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On Generalized Φ −Recurrent of Kenmotsu Type Manifolds

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Abstract:

The present paper studies the generalized Φ −recurrent of Kenmotsu type manifolds. This is done to determine the components of the covariant derivative of the Riemannian curvature tensor. Moreover, the conditions which make Kenmotsu type manifolds to be locally symmetric or generalized Φ −recurrent have been established. It is also concluded that the locally symmetric of Kenmotsu type manifolds are generalized Φ −recurrent under suitable condition and vice versa. Furthermore, the study establishes the relationship between the Einstein manifolds and locally symmetric of Kenmotsu type manifolds.

Keywords: Einstein manifolds, Generalized Φ −recurrent manifolds, Kenmotsu manifolds.

Introduction:

The locally Φ −symmetric property studied by Takahashi [1] for Sasakian manifolds is a weak version of the locally symmetric property. In 1993, Jiménez and Kowalski [2] classified the locally Φ −symmetric Sasakian manifolds. Conversely, the recurrent property studied by Walker [3] is also a weak version of the locally symmetric property, one of its generalization is Φ −recurrent property that was studied by Venkatesha [4] for the generalized Sasakian space forms. There are other generalizations and extensions of recurrent curvature studied by researchers like Tamássy and Binh [5], Venkatesha et al. [6] and Siddiqi et al. [7].

Preliminaries:

The notations M^{2n+1}, $X(M)$, d, ∇ and θ were used to denote the smooth manifold M of dimension $2n+1$, the Lie algebra of smooth vector fields of M, the exterior differentiation operator, the Riemannian connection and the Riemannian connection form with components θ^i_j respectively, where $i,j=0,1,\dots,2n$.

Definition 1 [8] A smooth manifold M^{2n+1} with the quadruple (Φ,ξ,η,g) is called an almost contact metric manifold or briefly ACR −manifold, where Φ is $(1,1)$ −tensor, ξ is a characteristic vector field, g is a Riemannian metric and $\eta(\cdot)=g(\cdot,\xi)$, such that the following conditions hold:

$$\Phi(\xi)=0;\quad \eta(\xi)=1;\quad \eta\circ\Phi=0;\quad \Phi^2=-id+\eta\otimes\xi;$$
$$g(\Phi X,\Phi Y)=g(X,Y)-\eta(X)\eta(Y);\quad \forall X,Y\in X(M).$$

In the present article, the components of the Riemannian metric g of ACR −manifold M^{2n+1} can be established as follows [8]:

$$g_{00}=1;\quad g_{a0}=g_{\hat{a}0}=g_{ab}=g_{\hat{a}\hat{b}}=0;\quad g_{\hat{a}b}=\delta^a_b;\quad g_{ij}=g_{ji}, \tag{1}$$

where $a,b=1,2,\dots,n$, and $\hat{a}=a+n$. Moreover, from [3] the components of the endomorphism Φ are given by

$$\Phi^0_0=\Phi^a_0=\Phi^{\hat{a}}_0=\Phi^a_{\hat{b}}=\Phi^{\hat{a}}_b=0;\quad \Phi^a_b=\sqrt{-1}\delta^a_b;\quad \Phi^{\hat{j}}_{\hat{i}}=-\Phi^{i}_{j},$$

where $\hat{\imath}=i$ and $\hat{0}=0$. So, for all $X,Y\in X(M)$, the following relations are attained:

$X=X^i\varepsilon_i$; $g(X,Y)=g_{ij}X^iY^j$; $\Phi(X)=\Phi^i_jX^j\varepsilon_i$, over A −frame $(p;\varepsilon_0=\xi,\varepsilon_1,\dots,\varepsilon_{2n})$ of M^{2n+1}, where $X^i\in C^\infty(M)$, $p\in M$, $\varepsilon_a=\frac{1}{\sqrt{2}}(id-\sqrt{-1}\Phi)e_a$, $\varepsilon_{\hat{a}}=\frac{1}{\sqrt{2}}(id+\sqrt{-1}\Phi)e_a$, and $\{e_1,\dots,e_n\}$ is a complex basis of the distribution $\ker(\eta)$. A set of all such A −frames given above is called an associated G −structure space (AG −structure space). For more detail, citations [8] and [9] may be referred to.

Definition 2 [10] An ACR −manifold such that the following identity:

$$\nabla_X(\Phi)Y - \nabla_{\Phi X}(\Phi)\Phi Y = -\eta(Y)\Phi X, \quad \forall\ X,Y \in X(M)$$

hold is called a Kenmotsu type manifold.

Suppose that $\{\omega, \omega^1, \ldots, \omega^{2n}\}$ are coframe over AG −structure space of Kenmotsu type manifold such that $\omega^{\hat{a}} = \omega_a = \overline{\omega^a}$, where $\overline{\alpha}$ is the complex conjugate of α and $\overline{\omega} = \omega$. Then from [10], the following theorem is obvious:

Theorem 1 Suppose that M^{2n+1} is the Kenmotsu type manifold, then the Cartan's first structure equations are given by

1. $d\omega = 0$;
2. $d\omega^a = -\theta^a_b \wedge \omega^b + B^{ab}{}_c\ \omega^c \wedge \omega_b - \omega^a \wedge \omega$;
3. $d\omega_a = \theta^b_a \wedge \omega_b + B_{ab}{}^c\ \omega_c \wedge \omega^b - \omega_a \wedge \omega$.

From [10], the components of the Riemannian connection form θ of the Kenmotsu type manifold are given by

$$\theta^0_0 = 0; \quad \theta^a_0 = \omega^a; \quad \theta^a_b; \quad \theta^a_{\hat{b}} = -B^{ab}{}_c\ \omega^c; \qquad (2)$$

and the others components can be established from the relations $\theta^i_j + \theta^{\hat{j}}_{\hat{\imath}} = 0$ and $\theta^i_j = \overline{\theta^{\hat{\imath}}_{\hat{j}}}$, where $c = 1, \ldots, n$, while $B^{ab}{}_c$ are the components of the first Kirichenko tensor that mentioned in [11] with the properties $B^{ab}{}_c = -B^{ba}{}_c$ and $\overline{B^{ab}{}_c} = B_{ab}{}^c$.

Theorem 2 [10] On the AG −structure space, the Kenmotsu type manifold M^{2n+1} has the following Cartan's second structure equations:

1. $d\theta^a_b = -\theta^a_c \wedge \theta^c_b + A^{ad}_{bc}\ \omega^c \wedge \omega_d + A^a_{bcd}\ \omega^c \wedge \omega^d + A^{acd}_b\ \omega_c \wedge \omega_d$;
2. $dB^{ab}{}_c = B^{ab}{}_d\ \theta^d_c - B^{db}{}_c\ \theta^a_d - B^{ad}{}_c\ \theta^b_d + B^{ab}{}_{cd}\ \omega^d + B^{abd}{}_c\ \omega_d - B^{ab}{}_c\ \omega$;
3. $dB_{ab}{}^c = -B_{ab}{}^d\ \theta^c_d + B_{db}{}^c\ \theta^d_a + B_{ad}{}^c\ \theta^d_b + B_{ab}{}^{cd}\ \omega_d + B_{abd}{}^c\ \omega^d - B_{ab}{}^c\ \omega$,

such that

$$A^{ad}_{[bc]} - B^{ad}{}_{[cb]} - B^{ah}{}_{[b}\ B_{|h|c]}{}^d = 0;$$

$$A^{acd}_b - B^{a[cd]}{}_b + B^{a[c}{}_h B^{|h|d]}{}_b = 0; \quad A^a_{[bcd]} = 0;$$

$$A^{[bc]}_{ad} + B_{ad}{}^{[cb]} + B_{ah}{}^{[b}\ B^{|h|c]}{}_d = 0;$$

$$A^b_{acd} + B_{a[cd]}{}^b - B_{a[c}{}^h B_{|h|d]}{}^b = 0; \quad A^{[bcd]}_a = 0;$$

where all indexes have range from 1 to n, and $[\cdot\,|\cdot|\,\cdot]$ denotes the anti-symmetric operator of the involving indexes except $|\cdot|$.

Denote R and r the Riemann curvature tensor with components R^i_{jkl} and Ricci tensor with components r_{ij} of the ACR −manifold respectively, where $k, l = 0,1, \ldots, 2n$.

Theorem 3 [10] The components of R for the Kenmotsu type manifold over the AG − structure space are given by

1. $R^a_{0c0} = -\delta^a_c$; $\quad R^a_{\hat{b}cd} = 2(B^{ab}{}_{[cd]} - \delta^a_{[c}\ \delta^b_{d]})$; $\quad R^a_{\hat{b}c\hat{d}} = B^{abd}{}_c - B^{ab}{}_h\ B^{hd}{}_c$;
2. $R^a_{bcd} = 2A^a_{bcd}$; $\quad R^a_{bc\hat{d}} = A^{ad}_{bc} - B^{ah}{}_c\ B_{bh}{}^d - \delta^a_c\ \delta^d_b$,

where $R(X,Y)Z = R^i_{jkl}X^kY^lZ^j\varepsilon_i$, and the remaining components of R are given by the first Bianchi identity or the conjugate (i.e. $R^i_{jkl} = \overline{R^{\hat{\imath}}_{\hat{j}\hat{k}\hat{l}}}$) to the above components or identical to zero.

Theorem 4 [10] The components of r for the Kenmotsu type manifold over the AG − structure space are given as follows:

1. $r_{00} = -2n$; $\quad r_{ab} = -2A^c_{abc} + B_{cab}{}^c - B_{ca}{}^h\ B_{hb}{}^c$;
2. $r_{a0} = 0$; $\quad r_{\hat{a}b} = -2(n\delta^a_b + B^{ca}{}_{[bc]}) + A^{ac}_{cb} - B^{ah}{}_b\ B_{ch}{}^c$,

where $r(X,Y) = r_{ij}X^iY^j$, $r_{ij} = r_{ji}$ and the remaining components of r are given by the property $r_{\hat{\imath}\hat{j}} = \overline{r_{ij}}$.

Definition 3 [10] An ACR −manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ with Ricci tensor r, is called

1. Einstein manifold, if $r_{ij} = \lambda g_{ij}$, where λ is an Einstein constant.
2. has Φ −invariant Ricci tensor, if $r_{a0} = r_{ab} = 0$.

Definition 4 [3, 4, 6] An ACR −manifold $(M^{2n+1}, \Phi, \xi, \eta, g)$ with R as Riemann curvature tensor, is called

1. Locally symmetric, if $\nabla_X(R)(Y,Z)W = 0$, where $X,Y,Z,W \in X(M)$.
2. Generalized Φ −Recurrent, if there are nonzero 1 −forms α and β such that

$$\Phi^2(\nabla_X(R)(Y,Z)W) = \alpha(X)R(Y,Z)W + \beta(X)\{g(Z,W)Y - g(Y,W)Z\}.$$

Locally symmetric Kenmotsu type manifolds and its weakened version:

In this section, the Cartan's second structure equations in Theorem 2 are differentiated exteriorly at the beginning. Then on AG −structure space of the Kenmotsu type manifold M^{2n+1}, suitable smooth functions exist such that:

$$\Delta A^a_{bcd} = A^a_{bcdh}\ \omega^h + A^{ah}_{bcd}\ \omega_h - 2A^a_{bcd}\ \omega; \qquad (3)$$

$$\Delta A^{ad}_{bc} = \tilde{A}^{ad}_{bch}\ \omega^h + \tilde{A}^{adh}_{bc}\ \omega_h - 2A^{ad}_{bc}\ \omega; \qquad (4)$$

$$\Delta B^{ab}{}_{cd} = B^{ab}{}_{cdh}\ \omega^h + B^{abh}{}_{cd}\ \omega_h - 2B^{ab}{}_{cd}\ \omega; \qquad (5)$$

where $h = 1, \ldots, n$, and

$$\Delta A^a_{bcd} = dA^a_{bcd} + A^h_{bcd}\ \theta^a_h - A^a_{hcd}\ \theta^h_b - A^a_{bhd}\ \theta^h_c - A^a_{bch}\ \theta^h_d;$$

$$\Delta A^{ad}_{bc} = dA^{ad}_{bc} + A^{hd}_{bc}\ \theta^a_h + A^{ah}_{bc}\ \theta^d_h - A^{ad}_{hc}\ \theta^h_b - A^{ad}_{bh}\ \theta^h_c;$$

$$\Delta B^{ab}{}_{cd} = dB^{ab}{}_{cd} + B^{hb}{}_{cdh}\ \theta^a_h + B^{ah}{}_{cd}\ \theta^b_h - B^{ab}{}_{hd}\ \theta^h_c - B^{ad}{}_{ch}\ \theta^h_d.$$

Now, the components of ∇R on ACR −manifold M^{2n+1} can be established with respect to the metric g from the following identity (12):

$$dR_{ijkl} - R_{tjkl}\ \theta_i^t - R_{itkl}\ \theta_j^t - R_{ijtl}\ \theta_k^t - R_{ijkt}\ \theta_l^t = R_{ijkl,t}\ \omega^t; \quad (6)$$

where $R(X,Y,Z,W) = g(R(Z,W)Y,X)$, $R_{ijkl} = R^{\hat{\imath}}_{jkl}$, $t = 0,1,\dots,2n$ and

$$R_{ijkl,t} = g(\nabla_{\varepsilon_t}(R)(\varepsilon_k,\varepsilon_l)\varepsilon_j,\varepsilon_i).$$

Theorem 5 The components of ∇R on AG −structure space of the Kenmotsu type manifold M^{2n+1} are given by

1. $R_{a0b0,0} = R_{a0b0,h} = R_{a0b0,\hat{h}} = 0;$
2. $R_{\hat{a}0b0,0} = R_{\hat{a}0b0,h} = R_{\hat{a}0b0,\hat{h}} = 0;$
3. $R_{a0bc,0} = R_{a0bc,h} = 0;\quad R_{a0bc,\hat{h}} = 2A^h_{abc};$
4. $R_{\hat{a}0bc,0} = 0;\quad R_{\hat{a}0bc,h} = -2A^a_{hbc};\quad R_{\hat{a}0bc,\hat{h}} = -2B^{ah}{}_{[bc]};$
5. $R_{a0\hat{b}c,0} = 0;\quad R_{a0\hat{b}c,h} = -2A^b_{cah};\quad R_{a0\hat{b}c,\hat{h}} = -A^{hb}_{ac} + B^{hd}{}_c\ B_{ad}{}^b;$
6. $R_{abcd,0} = R_{abcd,h} = 0;\quad R_{abcd,\hat{h}} = 4\{B_{f[a}{}^h\ A^f_{b]cd} + B_{f[c}{}^h\ A^f_{d]ab}\};$
7. $R_{\hat{a}bcd,0} = -4A^a_{bcd};\quad R_{\hat{a}bcd,h} = 2A^a_{bcdh};$
8. $R_{\hat{a}bcd,\hat{h}} = 2\{A^{ah}_{bcd} + B^{af}{}_{[cd]}\ B_{fb}{}^h + A^{af}_{b[c}\ B_{|f|d]}{}^h + B_{f[c}{}^h\ B^{a\tilde{f}}{}_{d]}\ B_{b\tilde{f}}{}^f\};$
9. $R_{\hat{a}bc\hat{a},0} = -2\{A^{ad}_{bc} - B^{af}{}_c\ B_{bf}{}^d\};$
10. $R_{\hat{a}bc\hat{a},h} = 2A^a_{bcf}\ B^{fd}{}_h - 2A^d_{cfb}\ B^{fa}{}_h - B^{af}{}_c\ B_{bfh}{}^d - B_{bf}{}^d\ B^{af}{}_{ch} + \tilde{A}^{ad}_{bch};$
11. $R_{\hat{a}bc\hat{a},\hat{h}} = \tilde{A}^{adh}_{bc} - B_{bf}{}^d\ B^{afh}{}_c - B^{af}{}_c\ B_{bf}{}^{dh} + 2A^{daf}_c\ B_{fb}{}^h - 2A^{afd}_b\ B_{fc}{}^h;$
12. $R_{\hat{a}\hat{b}cd,0} = -4B^{ab}{}_{[cd]};\quad R_{\hat{a}\hat{b}cd,h} = 2B^{ab}{}_{[cd]h} + 4B^{f[b}{}_h\ A^{a]}_{fcd};$
13. $R_{\hat{a}\hat{b}cd,\hat{h}} = 2B^{abh}{}_{[cd]} + 4B_{f[d}{}^h\ A^{fab}_{c]};$

Proof. The results follow from the equation (6) by taking

$$(i,j,k,l) = (a,0,b,0),(\hat{a},0,b,0),(a,0,b,c),$$

$$(\hat{a},0,b,c),(a,0,\hat{b},c),(a,b,c,d),$$

$$(\hat{a},b,c,d),(\hat{a},b,c,\hat{d}),(\hat{a},\hat{b},c,d);$$

and using the Theorem 3 and the equation (2). For instance, if $(i,j,k,l) = (a,0,b,0)$, then the equation (6) given by

$$dR_{a0b0} - R_{t0b0}\ \theta_a^t - R_{atb0}\ \theta_0^t - R_{a0t0}\ \theta_b^t - R_{a0bt}\ \theta_0^t = R_{a0b0,t}\ \omega^t.$$

The above equation can be simplified by using the Theorem 3 and the equation (2) as the following:

$$R_{a0b0,t}\ \omega^t = -R_{\hat{h}0b0}\ \theta_a^{\hat{h}} - R_{a0\hat{h}0}\ \theta_b^{\hat{h}}; = \delta_b^h\ \theta_a^{\hat{h}} + \delta_a^h\ \theta_b^{\hat{h}}; = \theta_a^{\hat{b}} + \theta_b^{\hat{a}} = 0.$$

So, $R_{a0b0,h}\ \omega^h + R_{a0b0,\hat{h}}\ \omega_h + R_{a0b0,0}\ \omega = 0$ and then

$$R_{a0b0,h} = R_{a0b0,\hat{h}} = R_{a0b0,0} = 0.$$

The same technique was used for the other cases, while for some cases, the equations (3), (4), or (5) must be used. For example, if $(i,j,k,l) = (\hat{a},b,c,d)$, then the equation (6) given by

$$dR_{\hat{a}bcd} - R_{tbcd}\ \theta_{\hat{a}}^t - R_{\hat{a}tcd}\ \theta_b^t - R_{\hat{a}btd}\ \theta_c^t - R_{\hat{a}bct}\ \theta_d^t = R_{\hat{a}bcd,t}\ \omega^t.$$

From the Theorem 3, it follows that

$$R_{\hat{a}bcd,t}\ \omega^t = 2dA^a_{bcd} - R_{\hat{h}bcd}\ \theta_{\hat{a}}^{\hat{h}} - R_{\hat{a}hcd}\ \theta_b^h - R_{\hat{a}\hat{h}cd}\ \theta_b^{\hat{h}} - R_{\hat{a}bhd}\ \theta_c^h - R_{\hat{a}b\hat{h}d}\ \theta_c^{\hat{h}} - R_{\hat{a}bch}\ \theta_d^h - R_{\hat{a}bc\hat{h}}\ \theta_d^{\hat{h}};$$

$$= 2\Delta A^a_{bcd} - R_{\hat{a}\hat{h}cd}\ \theta_b^{\hat{h}} - R_{\hat{a}b\hat{h}d}\ \theta_c^{\hat{h}} - R_{\hat{a}bc\hat{h}}\ \theta_d^{\hat{h}};$$

$$= 2\Delta A^a_{bcd} - R_{\hat{a}\hat{f}cd}\ \theta_b^{\hat{f}} + R_{\hat{a}bd\hat{f}}\ \theta_c^{\hat{f}} - R_{\hat{a}bc\hat{f}}\ \theta_d^{\hat{f}};$$

where $f = 1,2,\dots,n$. According to the equation (2), $\theta_b^{\hat{f}} = -B_{fb}{}^h\ \omega_h$. So regarding Theorem 3, the equation (3) and the previous results, the following is obtained:

$$R_{\hat{a}bcd,0} = -4A^a_{bcd};$$
$$R_{\hat{a}bcd,h} = 2A^a_{bcdh};$$
$$R_{\hat{a}bcd,\hat{h}} = 2\{A^{ah}_{bcd} + B^{af}{}_{[cd]}\ B_{fb}{}^h + A^{af}_{b[c}\ B_{|f|d]}{}^h + B_{f[c}{}^h\ B^{a\tilde{f}}{}_{d]}\ B_{b\tilde{f}}{}^f\}.$$

Now, since the proof of the remaining results becomes obvious, it is deleted.

Theorem 6 The Kenmotsu type manifold M^{2n+1} is locally symmetric if and only if, the following conditions hold:

$$A^a_{bcd} = 0;\quad B^{ab}{}_{[cd]} = 0;\quad A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d.$$

Proof. Suppose that M^{2n+1} is locally symmetric, then $\nabla_U(R)(Z,W)Y = 0$, thus giving:

$$g(\nabla_U(R)(Z,W)Y,X) = 0;\quad \forall\ X,Y,Z,W,U \in X(M).$$

Therefore, the components $R_{ijkl,t}$ are identically zero for all $i,j,k,l,t = 0,1,\dots,2n$. Regarding Theorem 5 gives $A^a_{bcd} = 0$; $B^{ab}{}_{[cd]} = 0$; and $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d$. Conversely, if $A^a_{bcd} = 0$; $B^{ab}{}_{[cd]} = 0$; $A^{ad}_{bc} = B^{ah}{}_c\ B_{bh}{}^d$, then $\Delta A^a_{bcd} = 0$; $\Delta B^{ab}{}_{[cd]} = 0$; and

$$\Delta A^{ad}_{bc} = \{B^{af}{}_c\ B_{bfh}{}^d + B_{bf}{}^d\ B^{af}{}_{ch}\}\omega^h + \{B_{bf}{}^d\ B^{afh}{}_c + B^{af}{}_c\ B_{bf}{}^{dh}\}\omega_h - 2A^{ad}_{bc}\ \omega.$$

Therefore, regarding the equations (3), (4), and (5) and Theorem 5, $R_{ijkl,t} = 0$ is obtained. Then M^{2n+1} is locally symmetric.

Theorem 7 The locally symmetric Kenmotsu type manifold M^{2n+1} is an Einstein manifold with

$\lambda = -2n$ if and only if the following condition holds:

$$B_{cab}^{\ \ c} = B_{ca}^{\ \ h}\ B_{hb}^{\ \ c}.$$

Proof. Suppose that M^{2n+1} is an Einstein manifold with $\lambda = -2n$, then, from Definition 3 and equation (1), the following is obtained:

$$r_{00} = -2n; \quad r_{a0} = r_{ab} = 0; \quad r_{\hat{a}b} = -2n\delta_b^a.$$

Since M^{2n+1} is locally symmetric Kenmotsu type manifold, Theorem 4 and 6 and the above relations lead to the following condition. Conversely, if the condition is valid, then the conditions of the Theorem 6 with the Theorem 4 lead to the result.

Corollary 1 The locally symmetric Kenmotsu type manifold M^{2n+1} is an Einstein manifold with $\lambda = -2\mathrm{n}$ if and only if M^{2n+1} has Φ −invariant Ricci tensor.

Proof. The assertion of this corollary follows from the Definition 3 and Theorem 7.

Now, suppose that $(M^{2n+1}, \Phi, \xi, \eta, g)$ is generalized Φ −recurrent ACR −manifold, then $\forall\ U, W, Y, Z \in X(M)$ gives the following:

$$\Phi^2(\nabla_U(R)(Z,W)Y) = \alpha(U)R(Z,W)Y + \beta(U)\{g(Y,W)Z - g(Y,Z)W\}.$$

Since for all $X \in X(M)$, the following holds true:

$$g(\Phi^2(\nabla_U(R)(Z,W)Y),X) = g(\nabla_U(R)(Z,W)Y, \Phi^2(X)); = -g(\nabla_U(R)(Z,W)Y,X) + \eta(X)g(\nabla_U(R)(Z,W)Y,\xi),$$

then the components of the generalized Φ −recurrent ACR −manifold are:

$$-R_{ijkl,t} + \eta_i\ R_{0jkl,t} = \alpha_t\ R_{ijkl} + \beta_t\{g_{ik}\ g_{jl} - g_{il}\ g_{jk}\}. \qquad (7)$$

So, if M^{2n+1} is the manifold of Kenmotsu type, then regarding Theorem 3 and equation (1), equation (7) becomes as follows:

1. $R_{a0b0,t} = 0;$
2. $R_{\hat{a}0b0,t} = \alpha_t\ \delta_b^a - \beta_t\ \delta_b^a;$
3. $R_{a0bc,t} = 0;$
4. $R_{\hat{a}0bc,t} = 0;$
5. $R_{a0\hat{b}c,t} = 0;$
6. $R_{abcd,t} = 0;$
7. $R_{\hat{a}bcd,t} = -2\alpha_t\ A_{bcd}^a;$
8. $R_{\hat{a}bc\hat{d},t} = \alpha_t(-A_{bc}^{ad} + B^{ah}_{\ \ c}\ B_{bh}^{\ \ d} + \delta_c^a\ \delta_b^d) - \beta_t\ \delta_c^a\ \delta_b^d;$
9. $R_{\hat{a}\hat{b}cd,t} = 2\alpha_t(-B^{ab}_{\ \ [cd]} + \delta_{[c}^a\ \delta_{d]}^b) - 2\beta_t\ \delta_{[c}^a\ \delta_{d]}^b.$

Regarding Theorem 5, from item 2 in the above discussion, $\alpha_t = \beta_t$ is obtained, implying that the 1-forms α and β must be equal. Moreover, combining the above items again with Theorem 5 leads to deducing the following theorem:

Theorem 8 The Kenmotsu type manifold M^{2n+1} is a manifold of generalized Φ −recurrent curvature if and only if the following conditions hold:

$$\alpha = \beta; \quad A_{bcd}^a = 0; \quad B^{ab}_{\ \ [cd]} = 0; \quad A_{bc}^{ad} = B^{ah}_{\ \ c}\ B_{bh}^{\ \ d}.$$

Corollary 2 The Kenmotsu type manifold M^{2n+1} is locally symmetric if and only if M^{2n+1} is a generalized Φ −recurrent with $\alpha = \beta$.

Proof. The result follows from Theorem 6 and Theorem 8.

Theorem 9 The Kenmotsu type manifold M^{2n+1} satisfies the following relations:

1. $g(\nabla_\xi(R)(Z,W)Y,X) = -2g(R(Z,W)Y + g(Y,W)Z - g(Y,Z)W, X);$
2. $g(\nabla_U(R)(Z,W)\xi,X) = -g(R(Z,W)U + g(U,W)Z - g(U,Z)W, X);$
3. $g(\nabla_U(R)(Z,\xi)Y,X) = -g(R(Z,U)Y + g(Y,U)Z - g(Y,Z)U, X).$

Proof. Since the components of $g(\nabla_\xi(R)(Z,W)Y,X)$, $g(\nabla_U(R)(Z,W)\xi,X)$ and $g(\nabla_U(R)\ (Z,\xi)Y,X)$ are $R_{ijkl,0}$, $R_{i0kl,t}$ and $R_{ijk0,t}$, respectively, the claim of the present theorem is achieved from Theorem 5, Theorem 3 and equation (1).

Conclusion:

The present paper concludes that the locally symmetric, $\Phi -$ recurrent and the generalized $\Phi -$ recurrent properties can be studied for any $ACR -$ manifold in a simple way, represented by $AG -$ structure space. Also, the axis of the present study required components of the Riemannian curvature tensor which computed on the $AG -$ structure space and the remaining results were established from them.

Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Basrah.

Authors' contributions statement:

MY Abass concluded Theorem 5, Theorem 6, and Theorem 9.

HM Abood deduced Theorem 7, Theorem 8, and their corollaries.

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حول تعميم متكرر – Φ لمنطويات من نوع كينموتسو

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الخلاصة:

البحث الحالي يدرس تعميم متكرر – Φ لمنطويات من النوع كينموتسو. هذا البحث عمل على تحديد مركبات المشتقة الاتجاهية لتنسر الانحناء الريماني. بالإضافة الى ذلك، استنتجت الشروط التي تجعل المنطويات من نوع كينموتسو متناظرة محلياً او تمتلك تعميم متكرر – Φ البحث ايضاً استنتج بان المنطويات من النوع كينموتسو المتناظرة محلياً تكون تعميم متكرر – Φ تحت شرط مناسب والعكس بالعكس. أضف الى ذلك، استنتاج الدراسة للعلاقة بين منطويات اينشتاين ومنطويات متناظرة محلياً من نوع كينموتسو.

الكلمات المفتاحية: منطويات اينشتاين، تعميم منطويات متكرر – Φ، منطويات كينموتسو.