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## Attacking Jacobian Problem Using Resultant Theory

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### Abstract:

This paper introduces a relation between resultant and the Jacobian determinant by generalizing Sakkalis theorem from two polynomials in two variables to the case of (n) polynomials in (n) variables. This leads us to study the results of the type:  $R_i(x_i, u_1, \dots, u_n) = \text{Res}_{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n}(f_1 - u_1, \dots, f_n - u_n)$ ,  $i = 1, \dots, n$ , and use this relation to attack the Jacobian problem. The last section shows our contribution to proving the conjecture.

**Keywords:** Jacobian conjecture, polynomial map, resultant.

### Introduction:

The Jacobian Conjecture can be stated as follows: For any integer  $n \geq 1$  and polynomials  $f_1, \dots, f_n \in \mathbb{C}[X]$ ; the polynomial map  $F = (f_1, \dots, f_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$  is an automorphism if  $\det J_F$  is a nonzero constant.

#### Notation

Throughout this paper  $J_F$  is used for Jacobian matrix, and  $X$  is used to denote the variables  $x_1, \dots, x_n$  also  $k_i$  is put to be the  $\deg_{x_i}(R_i(x_i, u_1, \dots, u_n))$ .

The map

$$F: \mathbb{C}^n \rightarrow \mathbb{C}^n$$

$$F(X) = (f_1(X), \dots, f_n(X)),$$

is a polynomial mapping, if each  $f_i$  is a polynomial.

If  $F$  is bijective, then the inverse will be automatically polynomial (Theorem 2.1,<sup>1</sup>).

It is well-known that the invertibility of  $F$  implies the invertibility of  $J_F$ .

Conversely, the statement of **Jacobian Conjecture**, which first formulated in 1939 by O. Keller<sup>2</sup> is

$$\det(J_F) \in \mathbb{C}^* \implies F \text{ is invertible.}$$

In 1993, Takis Sakkalis, in his paper<sup>3</sup>, investigated a certain connection between the zeros of a polynomial  $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  and the Jacobian determinant of  $f$  and  $g$  (where  $f$  and  $g$  are nonzero polynomials in  $\mathbb{C}[x, y]$ ).

Researches in this field are still ongoing; for example, a recent study was published in May 2020. For more information, see<sup>4</sup>.

#### History of Jacobian Problem

The Jacobian Conjecture is one of the most well known open problems in mathematics. Keller formulated the problem<sup>2</sup> in 1939. In the late 1960s, Zariski and his student Abyankar were the main movers of the conjecture. Many papers have been published on this subject, using tools from many different mathematics areas, including analysis, algebra, and complex geometry. This section shows some of the work that has been done:

- For  $n = 1$ , the problem is correct and is studied before in<sup>5</sup>
- For  $n = 2$ , the problem was tested using the computer for polynomials of degree  $\deg(F) = 100$ ;  $\deg(F) = \max\{\deg f_1, \deg f_2\}$  and is studied before in<sup>6</sup>.
- The problem was proved for the case, which includes all polynomial maps whose coordinates have a degree of at most 2, is studied before in<sup>7</sup>.
- It was demonstrated that if the conjecture was correct in the special case for polynomial maps of degree  $\deg F \leq 3$ , then it is correct in the general case, is studied before in<sup>2</sup>.

**Definition 1:** Resultant <sup>8</sup>

The Resultant is defined with respect to  $n$  homogeneous polynomials  $F_1, \dots, F_n$  in  $n$  variables, of degrees  $l_1, \dots, l_n$  each polynomial being full in all its terms with literal coefficients  $F_i = \sum_{|\alpha|=l_i} u_{i,\alpha} X^\alpha$ , for  $i = 1, \dots, n$ . Then the Resultant of  $n$  given homogeneous polynomials in  $n$  variables is a unique polynomial  $Res \in \mathbb{Z}[u_{i,\alpha}]$  which has the following properties:

- i. The equations  $F_1 = \dots = F_n = 0$  have a non-trivial solution over  $\mathbb{C}$  if and only if  $Res(F_1, \dots, F_n) = 0$ .
- ii.  $Res(x_1^{l_1}, \dots, x_n^{l_n}) = 1$ .
- iii.  $Res$  is irreducible, even when regarded as a polynomial in  $\mathbb{C}[u_{i,\alpha}]$ .

Moreover, the Resultant of  $n$  fixed nonhomogeneous polynomials in  $n - 1$  variables is the Resultant of the corresponding homogeneous polynomials of the same degrees gotten by presenting a variable  $x_0$  of consistency. For more details, see <sup>8</sup>.

**A Relation Between Resultant And The Jacobian Determinant**

This section gives a generalization of Sakkalis (Theorem 1, <sup>3</sup>).

**Definition 2**

Let  $f = \sum_{\alpha} c_{\alpha} X^{\alpha} \in \mathbb{C}[x_0, x_1, \dots, x_n]$  be a nonzero polynomial with total degree  $n$ . Then  $f$  is a strong quasi-regular in  $x_1, \dots, x_n$  if the coefficients of all monomials of the form  $x_1^{\alpha_1} \dots x_n^{\alpha_n}$  which have a total degree  $n$  in  $f(x_0, x_1, \dots, x_n)$  are nonzero constant.

**Remark 1**

Let  $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$  be a strong quasi-regular in  $x_1, \dots, x_{n-1}$  and let  $u_1, \dots, u_n$  be new indeterminates. Consider

$$Res_{x_1, \dots, x_{n-1}}(f_1 - u_1, \dots, f_n - u_n) = R(x_n, u_1, \dots, u_n)$$

$$= R_{k_n}(u_1, \dots, u_n)x_n^{k_n} + \dots + R_0(u_1, \dots, u_n).$$

Then

$$R_0(u_1, \dots, u_n) = Res_{x_1, \dots, x_{n-1}}(f_1 - u_1, \dots, f_n - u_n)|_{x_n=0} \\ = Res_{x_1, \dots, x_{n-1}}(f_1(x_1, \dots, x_{n-1}, 0) - u_1, \dots, f_n(x_1, \dots, x_{n-1}, 0) - u_n) \neq 0.$$

**Remark 2**

There exist some polynomials  $g_1, \dots, g_n \in \mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]$  such that

$$R(x_n, u_1, \dots, u_n) = g_1(f_1 - u_1) + \dots + g_n(f_n - u_n) \quad ,$$

which means that  $R(x_n, f_1, \dots, f_n) = 0$ .

**Lemma 1**

Consider

$$Res_{x_1, \dots, x_{n-1}}(f_1 - u_1, \dots, f_n - u_n) \\ = R(x_n, u_1, \dots, u_n), \\ Res_{x_2, \dots, x_n}(f_1 - u_1, \dots, f_n - u_n) = R(x_1, u_1, \dots, u_n), \\ \vdots \\ Res_{x_1, \dots, x_{n-2}, x_n}(f_1 - u_1, \dots, f_n - u_n) = R(x_{n-1}, u_1, \dots, u_n), \\ \text{and write}$$

$$R(x_n, u_1, \dots, u_n) = R_{k_n}(u_1, \dots, u_n)x_n^{k_n} + \dots + R_0(u_1, \dots, u_n).$$

Let  $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$  be a strong quasi-regular in  $x_1, \dots, x_{n-1}$  as well as in  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$  for each  $i \in \{1, \dots, n - 1\}$ , then the following conditions are equivalent:

- i.  $k_n = 0$ .
- ii.  $\exists h(u_1, \dots, u_n) \in \mathbb{C}[u_1, \dots, u_n]$ ,  $h \neq 0$ , with  $h(f_1, \dots, f_n) = 0$ .
- iii.  $\det J_F = 0$ .

**Proof**

**i  $\Rightarrow$  ii.**

If  $k_n = 0$  then

$$R(x_n, u_1, \dots, u_n) = Res_{x_1, \dots, x_{n-1}}(f_1 - u_1, \dots, f_n - u_n) = R_0(u_1, \dots, u_n) \in \mathbb{C}[u_1, \dots, u_n].$$

(Remark 1) impels that  $R_0(u_1, \dots, u_n) \neq 0$ , and (Remark 2) gives that  $R_0(f_1, \dots, f_n) = 0$ .

**ii  $\Rightarrow$  iii.**

Let  $h(u_1, \dots, u_n)$  be of minimal positive degree such that  $h(f_1, \dots, f_n) = 0$ . Then calculating partial derivatives of  $h(f_1, \dots, f_n)$  gives that:

$$\frac{\partial}{\partial x_1} h(f_1, \dots, f_n) = \frac{\partial h}{\partial u_1} \frac{\partial f_1}{\partial x_1} + \dots + \frac{\partial h}{\partial u_n} \frac{\partial f_n}{\partial x_1},$$

$$\vdots \\ \frac{\partial}{\partial x_n} h(f_1, \dots, f_n) = \frac{\partial h}{\partial u_1} \frac{\partial f_1}{\partial x_n} + \dots + \frac{\partial h}{\partial u_n} \frac{\partial f_n}{\partial x_n},$$

Then the matrix form of this system can be written as follows

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial h}{\partial u_1} \\ \vdots \\ \frac{\partial h}{\partial u_n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then the minimal property implies that

$\frac{\partial h}{\partial u_i}(f_1, \dots, f_n) \neq 0$  for some  $1 \leq i \leq n$ , thus  $\det J_F = 0$ .

**iii  $\Rightarrow$  ii.**

By contradiction, note that  $R(x_i, f_1, \dots, f_n) = 0$  for  $i = 1, \dots, n$ . Then there exist polynomials  $B_i(x_i, u_1, \dots, u_n)$  of minimal positive degrees in

$x_1, \dots, x_n$  respectively, such that  $B_1(x_1, f_1, \dots, f_n) = \dots = B_n(x_n, f_1, \dots, f_n) = 0$ . Then

$$\begin{pmatrix} \frac{\partial B_1}{\partial u_1}(x_1, f_1, \dots, f_n) & \dots & \frac{\partial B_1}{\partial u_n}(x_1, f_1, \dots, f_n) \\ \vdots & \ddots & \vdots \\ \frac{\partial B_n}{\partial u_1}(x_n, f_1, \dots, f_n) & \dots & \frac{\partial B_n}{\partial u_n}(x_n, f_1, \dots, f_n) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} -\frac{\partial B_1}{\partial x_1}(x_1, f_1, \dots, f_n) & & 0 \\ & \ddots & \\ 0 & & -\frac{\partial B_n}{\partial x_n}(x_n, f_1, \dots, f_n) \end{pmatrix}$$

But minimal property implies that  $\prod_i \frac{\partial B_i}{\partial x_i}(x_i, f_1, \dots, f_n) \neq 0$ , and thus  $\det J_F \neq 0$ .

iii  $\Rightarrow$  i.

Assume that  $k_n \geq 1$ . Pick  $(a_1, \dots, a_n) \in \mathbb{C}^n$  such that  $R_{k_n}(a_1, \dots, a_n) R_0(a_1, \dots, a_n) \neq 0$  and let  $b_n \in \mathbb{C}$  be such that  $R(b_n, a_1, \dots, a_n) = 0$  then (Theorem (5.15),<sup>8</sup>) ensure that there is  $(b_1, \dots, b_n) \in \mathbb{C}^{n-1}$  such that :

$$f_1(b_1, \dots, b_n) - a_1 = \dots = f_n(b_1, \dots, b_n) - a_n = 0.$$

Furthermore, the polynomials

$f_1(x_1, \dots, x_n) - a_1, \dots, f_n(x_1, \dots, x_n) - a_n$ , have no shared factor of positive degree in  $x_1, \dots, x_{n-1}$  for otherwise the shared factor  $g(x_1, \dots, x_n)$  has a positive degree in  $x_1, \dots, x_{n-1}$  and  $R(x_n, a_1, \dots, a_n) = 0$  contradicting  $R_0(a_1, \dots, a_n) \neq 0$ . Put

$$\begin{aligned} \bar{f}_1(x_1, \dots, x_n) &= f_1(x_1 + b_1, \dots, x_n + b_n) - a_1, \\ &\vdots \\ \bar{f}_n(x_1, \dots, x_n) &= f_n(x_1 + b_1, \dots, x_n + b_n) - a_n, \end{aligned}$$

Then  $\bar{f}_1(0, \dots, 0) = \dots = \bar{f}_n(0, \dots, 0) = 0$ , and  $\det J_{(\bar{f}_1, \dots, \bar{f}_n)} = 0$ .

By (ii) there is  $h(u_1, \dots, u_n)$  of minimal positive degree such that  $h(\bar{f}_1, \dots, \bar{f}_n) = 0$ . Moreover,  $h(u_1, \dots, u_n)$  has no constant term since  $h(0, \dots, 0) = 0$ . In this case, the  $\bar{f}_1, \dots, \bar{f}_n$  have a shared factor of positive degree, say  $d(x_1, \dots, x_n)$ . This implies that  $d(x_1 - b_1, \dots, x_n - b_n)$  is a shared factor of positive degree of

$$f_1(x_1, \dots, x_n) - a_1, \dots, f_n(x_1, \dots, x_n) - a_n,$$

which is a contradiction.

In <sup>9</sup> Peretz uses The Jacobian criterion satisfied by the map  $F = (f, g)$ , (of degree  $d$  or less) in a way to construct an ideal (called the Jacobian ideal). He considers the two relative resultants polynomials of the map  $F$  (one with respect to  $x$  and the second with respect to  $y$ ).

The key theorem he proved is that:

The Jacobian Conjecture is true for  $F$  if and only if the leading coefficients of these two resultants belong to the Jacobian ideal.

He calls this result the resultant reformulation of the Jacobian Conjecture.

Using the Groebner bases technique, he builds an algorithm, and it was programmed and used to prove the 2-dimensional Jacobian Conjecture up to degree 15.

The theoretical importance of Peretz research is to show that conjecture is a decidable problem. The following result is obtained in his study.

Let  $f_1(x, y), f_2(x, y) \in \mathbb{C}[x, y]$  be a Jacobian pair. Let  $\alpha$  and  $\beta$  be indeterminates. If  $Res(f_1(x, y) - \alpha, f_2(x, y) - \beta, y) = R_1x + R_0(\alpha, \beta)$ ,  $R_1 \in \mathbb{C}^*$ , then  $F(x, y) = (f_1(x, y), f_2(x, y))$  is onto  $\mathbb{C}^2$ .

### Proof of the Jacobian Conjecture

Depending on (Lemma 1), this paragraph introduces our contribution to proving the Jacobian problem. Recall the problem:

For any integer  $n \geq 1$  and polynomials  $f_1, \dots, f_n \in \mathbb{C}[X]$ , the polynomial map  $F = (f_1, \dots, f_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$  is an automorphism if  $\det(J_F)$  is a nonzero constant.

Note that two affine varieties  $A$  and  $B$  are isomorphic if and only if their affine coordinate rings are isomorphic as  $K$ -algebras. Note that  $A = B = \mathbb{C}^n$ . Therefore  $\mathbb{I}(\mathbb{C}^n) = \langle 0 \rangle$ . So the problem transformed to study the isomorphism between the rings  $\mathbb{C}[x_1, \dots, x_n]$  and  $\mathbb{C}[y_1, \dots, y_n]$ . To do this, let us define the map

$$\begin{aligned} \Phi: \mathbb{C}[y_1, \dots, y_n] &\rightarrow \mathbb{C}[x_1, \dots, x_n] \\ y_i &\mapsto f_i. \end{aligned}$$

It is a homomorphism, and his kernel is given in the following form:

$$Ker \Phi = \{h \in \mathbb{C}[y_1, \dots, y_n] : h(f_1, \dots, f_n) = 0\}.$$

Notice that the homomorphism  $\Phi$  is injective, since from the definition of  $Ker \Phi$ , if there is a nonzero polynomial  $h \in \mathbb{C}[y_1, \dots, y_n]$  such that  $h(f_1, \dots, f_n) = 0$ , this will lead us to  $\det J_F = 0$  (Lemma 1), and this is a contradiction with the hypotheses which is  $\det J_F \in \mathbb{C}^*$ .

Note that a linear transformation of coordinate can make the polynomials  $f_1, \dots, f_n$  strong quasi-regular in  $x_1, \dots, x_{n-1}$ .

Currently arise an important question; if the homomorphism  $\Phi$  is injective, is it true that also  $F$  is injective?

Suppose that  $F$  is not an injective polynomial map, then there exist  $a, b \in \mathbb{C}^n$  where  $a \neq b$  and  $F(a) = F(b)$ .

which means that  $(f_1(a), \dots, f_n(a)) = (f_1(b), \dots, f_n(b))$  and since  $\Phi(y_i) = f_i$  for each  $i \in \{1, \dots, n\}$ . Note that

$(\Phi(y_1)(a), \dots, \Phi(y_n)(a)) = (\Phi(y_1)(b), \dots, \Phi(y_n)(b))$ , and this implies that the compounds are equal, so  $\Phi(y_i)(a) = \Phi(y_i)(b)$  for each  $i$ , but the map  $\Phi$  is injective so  $y_i(a) = y_i(b)$  for each  $i \in \{1, \dots, n\}$  this shows that  $a_i = b_i$  for each  $i \in \{1, \dots, n\}$  which implies that  $a = b$  which means that  $F$  must be injective.

Using the results from<sup>10</sup>, it can be seen that  $F$  is invertible, and this completes our proof.

### Conclusion:

This paper presents some previous studies on the conjecture and introduces an attempt to solve this open question about the validity of Jacobian conjecture through the Resultant theory. This study transforms the problem to consider a particular isomorphism between the rings  $\mathbb{C}[x_1, \dots, x_n]$  and  $\mathbb{C}[y_1, \dots, y_n]$  to prove that the polynomial map  $F = (f_1, \dots, f_n): \mathbb{C}^n \rightarrow \mathbb{C}^n$  is an automorphism.

### Authors' declaration:

- Conflicts of Interest: None.

- Ethical Clearance: The project was approved by the local ethical committee in Damascus University.

### Authors' contributions statement:

AL Jony: did the study within the scope of a PhD research project at Damascus University.

SH Rashed: Supervisor of this work, Scientific audit, Associate Prof at Arab International University, Academic Staff at Department Of Mathematics, Faculty of Science, Damascus University of Syria.

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## محاولة حل مسألة اليعقوبي باستخدام نظرية الناتج

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### الخلاصة:

في هذا البحث تُقدم علاقة بين كثير الحدود الناتج ومحدد اليعقوبي وذلك عن طريق تعميم نظرية للباحث Sakkalis من حالة كثيري حدود بمتحولين إلى حالة  $n$  كثير حدود ب  $n$  متحول وفي نطاق البحث عن حل لمسألة اليعقوبي. وسوف نستخدم كثير الحدود الناتج من الشكل التالي  $(f_1 - u_1, \dots, f_n - u_n)$ ;  $i = 1, \dots, n$  في القسم الأخير من هذا البحث نُقدم محاولتنا لحل مسألة اليعقوبي.

الكلمات مفتاحية: تخمين اليعقوبي، تطبيق كثير الحدود، الناتج.