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# On Cohomology Groups of Four-Dimensional Nilpotent Associative Algebras

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**Abstract:**

The study of cohomology groups is one of the most intensive and exciting researches that arises from algebraic topology. Particularly, the dimension of cohomology groups is a highly useful invariant which plays a rigorous role in the geometric classification of associative algebras. This work focuses on the applications of low dimensional cohomology groups. In this regards, the cohomology groups of degree zero and degree one of nilpotent associative algebras in dimension four are described in matrix form.

**Keywords:** Cohomology group, Derivation, Inner derivation, Nilpotent associative algebra.

## Introduction:

Algebraic topology is one of the main areas in mathematics. This area uses fundamental ingredients from abstract algebra in the study of topological spaces. One of the essential techniques of algebraic topology is cohomology groups. It is a general term of a sequence of groups associated with a topological space which is defined from a cochain. On the other hand, one of the ancient areas of the modern algebra is the theory of finite-dimensional associative algebras, and it has been studied by many investigators like Pierce [1], Mazzola [2] and Basri [3]. In this regard, this study concentrates on cohomology groups for associative algebras.

The cohomology theory of groups has numerous applications in many sorts of mathematical and physical studies (see for instance [4,5] and [6]). The cohomology groups $H^p(A, D)$, with an action associative algebra A on A-bimodule D, were presented by Hochschild [7]. One of the important invariant in studying cohomological problems is derivations. The derivations are linear transformations on algebras and exactly the elements of $Z^1(A, D)$ 1-cocycles, while the so-called inner derivations are 1-coboundaries denoted by $B^1(A, D)$, [8, 9].

The present study concentrates on the description of low dimensional cohomology groups $H^i(A, A)$; $i = 0, 1$, because the dimension of the cohomology groups is considered as one of the important invariants to study the properties of algebras. Particularly, this invariant plays a rigorous role in the geometric classification of associative algebras. Using the classification result of nilpotent associative algebras, the description of cohomology groups of degree zero and degree one of nilpotent associative algebras in dimension four are given in matrix form.

## Preliminaries

This section provides some terminologies that are needed in this work.

**Definition 1** [10]**:** An associative algebra over a field K is a K -vector space A equipped with bilinear map $\lambda: A \times A \longrightarrow A$ satisfying the associative law:

$$\lambda(\lambda(x, y), z) = \lambda(x, (\lambda(y, z)), \quad \forall x, y, z \in A$$

**Definition 2** [10]**:** Let A be an associative algebra over K and define as:

$$A^1 = A;\ A^k = \lambda(A, A^{k-1})\quad (k > 1)$$

The series

$$A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots$$

is called the descending central series of A. If there exists an integer $s \in \mathbb{N}$,, such that

$$A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots A^s = \{0\}$$

then this algebra is called nilpotent.

**Definition 3** [10] **:** Let $A_1$ and $A_2$ be two associative algebras over $K$. A homomorphism between $A_1$ and $A_2$ is an $K$ -linear mapping $f : A_1 \longrightarrow A_2$ such that

$$f(xy) = f(x)f(y) \qquad \forall x, y \in A_1$$

**Definition 4** [10] **:** A linear transformation $d$ of associative algebra $A$ is called a derivation if

$$d(x \cdot y) = d(x)y + xd(y) \qquad \forall x, y \in A$$

The set of all derivations is denoted by $Der(A)$. An important special case of derivation mapping so-called inner derivation mapping is defined as follows:

**Definition 5** [10] **:** A linear transformation $ad_z$ of the associative algebra A is called an inner derivation if

$$ad_z(x) = xz - zx \quad \forall x \in A, \text{ and } z \in A$$

The set of all inner derivations is represented by $Inn(A)$.

**Definition 6** [10] **:** Let $A$ be associative algebra on a field $K$. An $n$-dimensional vector space $D$ over the same field $K$ is called a $A - A$ bimodule if $D$ is a right and a left $A$-module, such that

$$(xu)y = x(uy) \qquad \forall x, y \in A \text{ and } u \in D;$$

$$\alpha u = u\alpha \qquad \forall \alpha \in K \text{ and } u \in D.$$

To simplify terminology, we will use the expression $A$-bimodule instead of $A - A$ bimodule.

Let $\Phi$ be a multilinear mapping from an associative algebra $A^p$ to an $A$-bimodule $D$. The set of all these multilinear mappings is called cochain of $A$ in dimensional $p$ and represented by $C^p(A, D)$. It is convenient to identify $C^0(A, D)$with $D$ and $C^p(A, D)$ with $\{0\}$ for $p < 0$.

**Definition 7** [10] **:** The mapping $\delta^{(p)}$ between $C^p(A, D)$ and $C^{p+1}(A, D)$ is called coboundary homomorphism such that

$$(\delta^{(p)}\Phi)(x_1, x_2, \cdots, x_{p+1}) = x_1\Phi(x_2, \cdots, x_{p+1}) + \sum_{i=1}^{p}(-1)^i\ \Phi(x_1, \cdots, x_i x_{i+1}, \cdots, x_{p+1}) + (-1)^p + 1\Phi(x_1, \cdots, x_p)x_{p+1}$$

**Lemma 1**[10] **:** The operator $\delta^{(p)}: C^p(A, D) \longrightarrow C^{p+1}(A, D)$ is called a $K$-module homomorphism such that $\delta^{(p+1)}\delta^{(p)} = 0$

**Remark 1**[10] **:**

1-The elements of the kernel $Z^p(A, D)$ of the operator $\delta^{(p)}$ are known as cocycles in dimensional $p$ with values in $D$.

2- The elements of the image of $\delta^{(p+1)}$ represented by $B^p(A, D)$ are known as coboundaries in dimensional $p$ with values in $D$.

3- Based on Lemma 1, it is easy to see that $B^p(A, D) \subseteq Z^p(A, D)$ for $(p \geq 1)$.

4- The quotient space

$$H^p(A, D) = Z^p(A, D)/B^p(A, D)$$

is known as the cohomology group of $A$ in degree $p$.

Following, a particular case is considered that $D = A$ as $A$-bimodule and all algebras considered are over a complex field $\mathbb{C}$.

**Main Results:**

This section is devoted to computing the zero-cohomology groups $H^0(A, A)$ and first-cohomology groups $H^1(A, A)$ of four-dimensional nilpotent associative algebras.

The algebraic classification of all nilpotent associative algebras in dimensional four is constructed by [3] and is provided with the following theorem. Note that $As_n^q$ denotes $q^{th}$ isomorphism class of associative algebra in dimension $n$.

**Theorem 1.** Any complex nilpotent associative algebra structure on four dimensional is isomorphic to one of the following classes of algebras:

$As_4^1\ : e_1e_1 = e_3, \quad e_2e_2 = e_4;$

$As_4^2\ : e_1e_2 = e_3, \quad e_2e_1 = e_4;$

$As_4^3\ : e_1e_2 = e_4, \quad e_3e_1 = e_4;$

$As_4^4\ : e_1e_2 = e_3, \quad e_2e_1 = e_4, \quad e_2e_2 = -e_3;$

$As_4^5\ : e_1e_2 = e_3, \quad e_2e_1 = -e_3, \quad e_2e_2 = e_4;$

$As_4^6$ : $e_1e_2 = e_4, \quad e_2e_1 = -e_4,$
$e_3e_3 = e_4;$

$As_4^7$ $(\alpha)$: $e_1e_2 = e_4, \quad e_2e_1 = \dfrac{1+\alpha}{1-\alpha}e_4,$
$e_2e_2 = e_3;$

$As_4^8$ $(\mu)$: $e_1e_1 = e_4, \quad e_1e_2 = e_3,$
$e_2e_1 = -\mu e_4, \quad e_2e_2 = -e_3;$

$As_4^9$ : $e_1e_1 = e_3,$
$e_1e_3 = e_4, \quad e_2e_2 = -e_4,$
$e_3e_1 = e_4;$

$As_4^{10}$ : $e_1e_1 = e_4,$
$e_1e_2 = e_3, \quad e_2e_1 = -e_3,$
$e_2e_2 = -2e_3 + e_4;$

$As_4^{11}$ : $e_1e_1 = e_4,$
$e_1e_2 = e_4, \quad e_2e_1 = -e_4,$
$e_3e_3 = e_4;$

$As_4^{12}$ : $e_1e_1 = e_4,$
$e_1e_4 = -e_3, \quad e_2e_1 = e_3,$
$e_4e_1 = -e_3;$

$As_4^{13}$ : $e_1e_1 = e_4,$
$e_1e_4 = -e_3, \quad e_2e_1 = e_3,$
$e_2e_2 = e_3, \quad e_4e_1 = -e_3;$

$As_4^{14}$ $(\lambda)$ : $e_1e_1 = e_4,$
$e_1e_2 = \lambda e_4, \quad e_2e_1 = -\lambda e_4,$
$e_2e_2 = e_4, \quad e_3e_3 = e_4;$

$As_4^{15}$ : $e_1e_2 = e_4,\ e_1e_3 = e_4, \quad e_2e_1 = -e_4,\ e_2e_2 = e_4,\ e_3e_1 = e_4;$

$As_4^{16}$ : $e_1e_1 = e_2, \quad e_1e_2 = e_3,$
$e_1e_3 = e_4, \quad e_2e_1 = e_3,$
$e_2e_2 = e_4, \quad e_3e_1 = e_4;$

for all $\alpha \in \mathbb{C}\setminus\{1\}$ and $\lambda, \mu \in \mathbb{C}$.

Following, the classification results will be used which is already known from Theorem 1 to compute cohomology groups $H^p(A,A)$ for $p = 0, 1$.

**The Cohomology Groups of Degree Zero**

Based on the expression of cohomology group of A

$$H^0(A,A) = Ker(\delta^0) = \{ x \in A : \delta^0(x) = 0\}$$
$$\{ x \in A : \delta^0(x)(a) = ax - xa = 0, \forall a \in A\} = Z(A) \qquad (1)$$

Thus, $H^0(A,A)$ is the center of $A$.

Next the zero-cohomology group of complex nilpotent associative algebras in dimension four is described. Theorem 2 provides the following results:

**Theorem 2:** The cohomology group in degree zero of complex nilpotent associative algebra in dimension four has the following form:

$H^0(As_4^1, As_4^1) = span_{\mathbb{C}}\{e_1, e_2, e_3, e_4\};$
$H^0(As_4^2, As_4^2) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^3, As_4^3) = span_{\mathbb{C}}\{e_2, e_3, e_4\};$
$H^0(As_4^4, As_4^4) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^5, As_4^5) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^6, As_4^6) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^7, As_4^7) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^8, As_4^8) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^9, As_4^9) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^{10}, As_4^{10}) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^{11}, As_4^{11}) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^{12}, As_4^{12}) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^{13}, As_4^{13}) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^{14}, As_4^{14}) = span_{\mathbb{C}}\{e_3, e_4\};$

$H^0(As_4^{15}, As_4^{15}) = span_{\mathbb{C}}\{e_3, e_4\};$
$H^0(As_4^{16}, As_4^{16}) = span_{\mathbb{C}}\{e_1, e_2, e_3, e_4\}.$

**Proof.** Let $\{e_1, e_2, e_3, e_4\}$ be a basis in $A$ where $A$ is an 4-dimensional nilpotent associative algebra.

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$$

Where $x$ is a vector of $A$. Based on Theorem 1, the structure constants of four-dimensional nilpotent associative algebra are substituted in

$$\{ x \in A : \delta^0(x)(a) = ax - xa = 0, \forall a \in A\} = Z(A)$$

Then, the structure constants of $As_4^1$ are given as follows:

$$\gamma_{11}^3 = 1, \gamma_{11}^4 = 1$$

and the others are zeros. Based on (1), it leads

$e_1x = xe_1,\ e_2x = xe_2,\ e_3x = xe_3$ and $e_4x = xe_4$

Thus

$a_1e_3 = a_1e_3,\ a_2e_4 = a_2e_4, 0 = 0$ and $0 = 0$

Therefore, the span basis of zero cohomology group for $As_4^1$ is given as follows:

$$H^0(As_4^1, As_4^1) = span_{\mathbb{C}}\{e_1, e_2, e_3, e_4\}$$

The remaining parts of zero cohomology groups in dimension four can be done in a similar manner as shown above.

**The Cohomology Groups of Degree One**

$H^1(A,A) = Ker\ (\delta^1)/\ Im\ (\delta^0)$, where

$$Ker\ (\delta^1) = \{\ \Phi \in C^1(A,A) : \delta^1(\Phi)(x_1, x_2) = x_1\Phi(x_2) - \Phi(x_1,x_2) + \Phi(x_1)\, x_2 = 0,\ \ \forall\, x_1, x_2 \in A\}$$

and

$$Im\ (\delta^0) = \{\ \Phi \in C^1(A,A) : \Phi = \delta^0(x), x \in A\}$$
$$= \{\ \Phi_x \in C^1(A,A), x \in A : \Phi_{(a)} = ax - xa, \forall a \in A\}$$

It is easy to see that $Ker\ (\delta^1)$ has elements satisfying the derivation condition and $Im(\delta^0)$ has elements satisfying the inner derivation condition, respectively.

Therefore, the quotient space

$$H^1(A,A) = Ker(\delta^1)/Im(\delta^0) = Der(A)/Inn(A)$$

The 1-cocycles (derivations) of nilpotent associative algebras in dimension four are given in [9]. Consequently, the concept of 1-coboundaries (inner derivations) is looked into. The following section describes the procedure for finding 1-Coboundaries.

**Procedure for Finding 1-Coboundaries**

Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of associative algebra $A$ with dimension $n$ over a field $K$ and let $\Phi$ be an element in $B^1(A,A)$. Then

$$x = a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

where $x$ is a vector in $A$ such that $\delta^0(x) = \Phi$.

$$\Phi_x(e_i) = e_ix - xe_i \qquad \forall e_i A, i = 1,2,\dots,n. \qquad (2)$$

A linear transformation $\Phi_x$ of $A$ can be represented as a matrix form

$$\Phi_x = (a_{ij}), \qquad i,j = 1,2,\dots n.$$

$$\Phi_x(e_i) = \sum_{j=1}^{n} d_{ji}\, e_j \qquad i = 1,2,\dots,n \qquad (3)$$

Thus

$$\sum_{j=1}^{n} d_{ji}\, e_j = e_ix - xe_i \qquad (4)$$

Then

$$d_{ji} = \sum_{t=1}^{n} a_t\, \gamma_{it}^{j} - \sum_{t=1}^{n} a_t\, \gamma_{ti}^{j} \qquad (5)$$

The solutions to the system give the description of 1−coboundaries in matrix form. Following, our procedure is applied to obtain the group $B^1(A,A)$ of nilpotent associative algebras in dimension four where IC is represented isomorphism classes of algebras.

**Theorem 3:** The group of all 1−coboundaries elements for complex nilpotent associative algebras in dimension four $As_4^q$ has the following form:

| **IC** | **1−coboundaries** | **Dim$B^1(A,A)$** |
|---|---|---|
| $As_4^1$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 0 |
| $As_4^2$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^3$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 - a_3 & -a_1 & a_1 & 0 \end{pmatrix}$ | 2 |
| $As_4^4$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^5$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_2 & -2a_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^6$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_2 & -2a_2 & 0 & 0 \end{pmatrix}$ | 2 |

| | | |
|---|---|---|
| $As_4^7$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1-\frac{1+\alpha}{1-\alpha})a_2 & \frac{1+\alpha}{1-\alpha}a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^8$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ \mu a_2 & -\mu a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^9$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 0 |
| $As_4^{10}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 1 |
| $As_4^{11}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_2 & 2a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^{12}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^{13}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^{14}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\lambda a_2 & -2\lambda a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^{15}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a_2 & -2a_1 & 0 & 0 \end{pmatrix}$ | 2 |
| $As_4^{16}$ | $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ | 0 |

where = 1, ... ,16 .

**Proof**. If n=4 and q=1 then, the structure constants of $As_4^1$ are given as follows:

$$\gamma_{11}^3 = 1, \gamma_{22}^4 = 1$$

and the others are zeros. Based on condition (5), it leads

$$d_{11} = d_{12} = d_{13} = d_{14} = d_{21} = d_{22} = d_{23} = d_{24} = d_{31} = d_{32} = d_{33} = d_{34} = d_{41} = d_{42} = d_{43} = d_{44} = 0$$

Thus,

$$\Phi_x = (d_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If n=4 and q=2 then, the structure constants of $As_4^2$ are given as follows:

$\gamma_{12}^3 = 1, \gamma_{21}^4 = 1$.

Based on condition (5), it leads

$$d_{11} = d_{12} = d_{14} = d_{21} = d_{22} = d_{23} = d_{24} = d_{31} = d_{33} = d_{34} = d_{43} = d_{44} = 0$$

$$d_{31} = a_2,\ d_{32} = -a_1, d_{41} = -a_2,\ \ d_{42} = a_1$$

Thus, the 1-coboundary of $As_4^2$ is

$$\Phi_x = (d_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{pmatrix}$$

The remaining parts of 1-coboundary algebras in dimension four can be done in a similar manner as shown above.

Based on [9] and Theorem 3, the span bases and the dimensions of first cohomology groups will be found as shown in Corollary 1.

**Corollary 1:** Let $H^1(As_{\mathrm{n}}^{\mathrm{q}}, As_{\mathrm{n}}^{\mathrm{q}})$ be the cohomology group in degree one where $As_{\mathrm{n}}^{\mathrm{q}}$ denotes as $m^{th}$ isomorphism class of associative algebra in dimension $n$. If $n = 4$ and $q = 1, 2, \ldots 16$ then, the span bases and the dimensions cohomology group in degree one of complex nilpotent associative algebra has the following form:

$$H^1(As_4^1, As_4^1) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{22}}, \overline{E_{31}}, \overline{E_{32}}, \overline{E_{41}}, \overline{E_{42}}\};$$

$$H^1(As_4^2, As_4^2) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{22}}, \overline{E_{41}}, \overline{E_{42}}\};$$

$$H^1(As_4^3, As_4^3) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{22}}, \overline{E_{43}}\};$$

$$H^1(As_4^4, As_4^4) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{41}}, \overline{E_{42}}\};$$

$H^1(As_4^5, As_4^5)$
$= span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{22}}, \overline{E_{41}}, \overline{E_{42}}\};$

$H^1(As_4^6, As_4^6)$
$= span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{33}}, \overline{E_{43}}, \overline{E_{44}}\};$

$H^1(As_4^7, As_4^7)$
$= span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{12}}, \overline{E_{22}}, \overline{E_{31}}, \overline{E_{32}}\};$

$H^1(As_4^8, As_4^8) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{41}}, \overline{E_{42}}\};$

$H^1(As_4^9, As_4^9)$
$= span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{31}}, \overline{E_{41}}, \overline{E_{42}}\};$

$H^1(As_4^{10}, As_4^{10}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{32}}, \overline{E_{41}}, \overline{E_{42}}\};$

$H^1(As_4^{11}, As_4^{11}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{43}}\};$

$H^1(As_4^{12}, As_4^{12}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{41}}\};$

$H^1(As_4^{13}, As_4^{13}) = span_{\mathbb{C}}\{\overline{E_{21}}, \overline{E_{41}}\};$

$H^1(As_4^{14}, As_4^{14}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{43}}\};$

$H^1(As_4^{15}, As_4^{15}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{43}}\};$

$H^1(As_4^{16}, As_4^{16}) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{21}}, \overline{E_{31}}, \overline{E_{41}}\}.$

**Proof.** Let $\{\{\overline{E_{ij}}, i = 1, \dots, 4, j = 1, \dots, 4\}$ be a basis of the quotient space

$$H^1(A, A) = Der(A)/Inn(A)$$

If $n = 4$ and $q = 1$. Then, the derivation (1-cocycle) of $As_4^1$ was given in (6) in a matrix form as follows:

$$d = (d_{ij}) = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & 2d_{11} & 0 \\ d_{41} & d_{42} & 0 & 2d_{22} \end{pmatrix}$$

Thus,
$Der(As_4^1) =$
$span_{\mathbb{C}}\{E_{11}, E_{22}, E_{31}, E_{32}, E_{41}, E_{42}\}$. On other hand, the inner derivation (1-coboundarie) of $As_4^1$ is given in Theorem 3 in a matrix form as follows:

$$\Phi_x = (d_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, $H^1(As_4^1, As_4^1) =$
$span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{22}}, \overline{E_{31}}, \overline{E_{32}}, \overline{E_{41}}, \overline{E_{42}}\}.$

If $n = 4$ and $q = 2$.. Then, the derivation (1-cocycle) of $As_4^2$ was given as follows:

$$d = (d_{ij}) = \begin{pmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ d_{31} & d_{32} & d_{11} & 0 \\ d_{41} & d_{42} & 0 & d_{11} \end{pmatrix}$$

Hence,
$Der(As_4^2) =$
$span_{\mathbb{C}}\{E_{11}, E_{22}, E_{31}, E_{32}, E_{41}, E_{42}\}$. Based on Theorem 3 the inner derivation (1-coboundarie) of $As_4^2$ is

$$\Phi_x = (d_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_2 & -a_1 & 0 & 0 \\ -a_2 & a_1 & 0 & 0 \end{pmatrix}$$

Then, $Inn(As_4^2) = span_{\mathbb{C}}\{E_{31}, E_{32}\}$. Let $v \in Der(As_4^2)$. The vector $v$ can be written

$$v = a_1 E_{11} + a_2 E_{22} + a_3 E_{31} + a_4 E_{32} + a_5 E_{41} + a_6 E_{42}$$

$$= (a_3 E_{31} + a_4 E_{32}) + (a_1 E_{11} + a_2 E_{22} + a_5 E_{41} + a_6 E_{42})$$

Let $x \in H^1(As_4^2, As_4^2) = Der(As_4^2)/Inn(As_4^2)$ such that $x = \overline{v}$. The vector $x$ can be written

$$x = \overline{v} = (a_3 \overline{E_{31}} + a_4 \overline{E_{32}}) + (a_1 \overline{E_{11}} + a_2 \overline{E_{22}} + a_5 \overline{E_{41}} + a_6 \overline{E_{42}})$$

$$= a_1 \overline{E_{11}} + a_2 \overline{E_{22}} + a_5 \overline{E_{41}} + a_6 \overline{E_{42}}$$

Since $\overline{E_{31}}, \overline{E_{32}} \in H^1(As_4^2, As_4^2)$ vanishes. Thus,
$H^1(As_4^2, As_4^2) = span_{\mathbb{C}}\{\overline{E_{11}}, \overline{E_{22}}, \overline{E_{41}}, \overline{E_{42}}\}.$

The span bases of cohomology group in degree one of the remaining parts can be done in a similar manner as shown above.

## Conclusion:

The present work focuses on the applications of low dimensional cohomology groups $H^i(A, A), i = 0, 1$. The dimensions of zero cohomology groups and first cohomology groups for four-dimensional complex nilpotent associative algebras range between zero and four and between zero and six, respectively.

## Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

## Authors' contributions statement:

N. F. Mohammed conceived the idea of the manuscript, interpreted all the results and provided the revision and proofreading. S. G. Gasim collected an acquisition of all data.
A. S. Mohammed structured the drafting of the MS and the design for the manuscript

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# حول الزمر الكوهومولوجية للجبر التجميعي عديم القوى ذات البعد الرابع

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**الخلاصة:**

تعد دراسة الزمر الكوهومولوجية واحدة من أكثر البحوث المكثفة والمثيرة والتي قد نشأت من التبولوجيا الجبرية. حيث يعتبرالبعد للزمر الكوهومولوجية على وجه الخصوص ثابت في غاية الفائدة ويلعب دوراً هاماً في التصنيف الهندسي للجبر التجميعي. هذا العمل يركز على تطبيقات الزمر الكوهومولوجية ذات الابعاد المنخفضة. في هذا الخصوص، الزمر الكوهومولوجية المنخفضة للجبر التجميعي عديم القوى ذات البعد الرابع تم وصفهاعلى شكل مصفوفة.

**الكلمات المفتاحية**: الجبر التجميعي عديم القوى، الزمرة الكوهومولوجية، الاشتقاق، الاشتقاق الداخلي.