

## On Almost Quasi-Frobenius Fuzzy Rings

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### Abstract

In this paper, we introduce the concept of almost Quasi-Frobenius fuzzy ring as a generalization of Quasi-Frobenius ring. We give some properties about this concept with quotient fuzzy ring. Also, we study the fuzzy external direct sum of fuzzy rings.

### Introduction

Quasi-Frobenius rings have been investigated in [14] and [15]. In this paper, we introduce almost Quasi-Frobenius fuzzy ring of a ring by extending the (ordinary) notion Quasi-Frobenius rings. Zadeh introduced the concepts of fuzzy set [1] and introduced concepts of fuzzy ring of a ring and fuzzy ideal of a ring [2]. We shall investigate the notion of an almost Quasi-Frobenius fuzzy rings. We give some characterization of this concept. We study the quotient fuzzy ring of almost Quasi-Frobenius fuzzy ring. Also, we consider the behavior of a fuzzy ring if it is a quotient fuzzy ring is almost Quasi-Frobenius ring. Next, we prove that the fuzzy external direct sum of almost Quasi-Frobenius fuzzy rings is an almost Quasi-Frobenius fuzzy ring. Also, we prove that a direct sum of and of

almost Quasi-Frobenius fuzzy ring is an almost Quasi-Frobenius fuzzy ring. In our work and for easiness, we assume  $A$  is a fuzzy ideal in  $X$  if  $A$  is a fuzzy ideal of  $R$  and  $A \subseteq X$ , where  $X$  is a fuzzy ring of  $R$  and  $R$  is commutative ring with identity.

### S.1 Preliminaries:

In this section, we give some definitions and results which will be used later. Let  $S$  be a non empty set. A fuzzy subset of  $S$  is a function from  $S$  to  $[0,1]$ , ([3],[4]). Let  $A$  and  $B$  be fuzzy subsets of  $S$ . We write  $A \subseteq B$  if  $A(x) \leq B(x)$  for all  $x \in S$ . If  $A \subseteq B$  and there exists  $x \in S$  such that  $A(x) < B(x)$ , then we write  $A \subset B$  and we say  $A$  is a proper fuzzy subset of  $B$ , [3]. Note that  $A=B$  iff  $A(x)=B(x)$  for all  $x \in S$ , [1]. We let  $\text{Im}(A)$  denote the image of  $A$ . We say that  $A$  is a finite-valued if  $\text{Im}(A)$  is

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finite, ([3], [5]). For each  $t \in [0, 1]$ , the set  $A_t = \{x \in S \mid A(x) \geq t\}$  is called a level subset of  $S$ , ([3], [5]). We let  $\text{supp}(A)$  denotes  $\{x \in S \mid A(x) > 0\}$ , the support of  $A$  and  $A_* = A_{A(0)} = \{x \in S \mid A(x) = A(0)\}$ , [5]. If  $x \in S$  and  $t \in [0, 1]$ , we let  $x_t$  denote the fuzzy subset of  $S$  defined by  $x_t(y) = 0$  if  $x \neq y$  and  $x_t(y) = t$  if  $y = x$  for all  $y \in S$ .  $x_t$  is called a fuzzy singleton of  $S$  ([2], [5]). If  $x_t$  and  $y_s$  are fuzzy singletons of  $R$ , then  $x_t + y_s = (x + y)_\lambda$  and  $x_t \circ y_s = (x \cdot y)_\lambda$  where  $\lambda = \min\{t, s\}$ , ([1], [5]). We let  $\phi$  denote  $\phi(x) = 0$  for all  $x \in S$ , the empty fuzzy subset of  $S$ , ([1], [2]). Let  $A$  and  $B$  be two fuzzy subsets of  $R$ . The product  $A \circ B$  defined by:

$$A \circ B(x) = \begin{cases} \sup\{\min\{A(y), B(z)\} \mid x \text{ is expressible as } x = y \cdot z\} & \text{(a)} \\ 0 & \text{otherwise} \end{cases} \quad \text{(b)}$$

$y, z \in R$ , for all  $x \in R$ , ([2], [4]). If  $R$  is a ring with identity or more generally if  $R \cdot R = R$ , then the case (b) in the definition of  $(A \circ B)(x)$  does not arise, [4]. The addition  $A + B$  is defined by:  $(A + B)(x) = \sup\{\min\{A(y), B(z)\} \mid x = y + z\}$   $y, z \in R$ , for all  $x \in R$ , ([6], [7]). Note that  $A \circ B$  and  $A + B$  are fuzzy subsets of  $R$ . Let  $A$  be a non empty fuzzy subset of  $R$ ,  $A$  is called a fuzzy subgroup of  $R$  if for all  $x, y \in R$ ,  $A(x - y) \geq \min\{A(x), A(y)\}$  and  $A(-x) = A(x)$ , ([2], [7]). Note that  $A(0) \geq A(x)$  for all  $x \in R$ . We begin with the following definition of fuzzy ring and fuzzy ideal.

**Definition 1.2** <sup>([1], [2])</sup>

Let  $R$  be any ring and  $A \in I^R$ ,  $A \neq \phi$ . Then  $A$  is a **fuzzy ring of  $R$**  iff for all  $x, y \in R$ ;

1.  $A(x - y) \geq \min\{A(x), A(y)\}$
2.  $A(x \cdot y) \geq \min\{A(x), A(y)\}$ .

**Definition 1.3** <sup>([8])</sup>

A fuzzy ring  $A$  of  $R$  is said to be an integral domain if  $x \cdot y = 0$  and  $\min\{A(x), A(y)\} > 0$  implies that  $x = 0$  or  $y = 0$  for each  $x, y \in R$ .

**Definition 1.4** <sup>([5], [6])</sup>

A fuzzy subset  $A$  of a field  $F$  is a fuzzy field of  $F$  if:

1.  $A(1) = 1$ .
2.  $A(x - y) \geq \min\{A(x), A(y)\}$  for each  $x, y \in F$ .
3.  $A(x \cdot y^{-1}) \geq \min\{A(x), A(y)\}$  for each  $x, y \in F, y \neq 0$ .

Let  $A$  be a fuzzy field of  $F$ . If  $x \in F, x \neq 0$ , then

$$A(0) = A(1) \geq A(x) = A(-x) = A(x^{-1}).$$

**Definition 1.5** <sup>([2], [3])</sup>

A non empty fuzzy subset  $A$  of  $R$  is called a **fuzzy ideal of  $R$**  iff for all  $x, y \in R$

- (1)  $A(x - y) \geq \min\{A(x), A(y)\}$
- (2)  $A(x \cdot y) \geq \max\{A(x), A(y)\}$ .

**Remark 1.6** <sup>([9])</sup>

It is clear that every fuzzy ideal of  $R$  is a fuzzy ring of  $R$ , but the converse is not true.

**Proposition 1.7** <sup>([3], [5])</sup>

Let  $R$  be a ring and  $A$  be a fuzzy ideal of  $R$ . Then  $A_* = \{x \in R \mid A(x) = A(0)\}$  and  $\text{supp}(A)$  are ideals of  $R$ .

**Proposition 1.8** <sup>([10], [11])</sup>

Let  $A$  be a fuzzy ideal of  $R$ . If  $t \in [0, A(0)]$ , then  $A_t$  is an ideal of  $R$ .

**Proposition 1.9** <sup>([10], [11])</sup>

Let  $A$  be a fuzzy subset of  $R$ . If for all  $t \in \text{Im}(A)$ ,  $A_t$  is an ideal of  $R$ , then  $A$  is a fuzzy ideal of  $R$ .

**Proposition 1.10** <sup>([11])</sup> Let  $A$  and  $B$  be two fuzzy ideals of  $R$ . Then  $(A + B)$  is a fuzzy ideal of  $R$ .

Now, we give the concept of Annihilator of a fuzzy ideal  $A$  of a ring  $R$  and Annihilator fuzzy ideal  $A$  of a ring  $R$ .

**Definition 1.11** <sup>([11])</sup>

Let  $A$  be a nonempty fuzzy ideal of  $R$ . **The Annihilator of  $A$**  denoted by  $(F\text{-Ann}A)$  is defined by  $\{x_t \mid x \in R, x_t \in A$

$\subseteq 0_1\}$ ,  $t \in [0, A(0)]$  and  $(F\text{-Ann}A)(a) = \sup \{t : t \in [0, A(0)], a_t \subseteq 0_1\}$ ,  $a \in R$ .

**Definition 1.12**<sup>[11]</sup>

Let  $A$  be a nonempty fuzzy ideal of a ring  $R$ .  $A$  is called an Annihilator fuzzy ideal if and only if  $A = F\text{-Ann}A$ . Now we give the following definition:

**Definition 1.13**<sup>[11]</sup>

Let  $X$  be a fuzzy ring of  $R$  and  $A$  be a fuzzy ideal in  $X$ . Define  $X/A : R/A_* \rightarrow [0, 1]$  such that:

$$X/A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a+b)\} & \text{if } b \in A_*, a \notin A_* \end{cases}$$

for all  $a + A_* \in R/A_*$ .  $X/A$  is called a quotient fuzzy ring of  $X$  by  $A$ .

**Proposition 1.14**<sup>[11]</sup>

If  $X$  is a fuzzy ring of  $R$ ,  $A$  and  $B$  are fuzzy ideals in  $X$  such that  $A \subseteq B$ . Then  $B/A : R/A_* \rightarrow [0, 1]$  such that:

$$B/A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{B(a+b)\} & \text{if } b \in A_*, a \notin A_* \end{cases}$$

is a fuzzy ideal in  $X/A$ .

**Defination 1.15**<sup>[11]</sup>

Let  $X$  be a fuzzy ring of a ring  $R_1$  and  $Y$  be a fuzzy ring of a ring  $R_2$ . Define  $T : R_1 \oplus R_2 \rightarrow [0, 1]$  by  $T(a, b) = \min\{X(a), Y(b)\}$  for all  $(a, b) \in R_1 \oplus R_2$ .  $T$  is called a fuzzy external direct sum of  $X$  and  $Y$ , denoted by  $X \oplus Y$ .

**S.2 ALMOST QUASI-FROBENIUS FUZZY RING**

In this section, we define almost quasi-Frobenius fuzzy ring and we give some examples to explain this concept. Moreover, we give some characterizations and properties of this concept.

**Definition 2.1**

Let  $X$  be a fuzzy ring of  $R$ .  $X$  is called an almost quasi-Frobenius fuzzy ring if

and only if  $R$  is a Noetherian ring and every non empty fuzzy ideal  $A$  in  $X$  is an Annihilator fuzzy ideal.

**Examples and Remarks 2.2**

1. For any Noetherian ring  $R$ , the fuzzy

ring  $X : R \rightarrow [0, 1]$  such that:

$$X(a) = \begin{cases} c & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

where  $c \in (0, 1]$ .

$X$  is an almost quasi-Frobenius fuzzy ring of  $R$ , since any non empty fuzzy ideal  $A$  in  $X$  is of the form:

$$A(x) = \begin{cases} d & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $d \in (0, c]$ .

$A$  is an Annihilator fuzzy ideal by [11, corollary (2.3.7)].

2. Let  $R$  be a Noetherian ring,  $X$  be

a fuzzy ring of  $R$ . Then  $X$  is an almost quasi-Frobenius fuzzy

ring

if any non empty fuzzy ideal

$A$  in

$X$  is of the form:

$$A(x) = \begin{cases} c & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $c \in (0, X(0)]$ .

It follows directly by [11, theorem (2.3.4)] and definition (2.1)].

$$3. \text{ Let: } X(a) = \begin{cases} 1 & \text{if } a \in Z \\ 0 & \text{if } a \in Q \setminus Z \end{cases}$$

Then  $X$  is a fuzzy ring of  $Q$ . If  $A$  is a fuzzy ideal in  $X$ , then  $A_*$  is an ideal of  $Q$  and hence either  $A_* = Q$  or  $A_* = \{0\}$ .

But  $A_* \neq Q$  by definition of  $X$ , so  $A_* = \{0\}$ . Let  $A(0) = c$  where  $c \in (0, X(0)]$ . Note that  $A(x) = 0$  for all  $x \in Q \setminus Z$  (since  $A \subseteq X$ ).

Now, suppose that there exists  $a \in Z$  such that  $A(a) = d$  where  $0 < d < c$ .

On the other hand, there exists  $r \in Q$  such that  $r \cdot a \in Q \setminus Z$  and so  $A(r \cdot a) = 0$ . But  $A(r \cdot a) \geq \max \{A(r), A(a)\} \geq d$ . Hence  $0 \geq d$  which is a contradiction.

It follows that  $A(a) = 0$  for all  $a \in Z, a \neq 0$ . Thus:

$$A(a) = \begin{cases} c & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence  $X$  is an almost quasi-Frobenius fuzzy ring by (2.2, (2)).

4. Let  $X : Z_2 \rightarrow [0, 1]$  such that:

$$X(a) = \begin{cases} 1 & \text{if } a = 0 \\ 3/4 & \text{otherwise} \end{cases}$$

$X$  is a fuzzy ring of  $Z_2$ , but  $X$  is not an almost quasi-Frobenius fuzzy ring since there exists a fuzzy ideal  $A$  in  $X$  such that:

$$A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{otherwise} \end{cases}$$

But  $A$  is not an Annihilator fuzzy ideal by [11, (2.3.2, (6))].

5. Any fuzzy ring of non-Noetherian ring is not an almost quasi-Frobenius fuzzy ring.

6. A fuzzy field  $X$  of a field  $F$  may not be an almost quasi-Frobenius fuzzy ring see [11, (3.1.2, (6))].

7. Since every fuzzy field of a field  $F$  is a fuzzy integral domain [11, (3.1.2, (6))], we can use the same example in (6) to show that a fuzzy integral domain may not be an almost quasi-Frobenius fuzzy ring.

8. If  $X$  is an almost quasi-Frobenius fuzzy ring of  $R$ , then it is not necessarily that  $R$  is a quasi-Frobenius ring as the following example shows.

**Example**

Let  $X$  be a fuzzy ring of  $Z$  such that:

$$X(a) = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that  $X$  is an almost quasi-Frobenius fuzzy ring of  $Z$ , see (2.2,(1)), but  $Z$  is not a quasi-Frobenius ring.

9. Let  $X$  be a fuzzy ring of a ring  $R$ . If  $R$  is a quasi-Frobenius ring, then it is not necessarily that  $X$  is an almost quasi-Frobenius fuzzy ring of  $R$  as the following example shows.

**Example**

The ring  $Z_2$  is a quasi-Frobenius ring. However, the fuzzy ring  $X$  of  $Z_2$  such that:

$$X(a) = \begin{cases} 1 & \text{if } a = 0 \\ 3/4 & \text{otherwise} \end{cases}$$

is not an almost quasi-Frobenius fuzzy ring, see (2.2, (4)).

We shall give some characterizations of almost quasi-Frobenius fuzzy rings.

**Proposition 2.3**

Let  $X$  be a fuzzy ideal of  $R$ . If  $X$  is an almost quasi-Frobenius fuzzy ring, then  $X$  is of the form:

$$X(a) = \begin{cases} c & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $c \in (0, 1]$ .

Proof:

Let  $X(0) = c$ , suppose that there exists  $a \in R, a \neq 0$  such that  $X(a) = d \neq 0$ . Hence either  $d < c$  or  $d = c$ . If  $d < c$ , let:

$$A(x) = \begin{cases} c & \text{if } x \in \langle 0 \rangle \\ d & \text{if } x \in \langle a \rangle \setminus \langle 0 \rangle \\ 0 & \text{otherwise} \end{cases}$$

Then  $A$  is a fuzzy ideal in  $X$ , but  $A$  is not an Annihilator fuzzy ideal of  $R$  by [11, proposition (2.3.3)], which contradicts the hypothesis  $X$  is an almost quasi-Frobenius fuzzy ring.

If  $d = c$ , then:

$$A(x) = \begin{cases} c & \text{if } x \in \langle a \rangle \\ 0 & \text{otherwise} \end{cases}$$

It follows that, for any  $k < c$ , there exists a fuzzy ideal  $B$  of  $R$  such that  $B \subseteq A$  and

$$B(x) = \begin{cases} c & \text{if } x \in \langle 0 \rangle \\ k & \text{if } x \in \langle a \rangle \setminus \langle 0 \rangle \\ 0 & \text{otherwise} \end{cases}$$

Hence  $B$  is a fuzzy ideal in  $X$  and  $B$  is not an Annihilator fuzzy ideal, which is a contradiction.

Thus  $X(a) = 0$  for all  $a \in R, a \neq 0$  and so

$$X(a) = \begin{cases} c & \text{if } x \in \langle a \rangle \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 2.4**

Let X be a fuzzy ring of Artinian ring R. Then the following are equivalent:

1. X is an almost quasi-Frobenius fuzzy ring.
2. Every fuzzy ideal of R is a finite-valued and any fuzzy ideal in X is an Annihilator fuzzy ideal.
3. Every fuzzy ideal in X is an Annihilator of finite-valued fuzzy ideal of R.
4. Every finite-valued fuzzy ideal in X is an Annihilator fuzzy ideal.

Proof:

(1) ⇒ (2) R is artinian ring, so by [10, theorem 3.2], every fuzzy ideal of R is a finite-valued. But X is an almost quasi-Frobenius fuzzy ring, then any fuzzy ideal A in X is an Annihilator fuzzy ideal.

(2) ⇒ (3) Since any fuzzy ideal A in X is an Annihilator fuzzy ideal, then A = F-AnnAnnA. Let B = F-AnnA. Then B be a fuzzy ideal of R, so B is a finite-valued fuzzy ideal of R. Thus A is the Annihilator of finite-valued fuzzy ideal of R.

(3) ⇒ (4) Let A be a fuzzy ideal in X, then A = F-AnnB, where B is a finite-valued fuzzy ideal of R. F-AnnA = F-AnnAnnB and F-AnnAnnA = F-AnnB = A. Thus F-AnnAnnA = A, that A is an Annihilator fuzzy ideal in X.

Hence every finite-valued fuzzy ideal in X is an Annihilator fuzzy ideal.

(4) ⇒ (1) R is artinian ring, so by [10, theorem 3.2], every fuzzy ideal of R is a finite-valued. Thus every fuzzy ideal A in X is a finite-valued and every fuzzy ideal A in X is an Annihilator fuzzy ideal. But R is Artinian ring, so R is a Noetherian ring by [12, theorem 8.5].

Hence by definition (2.1), X is an almost quasi-Frobenius fuzzy ring

**Theorem 2.5**

Let X be an almost quasi-Frobenius fuzzy ring of R, A be a non empty fuzzy ideal in X. Then  $A_t = \{0\}$  for all  $t \in (0, A(0)]$ .

Proof:

Suppose there exists  $t \in (0, A(0)]$  such that  $A_t \neq \{0\}$ . That is  $A(a) \geq t$  for some  $a \in R, a \neq 0$ . Now, let  $B : R \rightarrow [0, 1]$  defined by:

$$B(x) = \begin{cases} A(0) & \text{if } x = 0 \\ k & \text{if } x \in A_t \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

where  $0 < k < t$ .

It follows that B is a fuzzy ideal of R and  $B \subseteq A$ , hence B is a fuzzy ideal in X.

But B is not an Annihilator fuzzy ideal of R, by [11, proposition (2.3.3)], which is a contradiction since X is an almost quasi-Frobenius fuzzy ring. Thus  $A_t = \{0\}$  for all  $t \in (0, A(0)]$ .

**Corollary 2.6**

Let X be an almost quasi-Frobenius fuzzy ring of R, A be a non empty fuzzy ideal in X. Then

- (1) A is of the form:

$$A(x) = \begin{cases} c & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $c \in (0, X(0)]$ .

- (2)  $A_* = \{0\}$ .

Proof:

It is easy and hence omitted.

**Remark 2.7**

The converse of theorem (2.5) and corollary (2.6) are not true see (2.2, (5)).

**Remark 2.8**

Let X be a fuzzy ring of a Noetherian ring R. If  $\text{Supp}(X) = \{0\}$ , then X is an almost quasi-Frobenius fuzzy ring.

Proof:

It follows directly by (2.2, (1)).

**Remark 2.9**

By (2.2, (3)) the converse of remark (2.8) is not true.

**S.3 Some Relationships Between Almost Quasi-Frobenius Fuzzy ring, Quotient Fuzzy Ring and External Direct Sum Fuzzy Ring.**

In this section, we shall study some relationships between almost quasi-frobenius fuzzy ring and quotient fuzzy ring and external direct sum fuzzy ring. Firstly, we consider the following questions:

1. Is the homomorphic image of almost quasi-Frobenius fuzzy ring almost quasi-Frobenius fuzzy ring ?.
2. If X is a fuzzy ring of R and A is a fuzzy ideal in X such that X/A is an almost quasi-Frobenius fuzzy ring of R/A\*. Is X almost quasi-Frobenius fuzzy ring of R?.

The answer of the first question is provided by the following proposition:

**Proposition 3.1**

The homomorphic image of almost quasi-Frobenius fuzzy ring X is an almost quasi-Frobenius fuzzy ring.

Proof:

X is an almost quasi-Frobenius fuzzy ring of R, so any non empty fuzzy ideal A in X has the form:

$$A(x) = \begin{cases} c & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $c \in (0, X(0)]$  by theorem (2.5). Thus  $A_* = \{0\}$ . But any fuzzy ideal B/A in X/A where B is a fuzzy ideal in X such that  $A \subseteq B$  is defined by B/A :  $R/A_* \rightarrow [0,1]$  such that:

$$B/A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{B(a+b)\} & \text{if } b \in A_*, a \notin A_* \end{cases} = \begin{cases} 1 & \text{if } a = 0 \text{ (i.e., } a + A_* \\ = A_* = \{0\} \\ \sup\{B(a)\} & \text{if } a \neq 0 \text{ (i.e., } a + A_* \neq A_*) \end{cases}$$

$$= \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a \neq 0 \end{cases}$$

Also, R is a Noetherian ring, then R/A\* is a Noetherian ring. Hence by (2.2, (2)), X/A is an almost quasi-Frobenius fuzzy ring of R/A\*.

We consider the second question. The answer is provided by the following remark:

**Remark 3.2**

Let X be a fuzzy ring of R and A be a fuzzy ideal in X. If X/A is an almost quasi-Frobenius fuzzy ring, then it is not necessarily that X is an almost quasi-Frobenius fuzzy ring as the following example shows:

**Example**

Let X :  $Z_4 \rightarrow [0,1]$  such that

$$X(a) = \begin{cases} 1 & \text{if } a \in \langle 2 \rangle \\ 0 & \text{otherwise} \end{cases}$$

X is a fuzzy ring of  $Z_4$ , but X is not an almost quasi-Frobenius fuzzy ring by [11, proposition (2.3.3)].

Let A :  $Z_4 \rightarrow [0,1]$  such that:

$$A(x) = \begin{cases} 1/2 & \text{if } x \in \langle 2 \rangle \\ 0 & \text{otherwise} \end{cases}$$

A is a fuzzy ideal in X, however  $A_* = \langle 2 \rangle$ . Then:

$$X/A(a + A_*) = \begin{cases} 1 & \text{if } \bar{a} = \bar{0} \\ 0 & \text{if } \bar{a} \neq \bar{0} \end{cases}$$

On the other hand,  $Z_4$  is a Noetherian ring, so  $Z_4/\langle 2 \rangle$  is a Noetherian ring.

Hence by (2.2, (1)), X/A is an almost quasi-Frobenius fuzzy ring of  $Z_4/\langle 2 \rangle$ .

**Proposition 3.3**

Let X be a fuzzy ring of R and A be a fuzzy ideal in X. If X/A is an almost quasi-Frobenius fuzzy ring and  $A_* = \{0\}$ , then X is an almost quasi-Frobenius fuzzy ring.

Proof:

By definition (1.13),

$$X/A(a + A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup\{X(a+b) \mid b \in A_*, a \notin A_*\} & \text{if } b \in A_*, a \notin A_* \end{cases}$$

But  $X/A$  is an almost quasi-Frobenius fuzzy ring, so

$\sup\{X(a+b) \mid b \in A_*\} = c$  for some  $c \in (0, 1)$ . Let  $B/A$  be a fuzzy ideal in  $X/A$  such that:  $B/A(a + A_*) =$

$$\begin{cases} 1 & \text{if } a \in A_* \\ d & \text{if } b \in A_*, a \notin A_* \end{cases}$$

where  $d \in (0, c]$ .

$B/A$  is not an Annihilator fuzzy ideal, see [11, (2.3.2, (6))], which contradicts the hypothesis  $X/A$  is an almost quasi-Frobenius fuzzy ring. Thus  $\sup\{X(a+b) \mid b \in A_*\} = 0$  which implies that  $X(a) = 0$  for all  $a \in R, a \neq 0$ . It follows that:

$$X(a) = \begin{cases} e & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases}$$

for some  $e \in (0, 1]$ .

Also,  $X/A$  is an almost quasi-Frobenius fuzzy ring implies that  $R/A_*$  is Noetherian ring, so  $R$  is a Noetherian ring.

Thus by (2.2, (1)),  $X$  is an almost quasi-Frobenius fuzzy ring of  $R$ .

Now, we study the following:

1. Fuzzy external direct sum of almost quasi-Frobenius fuzzy rings.
2. The behaviour of fuzzy rings if their direct sum is an almost quasi-Frobenius fuzzy ring.

**Theorem 3.4**

Let  $X$  be a fuzzy ring of a ring  $R_1$  and  $Y$  be a fuzzy ring of a ring  $R_2$ . Then  $T = X \oplus Y$  is an almost quasi-Frobenius fuzzy ring of  $R_1 \oplus R_2$  if and only if  $X$  and  $Y$  are almost quasi-Frobenius fuzzy rings of  $R_1$  and  $R_2$  respectively.

Proof:

If  $T$  is an almost quasi-Frobenius fuzzy ring of  $R_1 \oplus R_2$ , then  $R_1 \oplus R_2$  is a Noetherian ring which implies  $R_1$  and  $R_2$  are Noetherian rings by [13, Exc. 12, p.333].

Let  $A$  and  $B$  be any non empty fuzzy ideals in  $X$  and  $Y$  respectively. By [11, theorem (2.4.1.8)],  $A \oplus B$  is a fuzzy ideal in  $T$ . Thus

$$\begin{aligned} A \oplus B &= F\text{-AnnAnn}(A \oplus B) \\ &= F\text{-Ann}(F\text{-Ann } A \oplus F\text{-Ann } B), \quad [11, \text{by theorem (2.4.2.4)}] \\ &= F\text{-AnnAnn } A \oplus F\text{-AnnAnn } B, \quad [11, \text{by theorem (2.4.2.4)}] \end{aligned}$$

Then  $A = F\text{-AnnAnn } A$  and  $B = F\text{-AnnAnn } B$ . Hence  $X$  and  $Y$  are almost quasi-Frobenius fuzzy rings of  $R_1$  and  $R_2$  respectively.

Conversely, if  $X$  and  $Y$  are almost quasi-Frobenius fuzzy rings of  $R_1$  and  $R_2$  respectively, then  $R_1$  and  $R_2$  are Noetherian rings. By [13, Exc.12, p.333],  $R_1 \oplus R_2$  is a Noetherian ring. Let  $A$  be any non empty fuzzy ideal in  $T$ . By [11, theorem (2.4.1.9)], there exists  $B_1$  and  $B_2$  fuzzy ideals in  $X$  and  $Y$  respectively such that  $A = B_1 \oplus B_2$ .

But  $B_1 = F\text{-AnnAnn } B_1$  and  $B_2 = F\text{-AnnAnn } B_2$ , so:

$$\begin{aligned} A &= F\text{-AnnAnn } B_1 \oplus F\text{-AnnAnn } B_2 \\ &= F\text{-Ann}(F\text{-Ann } B_1 \oplus F\text{-Ann } B_2) \quad \text{by [11, theorem (2.4.2.4)]} \\ &= F\text{-AnnAnn}(B_1 \oplus B_2) \quad \text{by [11, theorem (2.4.2.4)]} \\ &= F\text{-AnnAnn } A. \end{aligned}$$

Hence  $T = X \oplus Y$  is an almost quasi-Frobenius fuzzy ring of  $R_1 \oplus R_2$ .

**Remark 3.5**

We can give another proof of theorem (3.4) by using corollary (2.6), and [11, theorem (2.4.1.8) and theorem (2.4.1.9)].

**References**

1. Zadeh L. A., 1965, "Fuzzy Sets", Information and Control, Vol.8, pp.338-353,.
2. Liu W. J., 1982, "Fuzzy Invariant Subgroups and Fuzzy Ideals", Fuzzy Sets and Systems, Vol.8, pp.133-139,.



3. Lee K. H. and Mordeson J. N., 1997, "Factorization of Fuzzy Ideals in Dedekind Domains", The Journal of Fuzzy Mathematics, Vol.5, No.3, pp.741-745,.
4. Mukherjee T. K. and Sen M. K., 1987, "On Fuzzy Ideals of a Ring I", Fuzzy Sets and Systems, Vol.21, pp.99-104,.
5. Alkhomees Y. and Mordeson J. N., 1998, "Fuzzy Principal Ideals and Fuzzy Simple Field Extensions", Fuzzy Sets and Systems, Vol.96, pp.247-253,.
6. Mordeson J. N., "Fuzzy Group Subalgebras", 1995, The Journal of Fuzzy Mathematics, Vol.3, No.1, pp.69-81,.
7. Malik D. S., Mordeson J. N. and Nair P. S., 1992, "Fuzzy Generators and Fuzzy Direct Sums of Abelian Groups", Fuzzy Sets and Systems, Vol.50, pp.193-199,.
8. Martinez L., "Prime and Primary L-Fuzzy Ideals of L-Fuzzy Rings, Fuzzy Sets and Systems, Vol.101, pp.489-494, (1999).
9. Hussein W. R., 1999, "Some Results of Fuzzy Rings", M.Sc. Thesis, University of Baghdad, College of Education-Ibn Al-Haitham,.
10. Malik D. S., 1990 "Fuzzy Ideals of Artinian Rings", Fuzzy Sets and Systems, Vol.37, pp.111-115,.
11. Abou-Dareb A. T. , 2000, "On Almost Quasi-Frobenius Fuzzy Rings", M.Sc. Thesis, University of Baghdad, College of Education-Ibn Al-Haitham,.
12. Atiah M. F. and Macdonald I. C., 1969, "Introduction to Commutative Algebra", Addison-Wesley Publishing Company,.
13. Quzi-Zameeruddin S. S., 1972, "Modern Algebra", Uikas Publishing House Putitd,.
14. Tachikawa H., 1973, "Quasi-Frobenius Rings and Generalizations", QF.3, QF.1 Rings. Lecture Notes in Mathematics, Springer-Verlage, Berlin,.
15. Atiah M. F. and Macdonald I. C., 1969, "Introduction to Commutative Algebra", Addison-Wesley Publishing Company,.

### حول الحلقات الضبابية شبه الفروبينية تقريباً

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#### المستخلص

في هذا البحث قدمنا مفهوم الحلقات الضبابية شبه الفروبينية تقريباً كتعميم لمفهوم الحلقات الضبابية شبه الفروبينية. أعطينا بعض الخواص لهذا المفهوم. كذلك درسنا العلاقة بين الحلقات الضبابية شبه الفروبينية تقريباً مع حلقات القسمة الضبابية وكذلك درسنا المجموع المباشر الخارجي الضبابي للحلقات الضبابية شبه الفروبينية تقريباً.