_2 \text{ have distinct secant distributions}\}.$$

So, for each $J \in X_0$, eliminate the points in $PG(2, 17)$ that lie on any 4-secant to J. Thus the set R_1 which is defined as

$$R_1 = \{PG(2, 17) \setminus \{\text{points on 4-secants}\}\}$$

is constructed. So, adding a point $Z_1 \epsilon R_1$ to J, the (8,4)-arc $S_2 = J \cup \{Z_1\}$ is constructed. Similarly, construct each (8,4)-arc. Now, construct ~~the~~ set:

X_1 = { $K \mid K$ is a (8, 4)-arc: K_1, $K_2 \in X_1$ K_1, K_2 have distinct secant distributions}.

In the same way that the set X_0 was constructed.

Iterate this method to construct the $(k, 4)$-arcs for k>8.

So, we can summarize the above method in the following steps.

Results:

From the review above, we describe the results of the classification of $(k$,4)-arcs in $PG(2, 17)$ in the following subsections. These subsections summarises the findings and contributions made in tables.

(7,4)-arcs

The number of (7,4)-arcs K_7 that have been constructed is 287. Among these arcs, the number of distinct (7,4)-arcs is four where each has the following class of secant distribution, that is $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$.

Table 3. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ **of** K_7

$K_7^{(i)}$, i=1,…,4	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_7^{(1)}$	{1, 1, 12, 95, 198}
$K_7^{(2)}$	{1, 2, 9, 98, 197}
$K_7^{(3)}$	{1, 3, 6, 101, 196}
$K_7^{(4)}$	{2, 0, 9, 100, 196}

(8,4)-arcs

In this process, the number of (8,4)-arcs K_8 that constructed is 1124. Here, the number of distinct (8,4)-arcs is nine and have the following classes:

Table 4. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ **of** K_8

$K_8^{(i)}$, i=1,…,9	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_8^{(1)}$	{1, 1, 19, 99, 187}
$K_8^{(2)}$	{1, 2, 16, 102, 186}
$K_8^{(3)}$	{1, 3, 13, 105, 185}
$K_8^{(4)}$	{1, 4, 10, 108, 184}
$K_8^{(5)}$	{1, 5, 7, 111, 183}
$K_8^{(6)}$	{2, 0, 16, 104, 185}
$K_8^{(7)}$	{2, 1, 13, 107, 184}
$K_8^{(8)}$	{2, 2, 10, 110, 183}
$K_8^{(9)}$	{2, 3, 7, 113, 182}

(9,4)-arcs

The number of (9,4)-arcs K_9 established is 2494. Here, there are 17 distinct (9,4)-arcs having the following distinct classes:

Table 5. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ **of** K_9

$K_9^{(i)}$, i=1,…,17	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_9^{(1)}$	{1, 1, 27, 101, 177}
$K_9^{(2)}$	{1, 2, 24, 104, 176}
$K_9^{(3)}$	{1, 3, 21, 107, 175}
$K_9^{(4)}$	{1, 4, 18, 110, 174}
$K_9^{(5)}$	{1, 5, 15, 113, 173}
$K_9^{(6)}$	{1, 6, 12, 116, 172}
$K_9^{(7)}$	{1, 7, 9, 119, 171}
$K_9^{(8)}$	{2, 0, 24, 106, 175 }
$K_9^{(9)}$	{2, 1, 21, 109, 174}
$K_9^{(10)}$	{2, 2, 18, 112, 173}
$K_9^{(11)}$	{2, 3, 15, 115, 172}
$K_9^{(12)}$	{2, 4, 12, 118, 171}
$K_9^{(13)}$	{2, 5, 9, 121, 170}
$K_9^{(14)}$	{3, 0, 18, 114, 172}
$K_9^{(15)}$	{3, 1, 15, 117, 171}
$K_9^{(16)}$	{3, 2, 12, 120, 170}
$K_9^{(17)}$	{3, 3, 9, 123, 169}

(10,4)-arcs

The number of (10,4)-arcs K_{10} constructed is 4600. Also, the number of distinct (10,4)-arcs is 29. The statistic of distinct classes of (10,4)-arcs is as follows:

Table 6. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{10}

$K_{10}^{(i)}$, i=1,…,29	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{10}^{(1)}$	{ 2, 2, 27, 112, 164}
$K_{10}^{(2)}$	{1, 1, 36, 101, 168}
$K_{10}^{(3)}$	{1, 2, 33, 104, 167 }
$K_{10}^{(4)}$	{1, 3, 30, 107, 166 }
$K_{10}^{(5)}$	{1, 4, 27, 110, 165}
$K_{10}^{(6)}$	{1, 5, 24, 113, 164}
$K_{10}^{(7)}$	{1, 6, 21, 116, 163}
$K_{10}^{(8)}$	{1, 7, 18, 119, 162}
$K_{10}^{(9)}$	{1, 8, 15, 122, 161}
$K_{10}^{(10)}$	{1, 9, 12, 125, 160}
$K_{10}^{(11)}$	{1, 10, 9, 128, 159}
$K_{10}^{(12)}$	{2, 0, 33, 106, 166 }
$K_{10}^{(13)}$	{2, 1, 30, 109, 165}
$K_{10}^{(14)}$	{2, 3, 24, 115, 163}
$K_{10}^{(15)}$	{2, 4, 21, 118, 162}
$K_{10}^{(16)}$	{2, 5, 18, 121, 161}
$K_{10}^{(17)}$	{2, 6, 15, 124, 160}
$K_{10}^{(18)}$	{2, 7, 12, 127, 159}
$K_{10}^{(19)}$	{2, 8, 9, 130, 158}
$K_{10}^{(20)}$	{3, 0, 27, 114, 163}
$K_{10}^{(21)}$	{3, 1, 24, 117, 162}
$K_{10}^{(22)}$	{3, 2, 21, 120, 161}
$K_{10}^{(23)}$	{3, 3, 18, 123, 160}
$K_{10}^{(24)}$	{3, 4, 15, 126, 159}
$K_{10}^{(25)}$	{3, 5, 12, 129, 158}
$K_{10}^{(26)}$	{4, 0, 21, 122, 160}
$K_{10}^{(27)}$	{4, 1, 18, 125, 159}
$K_{10}^{(28)}$	{4, 2, 15, 128, 158 }
$K_{10}^{(29)}$	{4, 3, 12, 131, 157}

(11,4)-arcs

Here, the number of (11,4)-arcs K_{11} established is 7697. Among the (11,4)-arcs, the number of distinct (11,4)-arcs is 47. The statistic of distinct classes of (11,4)-arcs is as follows:

Table 7. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{11}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{11}^{(i)}$, =1,…,47	$\zeta_4 \epsilon\{1,\ldots,5\}$,$\zeta_3\epsilon\{0,\ldots,13\}$, $\zeta_2 \epsilon\{10,\ldots,46\}$,$\zeta_1\epsilon\{99,\ldots,140\}$,$\zeta_0 \epsilon \{145,\ldots,160\}$

(12,4)-arcs

The number of (12,4)-arcs K_{12} is 12167 and the number of distinct (12,4)-arcs is 72. The statistic of distinct classes is written in Table 8.

Table 8. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{12}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{12}^{(i)}$, i=1,…,72	$\zeta_4 \epsilon \{1,\ldots,7\}$,$\zeta_3\epsilon\{0,\ldots,16\}$, $\zeta_2 \epsilon\{12,\ldots,57\}$,$\zeta_1\epsilon$95,…,150}, $\zeta_0 \epsilon \{133,\ldots,153\}$

(13,4)-arcs

The number of (13,4)-arcs K_{13} is 17988, among them the number of distinct (13,4)-arcs which is 104. The statistic of distinct classes is given in Table 9.

Table 9. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{13}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{13}^{(i)}$,i=1,…,104	$\zeta_4 \epsilon \{1,\ldots,9\}$,$\zeta_3 \epsilon \{0,\ldots,19\}$, $\zeta_2\epsilon\{12,\ldots,66\}$,$\zeta_1\epsilon \{92,\ldots,157\}$, $\zeta_0 \epsilon \{122,\ldots,146\}$

(14,4)-arcs

Here, the number of (14,4)-arcs K_{14} is 24984 and the number of distinct (14,4)-arcs is 139. The statistic of this process is shown in Table 10.

Table 10. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{14}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{14}^{(i)}$,i=1,…,139	$\zeta_4 \epsilon \{1,\ldots,10\}$,$\zeta_3 \epsilon \{0,\ldots,22\}$, $\zeta_2\epsilon\{13,\ldots,76\}$,$\zeta_1\epsilon\{89,\ldots,162\}$, $\zeta_0 \epsilon \{112,\ldots,139\}$

(15,4)-arcs

The number of (15,4)-arcs K_{15} is 32154 and the number of distinct (15,4)-arcs is 185. The statistic of this process is given in Table 11.

Table 11. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{15}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{15}^{(i)}$,i=1,…,185	$\zeta_4 \epsilon\{1,\ldots,12\}$,$\zeta_3 \epsilon \{0,\ldots,26\}$, $\zeta_2\epsilon\{15,\ldots,87\}$,$\zeta_2\epsilon\{84,\ldots,166\}$, $\zeta_0 \epsilon \{103,\ldots,133\}$

(16,4)-arcs

The number of (16,4)-arcs K_{16} is 41107 and the number of distinct (16,4)-arcs is 237. The statistic of this process is given in Table 12.

Table 12. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{16}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{16}^{(i)}$,i=1,…,237	$\zeta_4 \epsilon\{1,\ldots,14\}$,$\zeta_3 \epsilon \{0,\ldots,30\}$, $\zeta_2 \epsilon\{19,\ldots,96\}$,$\zeta_1 \epsilon\{78,\ldots,170\}$, $\zeta_0 \epsilon \{95,\ldots,128\}$

(17,4)-arcs

The number of (17,4)-arcs K_{17} is 50325 and the number of distinct (17,4)-arcs is 290. The statistic of this process is given in Table 13.

Table 13.Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{17}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{17}^{(i)}$,i=1,…,290	$\zeta_4\epsilon\{1,\ldots,15\}$,$\zeta_3\epsilon\{0,\ldots,34\}$, $\zeta_2 \epsilon\{19,\ldots,103\}$,$\zeta_1 \epsilon\{73,\ldots,173\}$,$\zeta_0 \epsilon \{87,\ldots,123\}$

(18,4)-arcs

In this process, the number of (18,4)-arcs K_{18} is 58845 also there are 353 distinct (18,4)-arcs. The statistic is given in Table 14.

Table 14. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{18}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{18}^{(i)}$,i=1,…,353	$\zeta_4 \epsilon\{1,\ldots,17\}$,$\zeta_3\epsilon \{0,\ldots,40\}$, $\zeta_2\epsilon\{21,\ldots,108\}$,$\zeta_1\epsilon\{69,\ldots,178\}$, $\zeta_0 \epsilon \{87,\ldots,118\}$

(19,4)-arcs

There are 424 distinct (19,4)-arcs. The statistic is given in Table 15.

Table 15. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{19}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{19}^{(i)}$,i=1,…,424	$\zeta_4 \epsilon \{1,\ldots,19\}$,$\zeta_3 \epsilon \{1,\ldots,44\}$, $\zeta_2\epsilon\{21,\ldots,114\}$,$\zeta_1 \epsilon\{65,\ldots,181\}$,$\zeta_0 \epsilon$ $\{70,\ldots,113\}$

(20,4)-arcs

The number of distinct (20,4)-arcs is 504. The related information is given in Table 16.

Table 16. Distinct classes $\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$ **of** K_{20}

Arcs	$\{\zeta_4, \zeta_3, \zeta_2, \zeta_1, \zeta_0\}$
$K_{20}^{(i)}$,i=1,…,504	$\zeta_4 \epsilon \{1,\ldots,21\}$,$\zeta_3 \epsilon \{1,\ldots,49\}$, $\zeta_2\epsilon\{25,\ldots,118\}$,$\zeta_1\epsilon\{60,\ldots,183\}$, $\zeta_0 \epsilon \{63,\ldots,109\}$

(21,4)-arcs

The number of distinct (21,4)-arcs is 581. The related information is given in Table 17.

Table 17. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{21}

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{21}^{(i)}$,i=1,…,581	$\zeta_4\epsilon\{1,\ldots,23\}$,$\zeta_3\epsilon\{2,\ldots,54\}$,$\zeta_2\epsilon\{27,\ldots,123\}$,$\zeta_1\epsilon\{56,..,185\}$,$\zeta_0\ \epsilon$ $\{56,\ldots,105\}$

(22,4)-arcs

The number of distinct (22,4)-arcs is 666. The related information is given in Table 18.

Table 18. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{22}

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{22}^{(i)}$,i=1,…,666	$\zeta_4\ \epsilon\ \{1,\ldots,26\}$,$\zeta_3\ \epsilon\ \{3,\ldots,59\}$, $\zeta_2\epsilon\{27,\ldots,126\}$,$\zeta_1\epsilon\{52,..,184\}$,$\zeta_0\ \epsilon$ $\{50,\ldots,101\}$

(23,4)-arcs

The number of distinct (23,4)-arcs is 749. The related information is given in Table 19.

Table 19. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{23}

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{23}^{(i)}$,i=1,…,749	$\zeta_4\ \epsilon\ \{1,\ldots,30\}$,$\zeta_3\ \epsilon\ \{4,\ldots,64\}$, $\zeta_2\ \epsilon\{28,\ldots,130\}$,$\zeta_1\epsilon\{47,..,181\}$, $\zeta_0\ \epsilon\ \{46,\ldots,98\}$

(24,4)-arcs

The number of distinct (24,4)-arcs is 832. The related information is given in Table 20.

Table 20. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{24}

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{24}^{(i)}$,i=1,…,832	$\zeta_4\ \epsilon\ \{1,\ldots,33\}$,$\zeta_3\ \epsilon\ \{5,\ldots,70\}$, $\zeta_2\epsilon\{30,\ldots,132\}$,$\zeta_1\ \epsilon\{43,..,178\}$, $\zeta_0\ \epsilon\ \{42,\ldots,94\}$

(25,4)-arcs

The number of distinct (25,4)-arcs is 907. The related information is given in Table 21.

Table 21. Distinct classes $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ of K_{25}

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{25}^{(i)}$,i=1,…,907	$\zeta_4\epsilon\{1,\ldots,35\}$,$\zeta_3\ \epsilon\{5,\ldots,76\}$,$\zeta_2\epsilon\{30,\ldots,135\}$,$\zeta_1\epsilon\{38,\ldots,173\}$,$\zeta_0\ \epsilon$ $\{38,\ldots,91\}$

Remark

The statistics of the $(k,4)$-arcs for the value of k = 26,…,48 are given in the following table:

Table 22. Statistics of $(k,4)$-arcs for k = 26,…,48

Arcs	$\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$
$K_{26}^{(i)}$,i=1,…,966	$\zeta_4\ \epsilon\ \{2,\ldots,38\}$,$\zeta_3\ \epsilon\ \{7,\ldots,84\}$, $\zeta_2\ \epsilon\{31,\ldots,136\}$,$\zeta_1\epsilon\{34,\ldots,169\}$, $\zeta_0\ \epsilon\ \{35,\ldots,89\}$
$K_{27}^{(i)}$,i=1,…,1002	$\zeta_4\ \epsilon\{3,\ldots,41\}$,$\zeta_3\ \epsilon\ \{10,\ldots,92\}$, $\zeta_2\ \epsilon\{30,\ldots,135\}$,$\zeta_1\epsilon\{31,\ldots,166\}$, $\zeta_0\ \epsilon\ \{33,\ldots,86\}$
$K_{28}^{(i)}$,i=1,…,1038	$\zeta_4\ \epsilon\{5,\ldots,44\}$,$\zeta_3\ \epsilon\ \{10,\ldots,97\}$, $\zeta_2\epsilon\{30,\ldots,135\}$,$\zeta_1\ \epsilon\{28,\ldots,161\}$, $\zeta_0\ \epsilon\ \{30,\ldots,83\}$
$K_{29}^{(i)}$,i=1,…,1053	$\zeta_4\ \epsilon\ \{6,\ldots,48\}$,$\zeta_3\ \epsilon\ \{12,\ldots,101\}$, $\zeta_2\ \epsilon\{28,\ldots,136\}$,$\zeta_1\ \epsilon\{25,\ldots,157\}$,$\zeta_0\ \epsilon\ \{28,\ldots,81\}$
$K_{30}^{(i)}$,i=1,…,1039	$\zeta_4\ \epsilon\{9,\ldots,51\}$,$\zeta_3\epsilon\ \{18,\ldots,106\}$, $\zeta_2\ \epsilon\{30,\ldots,132\}$,$\zeta_1\epsilon\{22,\ldots,152\}$, $\zeta_0\ \epsilon\ \{26,\ldots,79\}$
$K_{31}^{(i)}$,i=1,…,1025	$\zeta_4\ \epsilon\{12,\ldots,55\}$,$\zeta_3\epsilon\{22,\ldots,111\}$, $\zeta_2\ \epsilon\{30,\ldots,132\}$,$\zeta_1\epsilon\{20,\ldots,146\}$, $\zeta_0\ \epsilon\ \{25,\ldots,77\}$
$K_{32}^{(i)}$,i=1,…,995	$\zeta_4\ \epsilon\{16,\ldots,59\}$,$\zeta_3\epsilon\{24,\ldots,116\}$, $\zeta_2\ \epsilon\{31,\ldots,130\}$,$\zeta_1\epsilon\{18,\ldots,137\}$, $\zeta_0\ \epsilon\ \{25,\ldots,75\}$
$K_{33}^{(i)}$,i=1,…,946	$\zeta_4\ \epsilon\{21,\ldots,61\}$,$\zeta_3\epsilon\{28,\ldots,119\}$, $\zeta_2\ \epsilon\{33,\ldots,126\}$,$\zeta_1\epsilon\{17,\ldots,132\}$, $\zeta_0\ \epsilon\ \{23,\ldots,72\}$
$K_{34}^{(i)}$,i=1,…,876	$\zeta_4\epsilon\{25,\ldots,65\}$,$\zeta_3\ \epsilon\{32,\ldots,122\}$, $\zeta_2\ \epsilon\{30,\ldots,120\}$,$\zeta_1\epsilon\{16,\ldots,126\}$, $\zeta_0\ \epsilon\ \{24,\ldots,70\}$
$K_{35}^{(i)}$,i=1,…,757	$\zeta_4\ \epsilon\{31,\ldots,69\}$,$\zeta_3\epsilon\{37,\ldots,124\}$, $\zeta_2\ \epsilon\{31,\ldots,115\}$,$\zeta_1\epsilon\{17,\ldots,118\}$, $\zeta_0\ \epsilon\ \{25,\ldots,68\}$
$K_{36}^{(i)}$,i=1,…,627	$\zeta_4\ \epsilon\{36,\ldots,74\}$,$\zeta_3\ \epsilon\{41,\ldots,124\}$, $\zeta_2\epsilon\{33,\ldots,108\}$,$\zeta_1\epsilon\{16,\ldots,110\}$, $\zeta_0\ \epsilon\ \{24,\ldots,65\}$
$K_{37}^{(i)}$,i=1,…,477	$\zeta_4\ \epsilon\ \{4\ ,\ldots,74\}$, $\zeta_3\ \epsilon\{48,\ldots,123\}$, $\zeta_2\ \epsilon\{30,\ldots,105\}$,$\zeta_1\ \epsilon\{15,\ldots,98\}$, $\zeta_0\ \epsilon\ \{28,\ldots,63\}$
$K_{38}^{(i)}$,i=1,…,310	$\zeta_4\ \epsilon\{51,\ldots,78\}$,$\zeta_3\epsilon\ \{52,\ldots,119\}$, $\zeta_2\ \epsilon\{28,\ldots,94\}$,$\zeta_1\ \epsilon\{17,\ldots,88\}$, $\zeta_0\ \epsilon\ \{30,\ldots,61\}$
$K_{39}^{(i)}$,i=1,…,172	$\zeta_4\epsilon\{59,\ldots,78\}$,$\zeta_3\epsilon\ \{69,\ldots,114\}$, $\zeta_2\ \epsilon\{27,\ldots,90\}$,$\zeta_1\ \epsilon\{15,\ldots,73\}$, $\zeta_0\ \epsilon\ \{37,\ldots,60\}$
$K_{40}^{(i)}$,i=1,…,74	$\zeta_4\epsilon\{69,\ldots,84\}$,$\zeta_3\epsilon\ \{66,\ldots,105\}$, $\zeta_2\ \epsilon\{30,\ldots,84\}$,$\zeta_1\ \epsilon\{11,\ldots,65\}$, $\zeta_0\ \epsilon\ \{38,\ldots,60\}$
$K_{41}^{(i)}$,i=1,…,35	$\zeta_4\ \epsilon\ \{77,\ldots,87\}$,$\zeta_3\ \epsilon\ \{75,\ldots,99\}$, $\zeta_2\ \epsilon\{58,\ldots,76\}$,$\zeta_1\ \epsilon\{8,\ldots,22\}$, $\zeta_0\ \epsilon\ \{53,\ldots,59\}$
$K_{42}^{(i)}$,i=1,…,20	$\zeta_4\ \epsilon\ \{86,\ldots,93\}$,$\zeta_3\ \epsilon\ \{78,\ldots,96\}$, $\zeta_2\ \epsilon\{57,\ldots,69\}$,$\zeta_1\ \epsilon\{6,\ldots,16\}$, $\zeta_0\ \epsilon\ \{55,\ldots,58\}$
$K_{43}^{(i)}$,i=1,…,14	$\zeta_4\epsilon\{96,\ldots,101\}$,$\zeta_3\epsilon\ \{78,\ldots,91\}$, $\zeta_2\ \epsilon\{54,\ldots,63\}$,$\zeta_1\ \epsilon\{4,\ldots,13\}$, $\zeta_0\ \epsilon\ \{55,\ldots,58\}$
$K_{44}^{(i)}$,i=1,…,9	$\zeta_4\epsilon\{106,\ldots,110\}$,$\zeta_3\epsilon\{76,\ldots,86\}$, $\zeta_2\ \epsilon\{52,\ldots,58\}$,$\zeta_1\ \epsilon\{4,\ldots,10\}$, $\zeta_0\ \epsilon\ \{55,\ldots,57\}$
$K_{45}^{(i)}$,i=1,…,5	$\zeta_4\epsilon\{118,\ldots,120\}$,$\zeta_3\epsilon\{72,\ldots,77\}$, $\zeta_2\ \epsilon\{51,\ldots,54\}$,$\zeta_1\ \epsilon\{2,\ldots,7\}$, $\zeta_0\ \epsilon\ \{55,\ldots,57\}$
$K_{46}^{(i)}$,i=1,…,3	$\zeta_4\ \epsilon\{130,\ldots,131\}$,$\zeta_3\ \epsilon\{66,\ldots,69\}$,$\zeta_2\ \epsilon\{48,\ldots,51\}$,$\zeta_1\ \epsilon\{2,\ldots,4\}$,$\zeta_0\ \epsilon$ $\{55,\ldots,56\}$
$K_{47}^{(i)}$,i=1,…,1	{143,58,49,2,55}
$K_{48}^{(i)}$,i=1,…,1	{156,48,48,0,55}

Discussion:

The purpose of this research is to find the number of $(k,4)$-arcs and the number of distinct $(k,4)$-arcs in $PG(2,17)$. Also, establish a large size of complete arc. The method followed to satisfy these problems is a new systematic tool based on the approach of i-secant distributions of the arcs. The statistics of $(k,4)$-arcs in $PG(2,17)$ obtained by this method are given in Table 23.

For each k, the number N of inequivalent $(k,4)$-arcs and the stabiliser size is given. The symbol S is the type of stabiliser group and m is the number of groups for each stabiliser.

Table 23. Inequivalent $(k,4)$-arcs

$(k,4)$-arcs	N	$S:m$
(7,4)-arcs	4	I:4
(8,4)-arcs	9	I:6, Z_3: 3
(9,4)-arcs	17	I:14, Z_2: 2, Z_3: 1
(10,4)-arcs	29	I: 28, Z_2: 1
(11,4)-arcs	47	I: 46, Z_2: 1
(12,4)-arcs	72	I: 72
(13,4)-arcs	104	I: 102, Z_2: 2
(14,4)-arcs	139	I: 139
(15,4)-arcs	185	I: 182, Z_2: 1, Z_3: 2
(16,4)-arcs	237	I: 237
(17,4)-arcs	290	I: 290
(18,4)-arcs	353	I: 353
(19,4)-arcs	424	I: 424
(20,4)-arcs	504	I: 504
(21,4)-arcs	581	I: 579, Z_2: 1, Z_3: 1
(22,4)-arcs	666	I: 666
(23,4)-arcs	749	I: 748, Z_2: 1
(24,4)-arcs	832	I: 832
(25,4)-arcs	907	I: 907
(26,4)-arcs	966	I: 966
(27,4)-arcs	1002	I: 1002
(28,4)-arcs	1038	I: 1038
(29,4)-arcs	1053	I: 1053
(30,4)-arcs	1039	I: 1039
(31,4)-arcs	1025	I: 1025
(32,4)-arcs	995	I: 995
(33,4)-arcs	946	I: 946
(34,4)-arcs	876	I: 876
(35,4)-arcs	757	I: 757
(36,4)-arcs	627	I: 627
(37,4)-arcs	477	I: 477
(38,4)-arcs	310	I: 310
(39,4)-arcs	172	I: 172
(40,4)-arcs	74	I: 74
(41,4)-arcs	35	I: 35
(42,4)-arcs	20	I: 20
(43,4)-arcs	14	I: 14
(44,4)-arcs	9	I: 8, Z_2: 1
(45,4)-arcs	5	I: 4, Z_2: 1
(46,4)-arcs	3	I: 1, Z_2: 2
(47,4)-arcs	1	Z_2
(48,4)-arcs	1	GL(2, 3) $\rtimes$ Z_2

From the above table, the main result that have been obtained established a large size of $(k,4)$-arc. This size is k=48. The complete (48,4)-arc is as follows:

K= {1, 0, 0}, {0, 1, 0}, {0, 0, 1}, {1, 1, 1}, {10, 10, 1}, {1, 1, 0}, {4, 15, 1}, {8, 16, 1}, {11,10,1 },[3, 1, 0}, {9, 14, 1}, {5, 11, 1}, {16, 1, 1}, {14, 0, 1}, {5, 0, 1}, {16, 14, 1}, {0, 7, 1}, {13, 8, 1},[7, 15, 1}, {3, 4, 1}, {12, 1, 1}, {13, 3, 1}, {10, 9, 1}, {3, 11, 1}, {9, 5, 1}, {14, 12, 1}, {12, 9, 1},[11, 6, 1}, {7, 13, 1}, {5, 15, 1}, {10, 14, 1}, {1, 2, 1}, {2, 7, 1}, {2, 4, 1}, {6, 16, 1}, {3, 10, 1},[1, 5, 1}, {7, 16, 1}, {11, 8, 1}, {15, 11, 1}, {0, 4, 1}, {14, 2, 1}, {6, 3, 1}, {6, 8, 1}, {9, 7, 1}, {12, 3, 1}, {16, 9, 1}, {4, 13, 1}.

Remark:

The geometric and algebraic properties of the above arc are described as below.

(1) The i-secant distribution of K is $\{\zeta_4,\zeta_3,\zeta_2,\zeta_1,\zeta_0\}$ = {55, 0, 48, 48, 156}.

(2) This complete arc has the stabiliser group **GL (2, 3) $\rtimes$ $\boldsymbol{Z_2}$**. This group is of order 96. It is a semi-direct product of general linear group, GL (2, 3) of order 48 and the cyclic group, Z_2 of order 2.

(3) The action of **GL (2, 3) $\rtimes$ $\boldsymbol{Z_2}$** on the associated (48,4)-arc is a one orbit. This orbit is as follows:

Orb = { 1, 11, 72, 3, 2, 256, 202, 216, 293, 73, 244, 221, 111, 170, 124, 203, 121, 288, 169, 254, 258, 289, 191, 134, 209, 220, 13, 28, 299, 283, 65, 252, 236, 265, 176, 212, 21, 163, 255, 59, 264, 285, 218, 68, 239, 96, 219, 243}, where 1 ={1,0,0} $\in K$, 2 ={0,1,0} $\in K$, 220 ={10, 14, 1}$\in K$, and similarly for each point in the above (48,4)-arc.

So, from the above result, the following theorem is given.

Theorem

In $PG(2,17)$; there is at least one complete $(48,4)$-arc.

Conclusion and suggestion:

In this research, the number of $(k,4)$-arcs, the number of distinct $(k,4)$-arcs, and the large size of $(k,4)$-arc are discussed. These problems had an approach that addresses programming difficulties in terms of storing data and reducing the increasing numbers of arcs as well as the time of implementation. Therefore, The approach of the i-secant distributions is considered in order to assure that the best result would be satisfied.

From the previous calculation the following results are obtained :

1- The number of $(k,4)$-arcs in $PG(2,17)$ is classified.

2- The number of distinct $(k,4)$-arcs in $PG(2,17)$ is classified.
3- A large complete $(k,4)$-arc of size k=48 is established.

In addition, the stabiliser group of this arc is computed. This group is GL (2, 3) $\rtimes Z_2$. Also, the action of this group on the complete (48,4)-arc is described. The concept of secant distributions is a useful and substantial research which recommend to construct a large k. But, it still needs to improve especially when the parameters $k, n,$ and q are large. For further study, super computers are needed.

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Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

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القوس التام (4 ;48) في المستوي الاسقاطي للحقل المنتهي من الرتبة السابعة عشر

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الخلاصة:

يقدم البحث طريقة حسابية خاصة لبناء عدد الاقواس الهندسية المختلفة من الرتبة الرابعة في المستوي الاسقاطي لقيم n =4, q=17, k=7,…,48 في هذه الطريقة تم تطبيق نهج جديد لحساب عدد الاقواس الاسقاطية وكذلك عدد الاقواس المختلفة على التوالي. حيث ان هذا النهج اعتمد عل اختيار عدد الصفوف الغير متكافئة للقواطع في كل عملية. حيث ان اكبر حجم قد تم بناءه في هذه الطريقة هو 48. ان الطريقة المتبعة هي أداة جديدة للتعامل مع صعوبات البرمجة التي قد تؤدي في بعض الأحيان إلى مشاكل البرمجة التي تمثلها زيادة عدد الأقواس. لذا كان من الضروري اتباع هذه الطريقة لتقليل العدد المحدد للأقواس في كل بناء خاصة بالنسبة لقيم كبيرة k وبالتالي تقليل وقت التنفيذ للعمليات الحسابية ومن ثم تقليل من استخدام الذاكرة للعمليات الحسابية. تم تأكيد تقييم فعالية الطريقة الجديدة من خلال نتائج الحساب حيث تمكنا من بناء أكبر حجم كامل. $(k,4)$-قوس. ان نتائج البحث الجديد طورت استراتيجية المناهج الحسابية لفحص الأحجام الكبيرة من الأقواس في PG (2,q) حيث تولي مزيدًا من الاهتمام لدراسة عدد الفئات غير المتكافئة من قواطع-i من الاقواس الاسقاطية وهو جانب مثير للاهتمام. وبالتالي، يمكن استخدامه لتأسيس قيمة كبيرة لـ k.

الكلمات المفتاحية: القوس التام، توزيع القاطع المختلف، الزمرة، الفضاء الاسقاطي، القوس (k, n).