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Strong Subordination for $\mathcal{E}$ -valent Functions Involving the Operator Generalized Srivastava-Attiya

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Abstract:

Some relations of inclusion and their properties are investigated for functions of type "$\mathcal{E}$-valent that involves the generalized operator of Srivastava-Attiya by using the principle of strong differential subordination.

Key words: Differential subordination, Operator generalized Srivastava-Attiya, P-valent functions, Strong subordination, Univalent function.

Introduction:

Let Ap be the class of $\mathcal{E}$-valent and analytic functions defined on U={ $z \epsilon C$:| z |< 1} :

$$f(z) = z^{\mathcal{E}} + \sum_{\tau=\mathcal{E}+1}^{\infty} a_\tau z^\tau, \quad (a_\tau \geq 0, \mathcal{E} \epsilon N = \{1,2,\dots\}). \qquad (1)$$

Now," let $\Phi(z, s, a)$ denote the Hurwitz-Lerch zeta function defined as follows"(1):

$$\Phi(z, s, a) \sum_{\tau=0}^{\infty} \frac{z^\tau}{(\tau+a)^s}, \qquad (2)$$

$(a \in C \setminus \bar{z}_0 = \{0, -1, \dots\}; s \in C$ when $|z| < 1$; $Re\{s\} > 1$ $when$ $|z| = 1)$. "$L_{s,\mathbb{b}}: A_1 \to A_1$ is defined by Srivastava and Attiya (2), (3), $A_1 = A(1)$ in the form of

$$L_{s,\mathbb{b}} f(z) = G_{s,\mathbb{b}}(z) * f(z), \quad (z \in U; \mathbb{b} \in C \setminus \bar{z}_0; s \in C), \qquad (3)$$

where

$$G_{\mathcal{E},s,\mathbb{b}} = (\mathbb{b}+1)^s [\Phi(z, s, \mathbb{b}) - \mathbb{b}^{-s}] (z \in U)". \qquad (4)$$

Analogously to" $L_{s,\mathbb{b}}$, Liu(4),(5,6,7) defined the operator $\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}: A_{\mathcal{E}} \to A_{\mathcal{E}}$ by

$$\mathfrak{J}_{\mathcal{E},s,\mathbb{b}} f(z) = G_{p,s,\mathbb{b}}(z) * f(z), \quad (z \in U; \mathbb{b} \in C \setminus \bar{z}_0; s \in C; \mathcal{E} \in N), \qquad (5)$$

where $G_{\mathcal{E},s,\mathbb{b}} = (\mathbb{b}+1)^s [\Phi_{\mathcal{E}}(z, s, \mathbb{b}) - \mathbb{b}^{-s}]$, and

$$\Phi_{\mathcal{E}}(z, s, \mathbb{b}) = \frac{1}{\mathbb{b}^s} + \sum_{\tau=\mathcal{E}}^{\infty} \frac{z^\tau}{(\tau - \mathcal{E} + 1 + \mathbb{b})^s}. \qquad (6)$$

equations (5) and (6) yield

$$\mathfrak{J}_{\mathcal{E},s,\mathbb{b}} f(z) = z^p + \sum_{\tau=\mathcal{E}+1}^{\infty} \left(\frac{1+\mathbb{b}}{\tau - \mathcal{E} + 1 + \mathbb{b}} \right)^s a_\tau z^\tau. \qquad (7)$$

from (7) we get

$$z(\mathfrak{J}_{\mathcal{E},s,\mathbb{b}} f(z))' = (\mathbb{b}+1)\mathfrak{J}_{\mathcal{E},s-1,\mathbb{b}} f(z) - (\mathbb{b}+1-\mathcal{E})\mathfrak{J}_{\mathcal{E},s,\mathbb{b}} f(z). \qquad (8)$$

The function f is said to be subordinate to g, if there exists Schwarz function w analytic in U, also $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$. Such that $f(z) = g(w(z)), z \in U$. Can be written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$.

If $g(z)$ is univalent in U", then from [1] we have "$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$".

Definition (1):" ((8),cf.(1,9)) Let $H(z, €)$ be analytic in $Ux\bar{U}$ and let $f(z)$ be analytic and univalent in U. Then the function $H(z, €)$ is said to be strongly subordinate to $f(z)$, written $H(z, €) \prec\prec f(z)$ if for $€ \in \bar{U}$, $H(z, €)$ as a function of z is subordinate to $f(z)$.

We note that $H(z, €) \prec\prec f(z)$ if and only if $H(0, €) = f(0)$ and $(Ux\bar{U}) \subset f(U)$".

Definition (2):((9),cf.(10)) "Let $\varphi: C^3 x U x \bar{U} \to C$ and let $h(z)$ be univalent in U .If $p(z)$ is analytic in U and satisfies the (second-order) differential subordination

$$\varphi(p(z), z p'(z), z^2 p''(z); z, €) \prec\prec h(z). \qquad (9)$$

Then $p(z)$ is called a solution of the strong differential subordination.

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The univalent function $q(ƶ)$ is called a domainant of the solutions of the strong differential subordination, or more simply a dominant, if $p(ƶ) \prec q(ƶ)$ for all $p(ƶ)$ satisfying (9) .
A dominant $\overline{q}(ƶ)$ that satisfies $\overline{q}(ƶ) \prec q(ƶ)$ for all dominant $q(ƶ)$ of (9) is said to be the best dominant".

Definition (3): (9) "Let Ω be a set in $C, q(ƶ) \in Q$ and n be positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of these functions $\psi: C^3 xUx\overline{U} \to C$ that satisfy the admissibility condition.
$\psi(r,s,t;\ ƶ, €) \notin \Omega$, whenever $r = q(\zeta)$, $s = \tau\xi q'(\zeta)$ and

$$Re\left\{\frac{t}{s}+1\right\} \geq \tau Re\left\{\frac{€q''(\zeta)}{q'(\zeta)}+1\right\},$$

for $ƶ \in U, \zeta \in \partial U \setminus E(q), € \in \overline{U}$ and $\tau \geq n$ .We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$ ".

Theorem (1):(8) "Let $\psi \in \Psi_n[\Omega, q]$ with $(0) = a$. If $p \in H[a, n]$ satisfies
$\psi\big((p(ƶ), ƶ\, p'(ƶ), ƶ^2 p''(ƶ);\ ƶ, €\big) \in \Omega$. Then $p(ƶ) \prec q(ƶ)$."

2-" Strong subordination results(3)"

Now we prove the "subordination theorem involving with the generalized Srivastava-Attiya operator"(4) $Ɉ_{\mathcal{E},s,𝕓}$.

Definition (4): The class of admissible functions $\phi_Ɉ[\Omega, q]$ satisfy the admissibility condition if consists of those functions $\varphi: C^3 xUx\overline{U} \to C$ when $\Omega \in C, q \in Q_0 \cap M[0, \mathcal{E}], 𝕓 \in C - \{0, -1, \dots\}, s \in C$.
$\varphi(u_1, u_2, u_3;\ ƶ, €) \notin \Omega$, whenever

$u = q(\zeta)$, $v = \frac{\tau\zeta q'(\zeta) + (𝕓+1-\mathcal{E})q(\zeta)}{(𝕓+1)}$, $(\mathcal{E} \in N, b \in C - \{0, -1, \dots\})$ and

$$Re\left\{\frac{(1+𝕓)^2 u_3 - (1+𝕓-\mathcal{E})^2 u_1}{(1+𝕓)u_2 - (1+𝕓-\mathcal{E})u_1} - 2(𝕓+1 - \mathcal{E})\right\} \geq kRe\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\},$$

$\forall\ ƶ \in U, \zeta \in \partial U \setminus E(q)$ and $\tau \geq \mathcal{E}$.

Theorem (2): Let $\varphi \in \phi_Ɉ[\Omega, q]$ .If $f \in A\boldsymbol{\mathcal{E}}$ satisfies

$$\left\{\varphi\big(Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ), Ɉ_{\mathcal{E},s,𝕓}f(ƶ), Ɉ_{\mathcal{E},s-1,𝕓}f(ƶ);\ ƶ\big): ƶ \in U, € \in U\right\} \subset \Omega \qquad (10)$$

Then $Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ) \prec q(ƶ)$.

Proof: suppose
$F(ƶ) = Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ)$ (11)
equations (8) and (11) give

$$ƶ\,(Ɉ_{\mathcal{E},s,𝕓}f(ƶ))' = (𝕓+1)Ɉ_{\mathcal{E},s-1,𝕓}f(ƶ) - (𝕓+1-\mathcal{E})Ɉ_{\mathcal{E},s,𝕓}f(ƶ).$$

$$Ɉ_{p,s-1,𝕓}f(ƶ) = \frac{ƶ\,(Ɉ_{\mathcal{E},s,𝕓}f(ƶ))' + (𝕓+1-\mathcal{E})Ɉ_{\mathcal{E},s,𝕓}f(ƶ)}{(𝕓+1)}, \qquad (12)$$

$$Ɉ_{\mathcal{E},s,𝕓}f(ƶ) = \frac{ƶ\,(Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ))' + (𝕓+1-\mathcal{E})Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ)}{(𝕓+1)}, \qquad (13)$$

from (11) we get

$$Ɉ_{\mathcal{E},s,𝕓}f(ƶ) = \frac{ƶ\,F'(ƶ) + (𝕓+1-\mathcal{E})F(ƶ)}{(𝕓+1)}$$

$$Ɉ_{\mathcal{E},s-1,𝕓}f(ƶ) = \frac{ƶ^2 F''(ƶ) + [1+2(𝕓+1-\mathcal{E})]\,ƶ\,F'(z) + (𝕓+1-\mathcal{E})^2 F(ƶ)}{(𝕓+1)^2}$$

Let u_1, u_2 and u_3 take the transformation from C^3 to C by

$$u_1 = r, u_2 = \frac{s+(𝕓+1-\mathcal{E})r}{(𝕓+1)}, u_3 = \frac{t+[1+2(𝕓+1-\mathcal{E})]s+(𝕓+1-\mathcal{E})^2 r}{(b+1)^2}.$$

Assume that
$\psi(s,r,t;\ ƶ) = \varphi(u_1, u_2, u_3; ƶ)$

$$= \varphi\left(r, \frac{s+(𝕓+1-\mathcal{E})r}{(𝕓+1)}, \frac{t+[1+2(𝕓+1-\mathcal{E})]s+(𝕓+1-\mathcal{E})^2 r}{(𝕓+1)^2}; z, €\right). \qquad (14)$$

By using equation (11),(12),(13), from (14), we get

$$\psi\big(F(ƶ), zF'(ƶ), z^2F''(ƶ);\ ƶ, €\big) = \varphi\big(Ɉ_{\mathcal{E},1+s,𝕓}f(ƶ), Ɉ_{\mathcal{E},s,𝕓}f(ƶ), Ɉ_{\mathcal{E},s-1,𝕓}f(z);\ ƶ, €\big) \quad (15)$$

Therefore, (10) becomes
$\psi\big(F(ƶ), ƶ\,F'(ƶ), ƶ^2F''(ƶ);\ ƶ, €\big) \in \Omega$.
Note that

$$\frac{t}{s}+1 = \frac{(𝕓+1)^2 u_3 - (𝕓+1-\mathcal{E})^2 u_1}{(𝕓+1)u_2 - (𝕓+1-\mathcal{E})u_1} - 2(𝕓+1-\mathcal{E}),$$

and since the admissibity condition for $\varphi \in \phi_Ɉ[\Omega, q]$ is equivalent to the the admissibity condition for ψ by Definition(3) ,then $\psi \in \Psi[\Omega, q]$, and by Theorem(1), $F(ƶ) \prec q(ƶ)$. Or
$Ɉ_{\mathcal{E},1+s,𝕓}f(z) \prec q(z)$, and the proof is complete.

If $\Omega \neq C$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping h.
In case, the class $\phi_{ɟ}[h(U), q]$ is written as $\phi_{ɟ}[h, q]$.

Theorem (3): Let $\varphi \in \phi_{ɟ}[h, q]$. If $f \in A\mathcal{E}$ satisfies

$$\varphi(ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ); ᴢ, €) \prec\prec h(ᴢ). \quad (16)$$

Then $ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ) \prec q(ᴢ)$.

Corollary (1): Let $\Omega \subset C$ and q be univalent in U with $q(0) = 1$. Let $\varphi \in \phi_{ɟ}[\Omega, q_\rho]$ $\forall \rho \in (0,1)$ where $q_\rho(ᴢ) = q(\rho ᴢ)$ .If $f \in A\mathcal{E}$ satisfies

$$\varphi(ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ); ᴢ, €) \in \Omega .$$

Then $J_{p,s+1,\mathbb{b}}f(ᴢ) \prec q(ᴢ)$.

Proof: Theorem (2) yields then $ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ) \prec q_\rho(ᴢ)$. The result is $q_\rho(ᴢ) \prec q(ᴢ)$.□

Theorem (4): U contains the univalent h and q such that $q(0) = 0$ and set $q_\rho(ᴢ) = q(\rho ᴢ)$ and $h_\rho(ᴢ) = h(\rho ᴢ)$. Let $\varphi: C^3 x U x \overline{U} \to C$ satisfies one of the following conditions

1- $\varphi \in \phi_{ɟ}[\Omega, q_\rho]$ $\forall \rho \in (0,1)$, or

2- $\exists$ $\rho_0 \in (0,1)$ such that $\varphi \in \phi_{ɟ}[h_\rho, q_\rho]$ $\forall \rho \in (\rho_0, 1)$.

If $f \in Ap$ satisfies (16), then $ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ) \prec q(ᴢ)$.

Proof: "similar to Theorem 2.3d (11)".□

Theorem (5): Let U contains the univalent h and Let $\varphi: C^3 x U x \overline{U} \to C$.
The differential equation

$$\varphi\left(q(z), \frac{ᴢ q'(ᴢ) + (\mathbb{b}+1-\mathcal{E})q(ᴢ)}{(\mathbb{b}+1)}, \frac{ᴢ^2 q''(ᴢ) + [1+2(\mathbb{b}+1-\mathcal{E})] ᴢ q'(z) + (\mathbb{b}+1-\mathcal{E})^2 q(ᴢ)}{(\mathbb{b}+1)^2}\right)$$

$$h(ᴢ) \quad (17)$$

has a solution q with $q(0) = 0$ and satisfies one of the following

1- $q \in \varphi_0$ and $\varphi \in \phi_{ɟ}[h, q]$,

2- q is univalent in U and $\varphi \in \phi_{ɟ}[h, q_\rho]$ $\forall$ $\rho \in (0,1)$,or

3- q is univalent in U and $\exists \rho_0 \in (0,1)$ such that $\varphi \in \phi_{ɟ}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$. If $f \in A\mathcal{E}$ satisfies (16) and

U contains the analytic $\varphi(ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ); ᴢ, €)$,then $ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ) \prec q(ᴢ)$ and q is the best dominant.

Proof: By "Theorem 2.3e" (11) and from Theorem (3) and Theorem (4) q is dominant.
q is satisfies (17), it is also a solution of (16) so q is the best dominant.□

when $q(z) = Mz, M > 0$, the class of admissible functions $\phi_{ɟ}[\Omega, q]$, denoted by $\phi_J[\Omega, M]$, is described below .

Definition (5): Let Ω be a set in $C, s \in C, \mathbb{b} \in C - \{0, -1, ...\}$, and $M > 0$. The class of admissible functions $\phi_J[\Omega, M]$ consists of those functions $\varphi: C^3 x U x \overline{U} \to C$ such that

$$\varphi\left(Me^{i\theta}, \frac{(\tau+\mathbb{b}+1-\mathcal{E})Me^{i\theta}}{(\mathbb{b}+1)}, \frac{L+[(1+2(\mathbb{b}+1-\mathcal{E}))\tau+(\mathbb{b}+1-\mathcal{E})^2]Me^{i\theta}}{(\mathbb{b}+1)^2}; ᴢ, €\right) \notin \Omega, \quad (18)$$

whenever $ᴢ \in U, € \in \overline{U}, Re\{Le^{-i\theta}\} \geq (k-1)kM, \theta \in R, \tau \geq \mathcal{E}$.

Corollary (2): Let $\varphi \in \phi_{ɟ}[\Omega, M]$. If $f \in A\mathcal{E}$ satisfies

$$\varphi(ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ); ᴢ, €) \in \Omega$$

Then $ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ) \prec M ᴢ$.

Corollary (3): Let $\varphi \in \phi_{ɟ}[M]$. If $f \in A\mathcal{E}$ satisfies

$$\left|\varphi(ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ), ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ); ᴢ, €)\right| < M.$$

Then $\left|ɟ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)\right| < M$. For the special case $q(U) = \{w: |w| < M\}$, the class $\phi_{ɟ}[\Omega, M]$ is denoted by $\phi_{ɟ}[M]$.

Corollary (4): Let $M > 0$ and $\overline{U}$ contains the analytic function $C(€)$ with $Re\{€C(€)\} \geq 0$ $\forall € \in \partial U$. If $f \in A\mathcal{E}$ satisfies

$$\left|(\mathbb{b}+1)^2 \; ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ) - (\mathbb{b}+1)ɟ_{\mathcal{E},s,\mathbb{b}}f(ᴢ) - (\mathbb{b}+1-\mathcal{E})^2 ɟ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ) + C(€)\right| < (\mathbb{b}+1-\mathcal{E})M .$$

Then $\left|ɟ_{\mathcal{E},s+1,\mathbb{b}}f(ᴢ)\right| < M$.

Proof: From corollary (2) by taking $\varphi(u_1, u_2, u_3; ᴢ, €) = (\mathbb{b}+1)^2 u_3 - (\mathbb{b}+1)u_2 - (\mathbb{b}+1-p)^2 u_1 + C(€)$ and $\Omega = h(U)$, where $h(ᴢ) = (b+1-\mathcal{E})M ᴢ$. By corollary (2), to prove $\varphi \in \phi_{ɟ}[\Omega, M]$, that is admissible condition(18) is satisfied. We get

$$\left|\varphi\left(Me^{i\theta}, \frac{(\tau+\mathbb{b}+1-\mathcal{E})Me^{i\theta}}{(\mathbb{b}+1)}, \frac{L+[(1+2(\mathbb{b}+1-\mathcal{E}))\tau+(\mathbb{b}+1-\mathcal{E})^2]Me^{i\theta}}{(\mathbb{b}+1)^2}; z, €\right)\right|$$

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$$= \left|L + \left[\left(1 + 2(b+1-\mathcal{E})\right)\tau + (b+1-\mathcal{E})^2\right]Me^{i\theta} - (\tau + b + 1 - \mathcal{E})Me^{i\theta} - (\mathbb{b}+1-\mathcal{E})^2Me^{i\theta} + C(€)\right|$$
$$= \left|L + (\mathbb{b}+1-\mathcal{E})(2\tau-1)Me^{i\theta} + C(€)\right|$$
$$\geq (\mathbb{b}+1-\mathcal{E})(2\tau-1)M\tau + Re\left\{Le^{-i\theta} + Re\{C(€)e^{-i\theta}\}\right\}$$
$$\geq (\mathbb{b}+1-\mathcal{E})(2\tau-1)M + \tau(\tau-1)M + Re\{C(€)e^{-i\theta}\} \geq (\mathbb{b}+1-\mathcal{E})M$$

Definition (6): Let Ω be a set in, $q \in Q_1 \cap M[1,1]$. The class of admissible functions $\phi_{ɉ,2}[\Omega, q]$ contains the functions $\varphi: C^3 x U x \overline{U} \to C$ that satisfy the admissibility condition:
$\varphi(u_1, u_2, u_3; ᴢ, €) \notin \Omega$, whenever
$u_1 = q(\zeta)$, $u_2 = \frac{\tau\zeta q'(\zeta) + (\mathbb{b}+1)(q(\zeta))^2}{(\mathbb{b}+1)q(\zeta)}$, and

$$Re\left\{(\mathbb{b}+1)\left(\frac{(u_3-u_1)}{u_2-u_1}u_1 - (u_3 - 3u_1)\right)\right\} \geq \tau Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

for $ᴢ \in U$, $\zeta \in \partial U \backslash E(q)$; $€ \in \overline{U}$ and $\tau \geq \boldsymbol{\mathcal{E}}$.

Theorem (6): Let $\varphi \in \phi_{ɉ,2}[\Omega, q]$ .If $f \in A\boldsymbol{\mathcal{E}}$ satisfies

$$\left\{\varphi\left(\frac{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)}, \frac{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}, \frac{ɉ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)}; ᴢ\right): ᴢ \in U\right\} \subset \Omega. \quad (19)$$

Then $\frac{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)} \prec q(ᴢ)$.

Proof: define the function F in U

$$F(ᴢ) = \frac{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)}$$

from (20) and computations that, show that

$$\frac{zF'(ᴢ)}{F(ᴢ)} = \frac{z(ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ))'}{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)} - \frac{ᴢ\left(ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)\right)'}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)}, \quad (21)$$

by using the relation(8) ,we get

$$\frac{zɉ_{\mathcal{E},1+s,\mathbb{b}}(f(ᴢ))'}{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)} = \frac{ᴢ F'(ᴢ)}{F(ᴢ)} + (\mathbb{b}+1)F(ᴢ) - (\mathbb{b}+1-\mathcal{E}). \quad (22)$$

Therefore,

$$\frac{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},1+,s,\mathbb{b}}f(ᴢ)} = \frac{ᴢ F'(ᴢ) + (\mathbb{b}+1)\left(F(z)\right)^2}{(\mathbb{b}+1)F(ᴢ)}. \quad (23)$$

by computations show that

$$\frac{ɉ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)} = \frac{ᴢ^2F''(ᴢ) + [1 + 3(\mathbb{b}+1)F(ᴢ)]ᴢ F'(ᴢ) + (\mathbb{b}+1)^2\left(F(ᴢ)\right)^3}{(\mathbb{b}+1)ᴢ F'(ᴢ) + (\mathbb{b}+1)^2\left(F(ᴢ)\right)^2}. \quad (24)$$

Define C^3 to C by

$$u_1 = r, \ u_2 = \frac{s + (\mathbb{b}+1)r^2}{(\mathbb{b}+1)r}, \ u_3 = \frac{t + [1 + 3(\mathbb{b}+1)r]s + (\mathbb{b}+1)^2r^3}{(\mathbb{b}+1)s + (\mathbb{b}+1)^2r^2}.$$

Assume that

$$\psi(s, r, t; ᴢ) = \varphi(u_1, u_2, u_3; ᴢ)$$
$$= \varphi\left(r, \frac{s + (\mathbb{b}+1)r^2}{(\mathbb{b}+1)r}, \frac{t + [1 + 3(\mathbb{b}+1)r]s + (\mathbb{b}+1)^2r^3}{(\mathbb{b}+1)s + (\mathbb{b}+1)^2r^2}; ᴢ\right). \quad (25)$$

The proof by Theorem (1), using equation (20),(23),(24), from (25), we get

$$\psi\left(F(ᴢ), ᴢ F'(ᴢ), ᴢ^2F''(ᴢ); ᴢ, €\right) = \varphi\left(\frac{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)}, \frac{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}, \frac{ɉ_{\mathcal{E},s-1,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},s,\mathbb{b}}f(ᴢ)}; z, €\right). \quad (26)$$

Therefore, (19) becomes
$\psi\left(F(ᴢ), ᴢ F'(ᴢ), ᴢ^2F''(ᴢ); ᴢ, €\right) \in \Omega$
Note that

$$\frac{t}{s} + 1 = (b+1)\left(\frac{(u_3-u_1)}{u_2-u_1}u_1 - (u_3 - 3u_1)\right),$$

Because the admissibility $\forall \ \varphi \in \phi_{ɉ,2}[\Omega, q]$ is equivalent to the admissibility $\forall \ \psi$ as given in Definition(3), then $\psi \in \Psi[\Omega, q]$, and by Theorem(1), $F(ᴢ) \prec q(ᴢ)$. Or $\frac{ɉ_{\mathcal{E},1+s,\mathbb{b}}f(ᴢ)}{ɉ_{\mathcal{E},2+s,\mathbb{b}}f(ᴢ)} \prec q(ᴢ)$, and the proof is complete. □

For some conformal mapping $\Omega = h(U)$ of U onto, if $\Omega \neq C$ is a simply connected domain . i.e, the class $\phi_{\mathfrak{J},2}[h(U), q]$ written by $\phi_{\mathfrak{J},2}[h, q]$.
The prove of Theorem (7) is immediate by Theorem(6).

Theorem (7): Let $\varphi \in \phi_{\mathfrak{J},2}[h,q]$. If $f \in A\mathcal{E}$ satisfies

$$\varphi\left(\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s-1,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}; z, €\right) \prec\prec h(z). \quad (27)$$

Then $\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)} \prec q(z) \quad (z \in U)$.

If $q(z) = 1 + Mz, M > 0$ then by Definition (6) the admissible functions $\phi_{\mathfrak{J},2}[\Omega, q]$, denoted by $\phi_{\mathfrak{J},2}[\Omega, M]$, is described below.

Definition (7): .The class of admissible functions $\phi_{\mathfrak{J},2}[\Omega, M]$ contains the functions $\varphi: C^3 x U x \overline{U} \to C$ s.t

$$\varphi\left(\begin{matrix} 1 + Me^{i\theta}, 1 + \frac{\tau + 1 + Me^{i\theta}}{(\mathbb{b}+1)(1+Me^{i\theta})}Me^{i\theta}, \\ \frac{L + \tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})\left\{3\tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})^2\right\}}{(\mathbb{b}+1)[\tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})^2]}; z, € \end{matrix}\right) \notin \Omega, \quad (28)$$

whenever $z \in U, € \in \overline{U}, Re\{Le^{-i\theta}\} \geq (\tau - 1)\tau M, \theta \in R$ and $\tau \geq \mathcal{E}$.
if Ω be a set in $C, s \in C, b \in C - \{0, -1, ...\}$, and $M > 0$

Corollary (5): Let $\varphi \in \phi_{\mathfrak{J},2}[\Omega, M]$. If $f \in A\mathcal{E}$ satisfies

$$\varphi\left(\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s-1,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}; z, €\right) \in \Omega$$

Then $\left(\frac{J_{p,s+1,\mathbb{b}}f(z)}{J_{p,s+2,\mathbb{b}}f(z)}\right) - 1 \prec Mz$.

Corollary (6): Let $\varphi \in \phi_{\mathfrak{J},2}[M]$. If $f \in A\mathcal{E}$ satisfies

$$\left|\varphi\left(\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}, \frac{\mathfrak{J}_{\mathcal{E},s-1,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}; z, €\right) - 1\right| < M,$$

Then $\left|\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)} - 1\right| < M$.

For the special case $q(U) = \{u_3: |u_3 - 1| < M\}$, the class $\phi_{\mathfrak{J},2}[\Omega, M]$ is denoted by $\phi_{\mathfrak{J},2}[M]$.

Corollary (7): Let $M > 0$ and $\overline{U}$ contains the analytic function $C(\xi)$ and $Re\{€C(€)\} \geq 0$ $\forall € \in \partial U$. If $f \in A\mathcal{E}$ satisfies

$$\left|(\mathbb{b}+1)^2 \frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)} - (\mathbb{b}+1)\frac{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}f(z)} - (\mathbb{b}+1-\mathcal{E})^2 \frac{\mathfrak{J}_{\mathcal{E},s-1,\mathbb{b}}f(z)}{\mathfrak{J}_{\mathcal{E},s,\mathbb{b}}f(z)} + C(€) - 1\right| < (\mathbb{b}+1-\mathcal{E})M.$$

Then $\left|\frac{\mathfrak{J}_{\mathcal{E},1+s,\mathbb{b}}(z)}{\mathfrak{J}_{\mathcal{E},2+s,\mathbb{b}}f(z)} - 1\right| < M$.

Proof: From corollary (5) by taking $\varphi(u_1, u_2, u_3; z, €) = (\mathbb{b}+1)^2 u_3 - (\mathbb{b}+1)u_2 - (\mathbb{b}+1-\mathcal{E})^2 u_1 + C(€) - 1$ and $\Omega = h(U)$, where $h(z) = (\mathbb{b}+1-\mathcal{E})Mz$, and corollary (5),to prove $\varphi \in \phi_{\mathfrak{J},2}[\Omega, M]$, that is admissible condition (28) is satisfied. We get

$$\left|\varphi\left(\begin{matrix} 1 + Me^{i\theta}, 1 + \frac{\tau + 1 + Me^{i\theta}}{(\mathbb{b}+1)(1+Me^{i\theta})}Me^{i\theta}, \\ \frac{L + \tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})\left\{3\tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})^2\right\}}{(\mathbb{b}+1)[\tau Me^{i\theta} + (\mathbb{b}+1)(1+Me^{i\theta})^2]}; z, € \end{matrix}\right) - 1\right|$$

$$= \left|1 + Me^{i\theta}\frac{\{(\mathbb{b}+1)(1+Me^{i\theta}) + \tau\}}{(\mathbb{b}+1)(1+Me^{i\theta})} - 1 - Me^{i\theta}\right|$$

$$= \left|\frac{M\tau}{(\mathbb{b}+1)(1+Me^{i\theta})}\right| \geq \frac{M}{(\mathbb{b}+1)(1+M)}$$

Author's declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in Al-Karkh University of Science.

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التبعية التفاضلية للدوال متعددة التكافوء متضمنه لتعميم الموثرSrivastava-Attiya

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الخلاصة:

الموضوع المقدم في هذا البحث يتضمن التحري عن بعض العلاقات وبعض الخواص المهمة للدوال متعددة التكافوء التي تتعامل مع موثر(Srivastava-Attiya)المعمم بواسطة استخدام مبادئ التبعية التفاضلية القوية.

الكلمات المفتاحية:التبعية التفاضلية، تعميم موثر سرفستافة –اتاي، الدوال متعددة التكافوء، التبعية القوية، الدوال احادية التكافوء.