Influence of MHD on Some Oscillating Motions of a Fractional Burgers' Fluid

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Abstract:

This paper presents a study for the influence of magnetohydrodynamic (MHD) on the oscillating flows of fractional Burgers' fluid. The fractional calculus approach in the constitutive relationship model is introduced and a fractional Burgers' model is built. The exact solution of the oscillating motions of a fractional Burgers' fluid due to cosine and sine oscillations of an infinite flat plate are established with the help of integral transforms (Fourier sine and Laplace transforms). The expressions for the velocity field and the resulting shear stress that have been obtained, presented under integral and series form in terms of the generalized Mittag-Leffler function, satisfy all imposed initial and boundary conditions. Finally, the obtained solutions are graphically analyzed for variations of interesting flow parameters. While the MATHEMATICA package is used to draw the figures velocity components in the plane.

Key words: Oscillating Motion, Burgers' Fluid, Fractional Model in Burgers' Fluid

Introduction:

The interest for flow of non-Newtonian fluids has considerably grown in recent years because of the advance in technological applications. However, it is difficult to suggest a single model which exhibits all properties of non-Newtonian fluids as it done for Newtonian fluids. For this reason, a number of constitutive equations have been proposed. Among them the models of differential type and those of rate type have received much attention[1], [2]. A thermodynamic framework has been put into place to develop a rate type model known as Burgers' model that is used to describe the motion of earth's mantle. This model is also used to characterize diverse viscoelastic materials, such as asphalt in geomechanics and cheese in food products.

Fractional derivatives have been found to be quite flexible in describing viscoelastic behavior[3]. In general, the constitutive equations for generalized non-Newtonian fluids are modified from the well known fluid model by replacing the time derivative of an integer order with the so-called Riemann-Lioville fractional calculus operator[4]. A very good fit of experimental data is achieved when the constitutive equation with fractional derivative is used [5]. Recently, the Burgers' fluid models which form a subclass of the viscoelastic type have given attention. Khan [6] Studied the accelerated flows for a viscoelastic fluid governed by the fractional Burgers' model. The velocity field of the flow is described by a fractional partial differential equation. Hyder [7] discussed some unidirectional flows of a viscoelastic fluid between two parallel plates with fractional Burgers' fluid models. Khan [8] investigated some fractional Burgers' fluid models

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including oscillating flow. Liu [9] researched for the MHD flow and heat transfer of an incompressible generalized Burgers' fluid due to an exponential accelerating plate with the effect of radiation. The rest of the paper is organized as follows. In section 2 the governing equations are introduced. In sections 3-5, we first defined the problems and then present their exact solution. Section 6 is devoted to the results and discussion. The paper ends with drawing the figures of velocity component in the plane.

Governing Equation

The equation governing the transient flow of an incompressible fluid include the continuity equation and the momentum equation, in the absence of body forces, are

$$\nabla \cdot \mathbf{V} = 0, \qquad (1)$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{1}{\rho}(\nabla \cdot \mathbf{T}), \qquad (2)$$

Where ρ and $\mathbf{V}$ are respectively the fluid density and velocity vector and ∇ represents the gradient operator. Cauchy stress tensor $\mathbf{T}$ for an incompressible Burgers' fluid is related to the fluid motion by the following constitutive equation [10]

$$\mathbf{T} = -P\mathbf{I} + \mathbf{S},$$

$$\mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu(\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t}) \qquad (3)$$

Where -$P\mathbf{I}$ denotes the indeterminate spherical stress, S the extra stress tensor, $\boldsymbol{A} = (\nabla \boldsymbol{V}) + (\nabla \boldsymbol{V})^{\boldsymbol{T}}$ the first Rivlin-Ericksen tensor, μ the dynamic viscosity of the fluid, λ_1 and $\lambda_3 (< \lambda_1)$ the relaxation and retardation times, respectively, λ_2 a new material constant of the Burgers' fluid, and $\frac{\delta}{\delta t}$ the upper convicted fractional derivative defined by [11]

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} + (\mathbf{V} \cdot \nabla)\mathbf{S} - (\nabla \mathbf{V})\mathbf{S} - \mathbf{S}(\nabla \mathbf{V})^{\mathbf{T}},$$

$$\frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}(\frac{\delta \mathbf{S}}{\delta t}) \qquad (4)$$

In which $\frac{d}{dt}$ is the usual material derivative.

We shall consider unsteady flows wherein the velocity and stress field are of the form

$$\mathbf{V} = \mathbf{V}(y,t) = u(y,t)i \quad ,$$

$$\mathbf{S} = \mathbf{S}(y,t) \qquad (5)$$

Where i is the unit vector in the x-coordinate direction of the Cartesian coordinate system. For such flows the constraint of incompressibility is automatically satisfied.

Substituting Eqs. (5) into Eqs. (2) and (3) and taking account the initial conditions $\mathbf{S}(y,0) = \frac{\partial \mathbf{S}(y,0)}{\partial t} = \mathbf{0},$ we obtain the relevant equations

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y}, \qquad (6)$$

$$(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2})S_{xy} = \mu(1 + \lambda_3 \frac{\partial}{\partial t})\frac{\partial u}{\partial y} \qquad (7)$$

$$(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2})S_{xx} -$$
$$2S_{xy}[\lambda_1 + \lambda_2 \frac{\partial}{\partial t}]\frac{\partial u}{\partial y} - 2\lambda_2 \frac{\partial u}{\partial y}\frac{\partial S_{xy}}{\partial t} = -2\mu\lambda_3 \left(\frac{\partial u}{\partial y}\right)^2 \qquad (8)$$

Where S_{xx} is a normal stress and S_{xy} is the tangential stress which is deferent from zero.

The relevant equation corresponding to the same motions of a fractional Burgers' fluid are obtained from Eqs. (7) and (8) by substituting the time derivatives with fractional derivatives, defined by [3]

$$D_t^p[f(t)] = \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}d\tau \ ,$$
$$0 \le p \le 1 \qquad (9)$$

where $\Gamma(.)$ denotes the Gamma function. Consequently, the relevant equations of a fractional Burgers' fluid are

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})S_{xy}(y,t)$$
$$=\mu(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial u(y,t)}{\partial y} \qquad (10)$$

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})S_{xx}-$$
$$2S_{xy}[\lambda_1^{\alpha}+\lambda_2^{\alpha}D_t^{\alpha}]\frac{\partial u}{\partial y}-2\lambda_2^{\alpha}\frac{\partial u}{\partial y}D_t^{\alpha}S_{xy}$$
$$=-2\mu\lambda_3^{\beta}\left(\frac{\partial u}{\partial y}\right)^2 \qquad (11)$$

Where α and β are the fractional parameters such that $0\le\alpha\le\beta\le1$.

Consider that the conducting fluid is permeated by an imposed magnetic field B_0 which acts in the positive y-direction. In the low-magnetic Reynolds number approximation, the magnetic body force is representation by $\sigma B_0^2 u$. Then, the equation of motion (6) yields the following scalar equations:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial P}{\partial x}+\frac{\partial S_{xy}}{\partial y}-\sigma B_0^2 u \qquad (12)$$

Eliminating S_{xy} from Eqs. (10) and (12) , lead the following governing equation

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial u}{\partial t}=-\frac{1}{\rho}(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})$$
$$\frac{\partial P}{\partial x}+\upsilon(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial^2 u}{\partial y^2}-M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})u \qquad (13)$$

Where $\upsilon=\dfrac{\mu}{\rho}$ is the kinematics' viscosity of the fluid and $M=\dfrac{\sigma B_{\circ}^2}{\rho}$.

In the next sections, Eq.(13) will be solved analytically for some simple oscillating flows.

Flow Induced by a Rigid Oscillating Plate

Let us consider an incompressible fractional Burgers' fluid occupying the space above the flat plate situated in (x, z) plane. Initially the fluid as well as the plate are at rest and for time $t>0$ the plate starts oscillated in its own plane according to

$$\mathbf{V}(0,t)=V\cos(\omega t)\mathbf{i} \qquad or$$
$$\mathbf{V}(0,t)=V\sin(\omega t)\mathbf{i}; \qquad t>0 \qquad (14)$$

Where ω is the frequency and V the amplitude of the velocity of the plate. Due to the shear, the fluid is moved gradually and has the velocity of the form Eq.(5). The governing equation, in the absence of a pressure gradient in the flow direction is

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial u}{\partial t}$$
$$=\upsilon(1+\lambda_3^{\beta}D_t^{\beta})\frac{\partial^2 u}{\partial y^2}-$$
$$M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})u \qquad (15)$$

The associated boundary and initial condition are

$$u(0,t)=V\cos(\omega t) \qquad or$$
$$u(0,t)=V\sin(\omega t); \qquad t>0 \qquad (16)$$

$$u(y,t),\frac{\partial u(y,t)}{\partial y}\to 0 \;\; as \;\; y\to\infty \qquad (17)$$

and

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=\frac{\partial^2 u(y,0)}{\partial t^2}=0\;,\; y>0 \qquad (18)$$

In order to solve this problem, we shall use the Fourier sine and Laplace transforms. Consequently, multiplying both sides of Eq.(15) by $\sqrt{\frac{2}{\pi}}\sin(\xi y)$, integrate the result with respect to y from 0 to ∞ and taking into account corresponding initial and boundary conditions (16)-(18), we obtain [12]

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \frac{\partial U_S(\xi,t)}{\partial t} =$$
$$\upsilon(1+\lambda_3^{\beta} D_t^{\beta})\,(\xi V\sqrt{\frac{2}{\pi}}\cos(\omega t) - \xi^2 U_s(\xi,t))$$
$$- M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\,U_s(\xi,t) \qquad (19)$$

$$(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha}) \frac{\partial U_S(\xi,t)}{\partial t} =$$
$$\upsilon(1+\lambda_3^{\beta} D_t^{\beta})\,(\xi V\sqrt{\frac{2}{\pi}}\sin(\omega t) - \xi^2 U_s(\xi,t))$$
$$- M(1+\lambda_1^{\alpha} D_t^{\alpha} + \lambda_2^{\alpha} D_t^{2\alpha})\,U_s(\xi,t) \qquad (20)$$

respectively, where the Fourier sine transform $U_s(\xi,t)\, of\; u(y,t)$ has to satisfy the initial conditions

$$U_s(\xi,0) = \frac{\partial U_s(\xi,0)}{\partial t} = \frac{\partial^2 U_s(\xi,0)}{\partial t^2} = 0\,; \quad \xi > 0. \qquad (21)$$

To obtain exact analytic solution of Eqs.(19) and (20) corresponding to initial conditions (21), we apply the Laplace transform principle of fractional derivative [3], and obtain

$$\overline{U}_s(\xi,s) = \sqrt{\frac{2}{\pi}}\,V\left[\frac{\upsilon\xi(1+\lambda_3^{\beta} s^{\beta})}{(s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + \upsilon\xi^2 + \upsilon\xi^2\lambda_3^{\beta} s^{\beta} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha})}\right]\frac{s}{s^2+\omega^2} \qquad (22)$$

$$\overline{U}_s(\xi,s) = \sqrt{\frac{2}{\pi}}\,V\left[\frac{\upsilon\xi(1+\lambda_3^{\beta} s^{\beta})}{(s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + \upsilon\xi^2 + \upsilon\xi^2\lambda_3^{\beta} s^{\beta} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha})}\right]\frac{\omega}{s^2+\omega^2} \qquad (23)$$

respectively, where $\overline{U}_s(\xi,s)$ is the Laplace transform of $U_s(\xi,t)$ with respect to *t*.

In order to get $U_s(\xi,t)$ and to avoid lengthy calculations of residues and contour integrals, we apply discrete the inverse Laplace transform method [3]. However, for a more suitable presentation of final results, we rewrite Eqs. (22) and (23) in the equivalent forms

$$\overline{U}_s(\xi,s) = \sqrt{\frac{2}{\pi}}\,\frac{V}{\xi}\left[1-\frac{s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha}}{(s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + \upsilon\xi^2 + \upsilon\xi^2\lambda_3^{\beta} s^{\beta} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha})}\right]\frac{s}{s^2+\omega^2} \qquad (24)$$

$$\overline{U}_s(\xi,s) = \sqrt{\frac{2}{\pi}}\,\frac{V}{\xi}\left[1-\frac{s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha}}{(s+\lambda_1^{\alpha} s^{\alpha+1} + \lambda_2^{\alpha} s^{2\alpha+1} + \upsilon\xi^2 + \upsilon\xi^2\lambda_3^{\beta} s^{\beta} + M + M\lambda_1^{\alpha} s^{\alpha} + M\lambda_2^{2\alpha} s^{2\alpha})}\right]\frac{\omega}{s^2+\omega^2} \qquad (25)$$

respectively,

Now, by using[13], $(\frac{1}{z+a} = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{a^{k+1}})$ and $(\,(1+b)^k = \sum_{m=0}^{k}\frac{k!\,b^m}{m!(k-m)!})$, Eqs. (24) and (25) can be expressed in the form of double series as follows:

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\ \frac{s}{s^2+\omega^2}-\sqrt{\frac{2}{\pi}}\frac{V}{\xi}[\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}$$
$$\frac{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}{(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M))^{m+1}}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d\ m!\,s^{\delta}]\ \frac{s}{s^2+\omega^2} \qquad (26)$$

$$\overline{U}_s(\xi,s)=\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\ \frac{\omega}{s^2+\omega^2}-\sqrt{\frac{2}{\pi}}\frac{V}{\xi}[\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}$$
$$\frac{(s+\lambda_1^{\alpha}s^{\alpha+1}+\lambda_2^{\alpha}s^{2\alpha+1}+M+M\lambda_1^{\alpha}s^{\alpha}+M\lambda_2^{2\alpha}s^{2\alpha})}{(s^{\alpha+1}+\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M))^{m+1}}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d\ m!\,s^{\delta}]\ \frac{\omega}{s^2+\omega^2} \qquad (27)$$

where
$\delta = m + 2\alpha l - j - \alpha i + \beta d - \alpha d.$

To find the inverse Laplace transform of the last two equations we need to introduce the generalized Mittag-Leffler function, given by:

$$E_{\lambda,\mu}^{(k)}(z)=\frac{d^k}{dz^k}\,E_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{(n+k)!z^n}{n!\Gamma(\lambda n+\lambda k+\mu)}. \qquad (28)$$

and property [12],

$$(\ L^{-1}\{\frac{k!s^{\lambda-\mu}}{(s^{\lambda}\mp c)^{k+1}}\}=t^{\lambda k+\mu-1}E_{\lambda,\mu}^{(k)}\ (\pm ct^{\lambda}),$$
$$(\mathrm{Re}(s)>|c|^{1/\lambda}) \qquad (29)$$

According to Eq. (29) and the convolution of two functions, the inverse Laplace transform of Eqs. (26) and (27) has the form:

$$U_s(\xi,t)=\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\cos(\omega t)-\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}$$
$$\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d$$
$$\times[t^{(\alpha+1)m+(\alpha-\delta)-1}\ E_{(\alpha+1),(\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+\lambda_1^{\alpha}\ t^{(\alpha+1)m-\delta-1}\ E_{(\alpha+1),-\delta}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\,t^{\alpha+1})$$
$$+\lambda_2^{\alpha}\ t^{(\alpha+1)m+(-\alpha-\delta)-1}\ E_{(\alpha+1),(-\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\ t^{(\alpha+1)m+(\alpha+1-\delta)-1}\ E_{(\alpha+1),(\alpha+1-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_1^{\alpha}t^{(\alpha+1)m+(1-\delta)-1}\ E_{(\alpha+1),(1-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$
$$+M\lambda_2^{\alpha}t^{(\alpha+1)m+(1-\alpha-\delta)-1}\ E_{(\alpha+1),(1-\alpha-\delta)}^{(m)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})]*\cos(\omega t) \qquad (30)$$

$$U_s(\xi,t)=\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\sin(\omega t)-\sqrt{\frac{2}{\pi}}\frac{V}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}$$

$$\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d$$

$$\times[t^{(\alpha+1)m+(\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\,t^{\alpha+1})$$

$$+\lambda_1^{\alpha}\ t^{(\alpha+1)m-\delta-1}\ E^{(m)}_{(\alpha+1),-\delta}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\ t^{\alpha+1})$$

$$+\lambda_2^{\alpha}\ t^{(\alpha+1)m+(-\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\ t^{(\alpha+1)m+(\alpha+1-\delta)-1}\ E^{(m)}_{(\alpha+1),(\alpha+1-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\lambda_1^{\alpha}t^{(\alpha+1)m+(1-\delta)-1}\ E^{(m)}_{(\alpha+1),(1-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})$$

$$+M\lambda_2^{\alpha}t^{(\alpha+1)m+(1-\alpha-\delta)-1}\ E^{(m)}_{(\alpha+1),(1-\alpha-\delta)}(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)t^{\alpha+1})]*\sin(\omega t) \qquad (31)$$

Finally, inverting (30) and (31) by means of inverse Fourier transform and using Eq. (28), it found that, the velocity distribution is given by:

$$u(y,t)=V\cos(\omega t)-\frac{2}{\pi}V\int_0^{\infty}\int_0^{t}\frac{\sin(\xi y)}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\xi^2)^d$$

$$\times[\tau^{(\alpha+1)m+(\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\delta))}$$

$$+\lambda_1^{\alpha}\,\tau^{(\alpha+1)m-\delta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\delta))}$$

$$+\lambda_2^{\alpha}\,\tau^{(\alpha+1)m+(-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\delta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\delta))}$$

$$+M\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\upsilon\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\delta))}$$

$$+M\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\delta))}]\cos(\omega(t-\tau))\,d\tau\,d\xi \qquad (32)$$

$$u(y,t)=V\sin(\omega t)-\frac{2}{\pi}V\int_0^\infty\int_0^t\frac{\sin(\xi y)}{\xi}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{\,j-d}\ (\upsilon\xi^2)^d$$

$$\times[\tau^{(\alpha+1)m+(\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M)\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\delta))}$$

$$+\lambda_1^\alpha\,\tau^{(\alpha+1)m-\delta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\delta))}$$

$$+\lambda_2^\alpha\,\tau^{(\alpha+1)m+(-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\delta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\delta))}$$

$$+M\,\lambda_1^\alpha\,\tau^{(\alpha+1)m+(1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\upsilon\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\delta))}$$

$$+M\,\lambda_2^\alpha\tau^{(\alpha+1)m+(1-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\upsilon\xi^2+M\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\delta))}]\sin(\omega(t-\tau))\,d\tau\,d\xi \qquad (33)$$

Calculation of Shear Stress

To obtain the expression for the shear stress $S_{xy}(y,t)$ we first apply the Laplace transform to Eq. (10) and using the initial condition $S(y,0)=\frac{\partial S(y,0)}{\partial t}=0$, we obtain:

$$S_{xy}(y,s)=\mu\frac{(1+\lambda_3^\beta s^\beta)}{1+\lambda_1^\alpha s^\alpha+\lambda_2^\alpha s^{2\alpha}}\frac{\partial\overline{U}(y,s)}{\partial y} \qquad (34)$$

The solution of equation (34) satisfying initial conditions $S(y,0)=\frac{\partial S(y,0)}{\partial t}=0$, is found (for both cases) in the form;

$$S_{xy}(y,t)=-\frac{2}{\pi}V\mu\int_0^\infty\int_0^t\cos(\xi y)\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{p=0}^{i}\frac{r!}{p!(r-p)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda_1^{\alpha(r-p-m+i-d-1)}\lambda_2^{\alpha(p+l-i)}\lambda_3^{\beta d}M^{j-d}(\upsilon\xi^2)^d$$

$$\times[[\tau^{(\alpha+1)m+(\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\eta))}$$

$$+\lambda_1^\alpha\tau^{(\alpha+1)m-\eta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\eta))}$$

$$+\lambda_2^\alpha\tau^{(\alpha+1)m+(-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\eta))}$$

$$+M\lambda_1^\alpha\tau^{(\alpha+1)m+(1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\eta))}$$

$$+M\lambda_2^\alpha\tau^{(\alpha+1)m+(1-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\eta))}]$$

$$+\lambda_3^\beta[\tau^{(\alpha+1)m+(\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\beta-\eta))}$$

$$+\lambda_1^\alpha\tau^{(\alpha+1)m+(-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\beta-\eta))}$$

$$+\lambda_2^\alpha\tau^{(\alpha+1)m+(-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\beta-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\beta-\eta))}$$

$$+M\lambda_1^\alpha\tau^{(\alpha+1)m+(1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\beta-\eta))}$$

$$+M\lambda_2^\alpha\tau^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^\alpha}(\xi^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\beta-\eta))}]]\cos(\omega(t-\tau)d\tau d\xi \quad (35)$$

$$S_{xy}(y,t)=-\frac{2}{\pi}V\mu\int_0^\infty\int_0^t\cos(\xi y)\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{p=0}^{i}\frac{r!}{p!(r-p)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(r-p-m+i-d-1)}\ \lambda_2^{\alpha(p+l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\,(\upsilon\xi^2)^d$$

$$\times[[\tau^{(\alpha+1)m+(\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\eta))}$$

$$+\lambda_1^{\alpha}\,\tau^{(\alpha+1)m-\eta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\eta))}$$

$$+\lambda_2^{\alpha}\,\tau^{(\alpha+1)m+(-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\eta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\eta))}$$

$$+M\,\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\eta))}$$

$$+M\,\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\eta))}]$$

$$+\lambda_3^{\beta}[\tau^{(\alpha+1)m+(\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\beta-\eta))}$$

$$+\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\beta-\eta))}$$

$$+\lambda_2^{\alpha}\,\tau^{(\alpha+1)m+(-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\beta-\eta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\beta-\eta))}$$

$$+M\,\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\xi^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\beta-\eta))}$$

$$+M\,\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\xi^2+M\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\beta-\eta))}]]\sin(\omega(t-\tau)d\tau d\xi \qquad (36)$$

where $\eta=\delta+\alpha(r+p)$.

The Flow Between Two Parallel Plates, One of Which is Oscillating

Here we consider an incompressible fractional Burgers' fluid at rest between two infinite parallel plates. At time $t=0$ the lower plat at $y=0$ begins oscillating while the upper plat at $y=d$ remains stationary. The governing equation and the initial condition are given by Eqs. (15) and (17) while the associated boundary conditions are

$$u(0,t)=V\cos(\omega t) \quad or \quad u(0,t)=V\sin(\omega t); \quad t>0 \quad (37)$$

$$u(d,t)=0\ ; \quad t>0 \quad (38)$$

In order to determine the analytic solution, multiplying Eq. (15) by $\sin(\gamma_n y)$ where $\gamma_n = n\pi/d$, integrating with respect to y from 0 to d and bearing in mind the boundary condition (37) and (38), we obtain

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial U_{Sn}(n,t)}{\partial t}= \upsilon(1+\lambda_3^{\beta}D_t^{\beta})(\gamma_n V\cos(\omega t)-\gamma_n{}^2U_{ns}(n,t)) -M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})U_{sn}(n,t) \quad (39)$$

$$(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})\frac{\partial U_{Sn}(n,t)}{\partial t} =\upsilon(1+\lambda_3^{\beta}D_t^{\beta})(\gamma_n V\sin(\omega t)-\gamma_n{}^2U_{ns}(n,t)) -M(1+\lambda_1^{\alpha}D_t^{\alpha}+\lambda_2^{\alpha}D_t^{2\alpha})U_{sn}(n,t) \quad (40)$$

respectively, where the finite Fourier sine transform $U_{sn}(n,t)\, of\ u(y,t)$ has to satisfy the conditions [13]

$$U_{sn}(0)=\frac{\partial U_{sn}(0)}{\partial t}=\frac{\partial^2 U_{sn}(0)}{\partial t^2}=0\,; \quad (41)$$

Further, solving the above problems by means of Laplace transform and adopting a similar procedure as previously discussed, we obtain the expressions for the velocity field and shear stress in the form

$$u(y,t)=V(1-\frac{y}{d})\cos(\omega t)-\frac{2}{d}V\sum_{n=1}^{\infty}\frac{\sin(\gamma_n y)}{\gamma_n}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\ \lambda_1^{\alpha(-m+i-d-1)}\ \lambda_2^{\alpha(l-i)}\ \lambda_3^{\beta d}\ M^{j-d}\ (\upsilon\gamma_n{}^2)^d$$

$$\int_0^t\cos(\omega(t-\tau))\times[\tau^{(\alpha+1)m+(\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\delta))}$$

$$+\lambda_1^{\alpha}\tau^{(\alpha+1)m-\delta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\delta))}$$

$$+\lambda_2^{\alpha}\tau^{(\alpha+1)m+(-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\delta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\delta))}$$

$$+M\,\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\upsilon\gamma_n{}^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\delta))}$$

$$+M\,\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\delta))}])\,d\tau \qquad (42)$$

$$u(y,t)=V(1-\frac{y}{d})\sin(\omega t)-\frac{2}{d}V\sum_{n=1}^{\infty}\frac{\sin(\gamma_n y)}{\gamma_n}\sum_{m=0}^{\infty}(-1)^m\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\,\lambda_1^{\alpha(-m+i-d-1)}\,\lambda_2^{\alpha(l-i)}\,\lambda_3^{\beta d}\,M^{\,j-d}\,(\upsilon\gamma_n{}^2)^d$$

$$\int_0^t\sin(\omega(t-\tau)\times[\tau^{(\alpha+1)m+(\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\delta))}$$

$$+\lambda_1^{\alpha}\,\tau^{(\alpha+1)m-\delta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\delta))}$$

$$+\lambda_2^{\alpha}\,\tau^{(\alpha+1)m+(-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\delta))}$$

$$+M\,\tau^{(\alpha+1)m+(\alpha+1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\delta))}$$

$$+M\,\lambda_1^{\alpha}\,\tau^{(\alpha+1)m+(1-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\upsilon\gamma_n{}^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\delta))}$$

$$+M\,\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\delta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\upsilon\gamma_n{}^2+M\,\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\delta))}])\,d\tau \qquad (43)$$

$$S_{xy}(y,t)=-\frac{\mu V}{d}\sum_{r=0}^{\infty}(-1)^r\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\lambda_1^{\alpha(r-p)}\lambda_2^{\alpha p}\int_0^t(\frac{\tau^{-1-(r+p)\alpha}}{\Gamma(-(r+p)\alpha)}+$$

$$\lambda_3^{\beta}\frac{\tau^{-1-(r+p)\alpha-\beta}}{\Gamma(-(r+p)\alpha-\beta)})\times\cos(\omega(t-\tau))-\frac{2}{d}V\mu\sum_{n=1}^{\infty}\cos(\gamma_n y)\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+r}\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{p=0}^{i}\frac{r!}{p!(r-p)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda_1^{\alpha(r-p-m+i-d-1)}\lambda_2^{\alpha(p+l-i)}\lambda_3^{\beta d}M^{j-d}(\upsilon\gamma_n^{\ 2})^d$$

$$\int_0^t\cos(\omega(t-\tau)\times[[\tau^{(\alpha+1)m+(\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\eta))}$$

$$+\lambda_1^{\alpha}\tau^{(\alpha+1)m-\eta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\eta))}$$

$$+\lambda_2^{\alpha}\tau^{(\alpha+1)m+(-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\eta))}$$

$$+M\lambda_1^{\alpha}\tau^{(\alpha+1)m+(1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\gamma_n^{\ 2}+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\eta))}$$

$$+M\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\eta))}]$$

$$+\lambda_3^{\beta}[\tau^{(\alpha+1)m+(\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\beta-\eta))}$$

$$+\lambda_1^{\alpha}\tau^{(\alpha+1)m+(-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\beta-\eta))}$$

$$+\lambda_2^{\alpha}\tau^{(\alpha+1)m+(-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\beta-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\beta-\eta))}$$

$$+M\lambda_1^{\alpha}\tau^{(\alpha+1)m+(1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\gamma_n^{\ 2}+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\beta-\eta))}$$

$$+M\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n^{\ 2}+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\beta-\eta))}]]\ d\tau \qquad (44)$$

$$S_{xy}(y,t) = -\frac{\mu V}{d}\sum_{r=0}^{\infty}(-1)^{r}\sum_{p=0}^{r}\frac{r!}{p!(r-p)!}\lambda_1^{\alpha(r-p)}\lambda_2^{\alpha p}\int_0^t\Big(\frac{\tau^{-1-(r+p)\alpha}}{\Gamma(-(r+p)\alpha)} +$$

$$\lambda_3^{\beta}\frac{\tau^{-1-(r+p)\alpha-\beta}}{\Gamma(-(r+p)\alpha-\beta)}\Big)\times\sin(\omega(t-\tau)) - \frac{2}{d}V\mu\sum_{n=1}^{\infty}\cos(\gamma_n y)\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+r}\sum_{l=0}^{m}\frac{1}{l!(m-l)!}$$

$$\sum_{p=0}^{i}\frac{r!}{p!(r-p)!}\sum_{j=0}^{l}\frac{l!}{j!(l-j)!}\sum_{i=0}^{j}\frac{j!}{i!(j-i)!}\sum_{d=0}^{i}\frac{i!}{d!(i-d)!}\lambda_1^{\alpha(r-p-m+i-d-1)}\lambda_2^{\alpha(p+l-i)}\lambda_3^{\beta d}M^{j-d}(\upsilon\gamma_n{}^2)^d$$

$$\int_0^t\sin(\omega(t-\tau)\times[[\tau^{(\alpha+1)m+(\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\eta))}$$

$$+\lambda_1^{\alpha}\tau^{(\alpha+1)m-\eta-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\eta))}$$

$$+\lambda_2^{\alpha}\tau^{(\alpha+1)m+(-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\eta))}$$

$$+M\lambda_1^{\alpha}\tau^{(\alpha+1)m+(1-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\gamma_n{}^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\eta))}$$

$$+M\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\eta))}]$$

$$+\lambda_3^{\beta}[\tau^{(\alpha+1)m+(\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha-\beta-\eta))}$$

$$+\lambda_1^{\alpha}\tau^{(\alpha+1)m+(-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\beta-\eta))}$$

$$+\lambda_2^{\alpha}\tau^{(\alpha+1)m+(-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(-\alpha-\beta-\eta))}$$

$$+M\tau^{(\alpha+1)m+(\alpha+1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M)\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(\alpha+1-\beta-\eta))}$$

$$+M\lambda_1^{\alpha}\tau^{(\alpha+1)m+(1-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda}(\gamma_n{}^2+M)\sigma^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\beta-\eta))}$$

$$+M\lambda_2^{\alpha}\tau^{(\alpha+1)m+(1-\alpha-\beta-\eta)-1}\sum_{n=0}^{\infty}\frac{(n+m)!(-\frac{1}{\lambda_1^{\alpha}}(\gamma_n{}^2+M\tau^{\alpha+1})^n}{n!\Gamma((\alpha+1)n+(\alpha+1)m+(1-\alpha-\beta-\eta))}]]\,d\tau \qquad (45)$$

Results and Discussion:

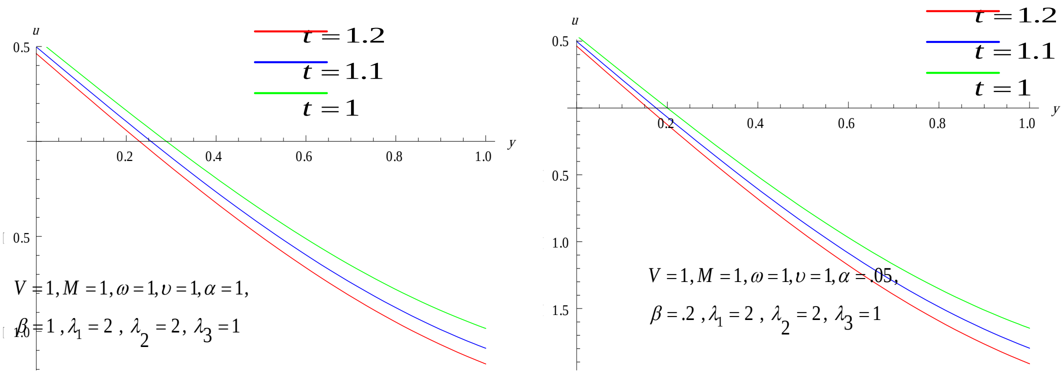
In the previous sections, we have presented the analytical solutions for two oscillatory flow problems of a fractional Burgers' fluid. In order to capture the relevant physical effects of the obtained results, several graphs are depicted in this section. The results illustrate the velocity profiles for the flow induced by a rigid oscillating plate. We interpret these results with respect to the variations of emerging parameters of interest.

In **Fig.(1),** the variation of the velocity filed is plotted at different times for a comparison between fractional Burgers' fluid (when $\alpha = .05$ and $\beta = .2$) and ordinary Burgers' fluid (when $\alpha = \beta = 1$). From this figure, it can be seen that the amplitude of the fluid oscillation decays away from the plate and approaches to zero. A comparison shows that the decay of the amplitude of oscillation in case of fractional Burgers' fluid is faster than ordinary Burgers' fluid.

Figs.(2,3) are plotted to illustrate the effect of parameters λ_1 and λ_3. We observed that they have the same effect with exception that they have different effect on the velocity value. **Figs.(4,5)** demonstrate the influence of the fractional parameters α and β on the motion of the fluid when the other parameters are fixed. small values of α, lead to the more slow velocity decays with the flow. However, one can see that an increase in the material parameter β has quite the opposite effect to that of α. Thus, it is obvious that the velocity fields are influenced by relaxation and retardation times, and the others of time fractional derivative.

Fig.(6) illustrates the variation of the velocity field for different values of the rheological parameter λ_2 of the Burgers' fluid. It appears that the velocity is a strong function of λ_2. As λ_2 increases, the amplitude of the fluid oscillation away from the plate is also reduced and reducing the velocity.

Fig.(7) shows the influence of a magnetic field on the velocity when the other parameters are fixed. We can see that the magnetic body force is favorable to the velocity decays, and the more value of M, the more rapidly the velocity decays.



(a) Ordinary model **(b) Fractional model**

Fig.(1) Velocity u(y,t) versus y for different value of t when the other parameters are fixed.

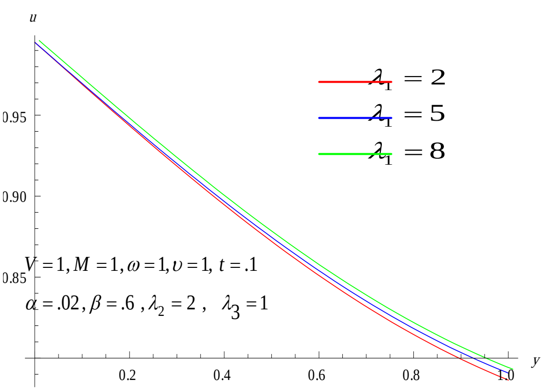


Fig.(2) Velocity u(y,t) versus y for different value of λ_1 when the other parameters are fixed.

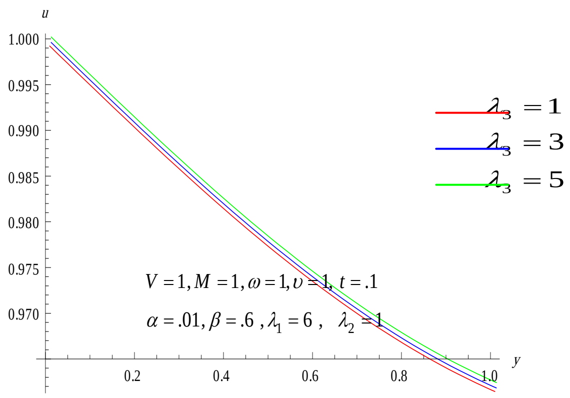


Fig.(3) Velocity u(y,t) versus y for different value of λ_3 when the other parameters are fixed.

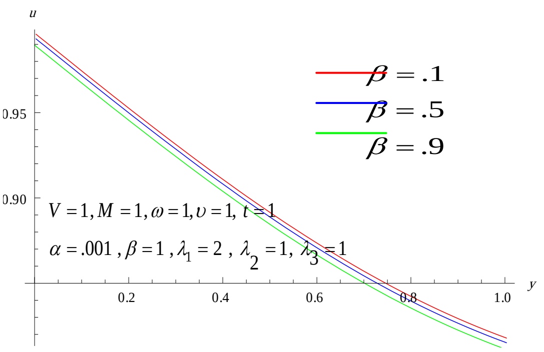


Fig.(4) Velocity u(y,t) versus y for different value of β when the other parameters are fixed.

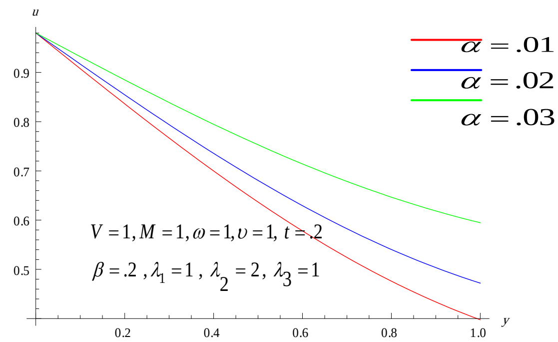


Fig.(5) Velocity u(y,t) versus y for different value of α when the other parameters are fixed.

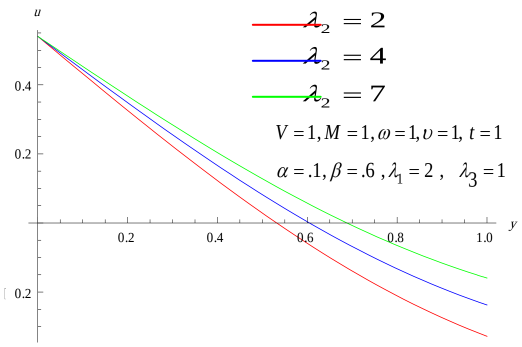


Fig.(6) Velocity u(y,t) versus y for different value of λ_2 when the other parameters are fixed.

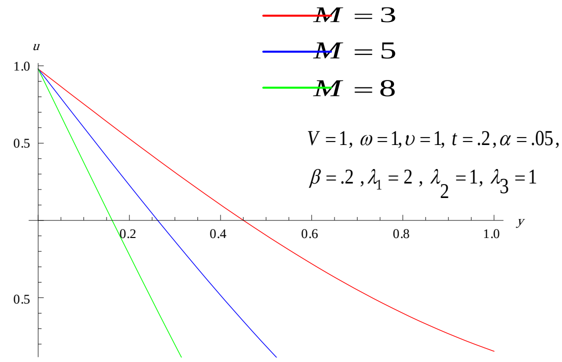


Fig.(7) Velocity u(y,t) versus y for different value of M when the other parameters are fixed.

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تأثيـــر المجال المغناطيسي الهيدروديناميكـــي على بعض الحركات التذبذبية لمائع بيركـــــر للمشتقات الكسرية

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الخلاصة:

تمت دراسة تأثير المجـال المغناطيســي الهيدروديناميكــي (MHD) علـى الجريان المتذبذب لمائع من النمط بيركر الكسري. تم استخدام التفاضل والتكامل الكسري في كتابة معادلات الحركة لنموذج بيركر الكسري. الحل الدقيق للحركات التذبذبية لمائع بيركر الكسري والناتج عن ذبذبات جيبية وجيب تمامية للوح مسطح لانهائي قد تم ايجاده باستخدام تحويلات فورير ولابلاس. وقد اوجدنا حقل السرعة واجهاد القص بشكل تكاملات ومتسلسلات وباستخدام دالة ميتاج-لفلر(Mittag-Leffler) المعممة وهذه الحلول تحقق جميع الشروط الابتدائية والحدودية في المسالة. واخيراً اعطينا تحليلا للجريان من خلال رسوم بيانية بأستخدام برنامج ماثيماتكا، حيث تم دراسة تاثير كل معلمه تظهر في حقل السرعة.