Third Order Differential Subordination for Analytic Functions Involving Convolution Operator

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Received 7/10/2020, Accepted 25/2/2021, Published Online First 20/11/2021

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Abstract:
In the present paper, by making use of the new generalized operator, some results of third order differential subordination and differential superordination consequence for analytic functions are obtained. Also, some sandwich-type theorems are presented.

Keywords: Analytic functions, Differential subordination, Hadamard product, Univalent functions.

Introduction:
Let $U$ be the open unit disk $U = \{ w : w \in \mathbb{C}, |w| < 1 \}$, and let $Y(U)$ be the class of analytic functions in $U$. For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $b \in \mathbb{C}$ define the subclass $Y(U)$ by

$$Y[b, n] = \left\{ f \in \mathcal{A} : f(w) = b + b_n w^n + b_{n+1} w^{n+1} + \ldots \right\},$$

with $Y_1 = [1, 1]$ and $Y_0 = [0, 1]$. Let $\mathcal{A}$ denote the class of all analytic functions in $U$ to satisfy the condition $f(0) = f'(0) - 1 = 0$ and write in the form

$$f(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad (w \in U). \quad (1)$$

For a function $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ are defined by

$$g(w) = w + \sum_{n=2}^{\infty} c_n w^n \quad (w \in U). \quad (2)$$

The convolution (or Hadamard product) of $f(w)$ and $g(w)$ is defined by

$$(f * g)(w) = w + \sum_{n=2}^{\infty} b_n c_n w^n = (g * f)(w) \quad (w \in U).$$

For two functions $f$ and $g$ are analytic in $U$. A function $f$ is subordinate to $g$ (or $g$ superordinate to $f$), written as:

$$f < g \text{ in } U \text{ or } g(w) < f(w), \quad (w \in U),$$

if there exists a Schwarz function $\hat{k}(w)$ which (by definition) is analytic in $U$ satisfies the following conditions (see 19),

$$\hat{k}(0) = 0 \text{ and } |\hat{k}(w)| < 1 \text{ for all } (w \in U),$$

such that $f(w) = g(\hat{k}(w)) \quad (w \in U)$. Indeed it is known that, $f(w) < g(w)$,

$$(w \in U) \implies f(0) = g(0) \text{ and } f(U) \subset g(U).$$

In a special case, if $g$ is a univalent function in open unit disk $U$, then the reverse implication also holds (see 10-11),

$$f(w) < g(w), (w \in U) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1. Let $\mu(w)$ is the univalent function in $U$, let $\psi : \mathbb{C} \times U \rightarrow \mathbb{C}$. If $\lambda(w)$ be an analytic function in open unit disk $U$ that satisfies the next third order differential subordination:

$$\psi(\lambda(w), w\lambda'(w), w^2\lambda''(w), w^3\lambda'''(w)); w < \mu(w). \quad (3)$$

From (3), $\lambda(w)$ is namely a solution of the differential subordination. Moreover, $q(w)$ is a univalent function which is namely a dominant of the solution of the differential subordination (3), or more simply, a dominant if $\lambda(w) < q(w)$ for all $\lambda(w)$ satisfying (3). A dominant $\hat{q}(w)$ satisfies $\hat{q}(w) < q(w)$ for all dominates $q(w)$ of (3) is called the best dominant. Note that the best dominant is unique up to a rotation of $U$. 

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Definition 2. (see\textsuperscript{12}). Let $\mu(w)$ is a univalent function in the open unit disk, let $\psi : \mathbb{C} \times U \rightarrow \mathbb{C}$. If the function $\lambda(w)$ is analytic in $U$ that satisfies the next third order differential subordination:

$$\mu(w) < \psi(\lambda(w), w'\lambda(w), w^2'\lambda''(w), w^3'\lambda'''(w); w).$$

(4)

Then $\lambda(w)$, is namely a solution of the differential superordination given by (4). Moreover, an analytic function $q(w)$ is namely a subordinates of the solutions of the differential superordination provided by (4), or more simply a subordinate if $q(w)$ subordinated $\lambda(w)$ for all $\lambda(w)$ it should be satisfy (4). A univalent subordinate $\tilde{q}(w)$ that satisfies $q(w) < \tilde{q}(w)$ for subordination $q(w)$ of (4) is called the best subordinate of the differential superordination given by (4). Note that the best subordinate is unique up to a rotation off. The process of admissible functions (also known as the differential subordinations method) was first introduced by Miller and Mocanu in 1978 (see\textsuperscript{13}), and the theory started to improve in 1981 (see\textsuperscript{14}). For more details, (see\textsuperscript{15}).

Definition 3. (see\textsuperscript{16}). For $f \in \mathcal{A}$, the generalized derivative operator $T_{s,n}^{m} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by

$$T_{s,n}^{m} f(w) = w + \sum_{s=1}^{\infty} [1 + \eta(n - 1)]^{s} C(s, n) b_{n} w^{n}, \; w \in U$$

(5)

where $s \in N_{0} = \{0,1,2,\ldots\}$, $\eta > 0$ and

$$C(s, n) = \frac{\binom{\eta + n - 1}{s}}{n!}.$$

It would be easily to see that $T_{0,0}^{0} f(w) = f(w)$ and

$$T_{0,1}^{0} f(w) = w f'(w)$$

For $m, s \in N_{0} = N \cup \{0\}$, $\eta > 0$, it is easy from (5), that

$$T_{s,n}^{m+1} f(w) = (1 - \eta) T_{s}^{m} f(w) + \eta w (T_{s}^{m} f(w))'$$

(6)

and

$$w (T_{s,n}^{m} f(w))' = (1 + s) T_{s+1,n}^{m} f(w) - s T_{s,n}^{m} f(w)$$

(7)

Definition 4. (see\textsuperscript{17}). For $f \in \mathcal{A}$, the Srivastava – Attiya operator is defined by

$$N_{k,c} = w + \sum_{m=1}^{\infty} \frac{c^{m+1}}{c^{m+1}!} b_{m} w^{m}, \; w \in U,$$

(8)

where $c \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\mathbb{Z}_{0}$. From (8) it is easy that

$$w (N_{k+1,c} f(w))' = (1 + c) N_{k,c} f(w) - c N_{k+1,c} f(w).$$

(9)

Definition 5. For $f \in \mathcal{A}$, the operator $N T_{s,n}^{m} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by convolution of the Srivastava – Attiya operator $N_{k,c}$ and the generalized operator $T_{s,n}^{m}$

$$N T_{s,n}^{m} f(w) = (N_{k,c} \ast T_{s,n}^{m}) f(w), \; (w \in U)$$

and

$$N T_{s,n}^{m} f(w) = w + \sum_{n=2}^{\infty} \frac{(c + 1)^{k}}{c + n} \left[ 1 + \eta(n - 1)]^{m} C(s, n) b_{n}^{2} w^{n}, \; w \in U$$. (10)

From (10), the following identity relations can be obtained:

$$w (N T_{s,n}^{m} f(w))' = (1 + s) N T_{s+1,n}^{m} f(w) - s N T_{s,n}^{m} f(w),$$

(11)

also

$$N T_{s,n}^{m+1} f(w) = (1 - \eta) N T_{s,n}^{m} f(w) + \eta w (N T_{s,n}^{m} f(w))'$$

(12)

and

$$w (N T_{s,n}^{m+1} f(w))' = (1 + c) N T_{s,n}^{m} f(w) - c N T_{s,n}^{m+1} f(w).$$

(13)

Note that, the following are special cases of operator $N T_{s,n}^{m}$

1. When $T_{s,n}^{m} = 1$ include the Srivastava-Attiya operator $N_{k,c}$ (see\textsuperscript{17}).

2. When $N_{0,c}$ include the generalized derivative operator $T_{s,n}^{m}$ (see\textsuperscript{18}).

3. When $m = 0, \eta = 1$, $T_{s,n}^{m}$ reduces to $T_{s}^{0}$ which is introduced by Ruscheweyh derivative operator (see\textsuperscript{19}).

4. When $m = 0, \eta = 1$, $T_{s,n}^{m}$ reduces to $T_{s}^{0}$ which is introduced by Salagean derivative operator (see\textsuperscript{20}).

5. When $s = 0$, $T_{s,n}^{m}$ reduces to $T_{s}^{0}$ which is introduced by generalized Salagean derivative operator (see\textsuperscript{21}).

6. When $m = 0$, $T_{s,n}^{m}$ reduces to $T_{s}^{0}$ which is introduced by generalized Ruscheweyh derivative operator (see\textsuperscript{22}).

7. When $s = 0$, $T_{s,n}^{m}$ reduces to $T_{s}^{0}$ which is introduced by Srivastava-attiya derivative operator (see\textsuperscript{23}).

8. When $k = 1, c = 0$, $N_{k,c}$ reduces to $N_{1,0}$ which is introduced by Alexander integral operator (see\textsuperscript{23}).

9. When $k = 1, c = \lambda$, $N_{k,c}$ reduces to $N_{1,1}$ which is introduced by Bernardi integral operator (see\textsuperscript{24}).

10. When $k = \sigma, c = 1$, $N_{\sigma,1}$ reduces to $N_{1,0}$ which is introduced by Jung-Kim-Srivastava integral operator (see\textsuperscript{25}).

Definition 6. (see\textsuperscript{26}) is symbolized by $Q$, a collection of all function $f$ that is univalent and
analytic on closed unit disk except $E(q)$ and denote $\bar{U} / E(q)$ where $\bar{U}$ is the closed unit disk $\bar{U} = U \cup \{w \in U : |z| \leq 1\}$

$$\text{and} \ E(q) = \left\{ \zeta : \zeta \in \partial U : \lim_{w \to \zeta} q(w) = \infty \right\}.$$  

(14)

Such that $\min|q'(\zeta)| = \alpha > 0$ for $\zeta \in \partial U / E(q)$ .

$Q(b)$ denote the subclass of $Q$ for which $q(0) = b$ with $Q_0 = Q_b$ and $Q_1 = Q_1$.

**Definition 7.** (see 6). Let $\Omega$ be a set in complex plane $\mathbb{C}$. Also, let $q \in Q$ and $n \in N\{1\}$, $N$ be the set of positive integers. The class of admissible function $\psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the next admissibility conditions:

$\psi(v, u, t, r, w) \in \Omega$, whenever

$v = q(\zeta)$, $u = \kappa \zeta q'(\zeta)$

$$\text{Re}\left(1 + \frac{t}{u}\right) \geq \kappa \text{Re}\left(1 + \frac{q'(\zeta)}{q'(\zeta)}\right)$$

and

$$\text{Re}\left(\frac{t}{u}\right) \geq \kappa^2 \text{Re}\left(\frac{q''(\zeta)}{q'(\zeta)}\right),$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq n$

**Definition 8.** (see 22). Let $\Omega$ be a set in complex plane $\mathbb{C}$, let $q \in Y[b, n]$ be in the subclass and $q'(w) \neq 0$. The class $\psi_n[\Omega, q]$ of admissible function $\psi_n[\Omega, q]$ consists of function $\psi : \mathbb{C}^4 \times \bar{U} \to \mathbb{C}$ that satisfies the next admissibility conditions:

$\psi(v, u, t, r, \zeta) \in \Omega$, whenever

$v = q(w), u = \frac{wq'(w)}{j}$

$$\text{Re}\left(1 + \frac{t}{u}\right) \geq \frac{1}{j} \text{Re}\left(1 + \frac{q'(w)}{q'(w)}\right)$$

and

$$\text{Re}\left(\frac{t}{u}\right) \leq \frac{1}{j^2} \text{Re}\left(\frac{wq''(w)}{q'(w)}\right),$$

where $w \in U$, $\zeta \in \partial U$ and $j \geq n \geq 2$.

**Lemma 1.** (see 6). Let $p \in Y[b, n]$ with $n \geq 2$ and $q \in Q(b)$ satisfy the next condition:

$$\text{Re}\left(\frac{q'(\zeta)}{q'(\zeta)}\right) \geq 0 \quad \text{and} \quad \left|\frac{wq'(w)}{q'(\zeta)}\right| \leq m$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq n$. If $\Omega$ is a set in complex plane $\mathbb{C}$, $\psi \in \psi_n[\Omega, q]$ and $\psi(\lambda(w), w^3\lambda'(w), w^2\lambda^3(w); w) \in \Omega$, then $\lambda(w) < q(w)$.

**Lemma 2.** (see 6). Let $\lambda \in Y[b, n]$ be in this subclass with $\psi \in \psi_n[\Omega, q]$. If $\psi(\lambda(w), w^3\lambda'(w), w^2\lambda^3(w); w) \in \Omega$, be univalent function in open unit disk $U$ and $\lambda \in Q(b)$ satisfying the next condition

$$\text{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0 \quad \text{and} \quad \left|\frac{w^3\lambda'(w)}{q'(w)}\right| \leq j,$$

where $w \in U$, $\zeta \in \partial U$ and $j \geq n \geq 2$, then,

$\Omega \subset \{\psi(\lambda(w), w^3\lambda'(w), w^2\lambda^3(w); w) : w \in U\}$,

it means that $q(w) < \lambda(w)$ ($w \in U$).

**Third-Order Differential Subordination Results Involving the Operator:** $N \mathcal{T}_{s, n, c}^k f(w)$

In this section, some results of differential subordination are obtained. Also studying the class of admissible functions involving the generalized derivative operator and Srivastava – Attiya operator defined by (10)

**Definition 9.** Let $\Omega$ be a set in complex plane $\mathbb{C}$and $q \in Q_0 \cap Y[0,1]$. The class $\Phi_n[\Omega, q]$ of admissible function consists of those function $\Phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy next admissibility conditions:

$\Phi(a, b, d, e ; w) \notin \Omega$, whenever

$a = q(\zeta), b = \kappa q'(\zeta) + sq(w)$,

$$\text{Re}\left(\frac{(s+1)^2d - s^2a}{(b+1)-ba} - 2s\right) \geq \kappa \text{Re}\left(\frac{q'(\zeta)}{q'(\zeta)}\right)$$

and

$$\text{Re}\left(\frac{(s+1)^2d - s^2a}{(b+1)-ba} - 2s\right) \geq \kappa \text{Re}\left(\frac{q'(\zeta)}{q'(\zeta)}\right),$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq 2$.

**Theorem 1.** Let $\Phi \in \Phi_n[\Omega, q]$be in this class. If the function $f$ belongs to $\mathcal{A}$ and $q$ belongs to $Q_0$ satisfying the following condition:

$$\text{Re}\left(\frac{q'(\zeta)}{q'(\zeta)}\right) \geq 0 \quad \text{and} \quad \frac{N_{s+3, n, c} f(w)}{q'(\zeta)} \leq \kappa,$$

(15)

then

$$N_{s+3, n, c} f(w) < q(w).$$

**Proof:** From the relation between (10) and (11), gives...
\[NT^m_k \cdot f(w) = \frac{w(NT^m_k \cdot f(w))'}{s} + sNT^m_k \cdot f(w) \quad (s+1) \]

Let \( g(w) \) be analytic in open unit disk defined by \( g(w) = NT^m_k \cdot f(w) \) \( (18) \)

Then
\[NT^m_k \cdot f(w) = \frac{w g'(w) + sg(w)}{(s+1)} \quad (19)\]
Based on that
\[NT^m_k \cdot f(w) = \frac{w^2 g''(w) + (2s+1)wg'(w) + s^2 g(w)}{(s+1)^2}\]
and
\[NT^m_k \cdot f(w) = \frac{w^3 g'''(w) + (3s+1)w^2 g''(w) + (3s^2 + 3s + 1)wg'(w) + s^3 g(w)}{(s+1)^3}\]

\[\quad \cdot (21)\]

Now, the transformation from \( C \) to \( \Omega \) defined by
\[\psi (\lambda (w), w^2 \lambda ''(w), w^3 \lambda '''(w); w) = \\]

Therefore, (16) gives
\[\psi (g(w), w g'(w), w^2 g''(w), w^3 g'''(w); w) \in \Omega \]

Such that
\[1 + \frac{s^2}{u} = \frac{(s+1)^2 d - s^2 a}{(b(s+1) - ba) - 2s} \quad \text{and} \quad r = \frac{(s+1)[(s+1)e - 3(s+1)d] + (2s^3a + 3s^2a)]}{u(s(b-a) + b)} \quad \text{in} \ \Omega \]

The admissibility condition for \( \psi \in \Phi_2[\Omega, q] \) given in Definition 7 with \( n = 2 \).
Thus, using (15) and Lemmal 1, implies that
\[NT^m_k \cdot f(w) < q(w) \]

This complete proof Theorem 1.

\[\varnothing (NT^m_k \cdot f(w), NT^m_k \cdot f(w), NT^m_k \cdot f(w), \Omega) \]

Then \( NT^m_k \cdot f(w) < q(w) \).

Proof: Since \( q_\rho \) is a univalent function in \( U \) therefore, \( E(\rho U) = \varnothing \) and \( q_\rho \in Q_0 \).

The class \( \varnothing = \Phi_n[\Omega, q_\rho] \) is an admissible class and from Theorem 1 yields
\[NT^m_k \cdot f(w) < q_\rho (w) \quad w \in U \]
The consequences certain by Corollary 1 is just a summary from the next subordination quality
\[q_\rho (w) < q(w) \quad w \in U \quad \text{Since} \quad NT^m_k \cdot f(w) < q_\rho (w) \quad \text{get} \quad NT^m_k \cdot f(w) < q(w) \quad w \in U \]
This is the end of the proof Corollary 1.

The next consequence is expansion of Theorem 1 for the case where the attitude of \( q(w) \) on \( \partial U \) is unknown.

**Corollary 1.** Let \( q(w) \) be an univalent function in open unit disk \( U \) with \( q(0) = 0 \) and let \( \Omega \subset C \).

Let \( \varnothing \in \Phi_n[\Omega, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho (w) = q(pw) \).
If the function \( f(w) \) belongs to \( \mathcal{A} \) and \( q_\rho \) satisfies :
\[\text{Re} \left( \frac{q''(\zeta)}{q'(\zeta)} \right) \geq 0 \quad \text{and} \quad \left| \frac{NT^m_k \cdot f(w)}{q'(\zeta)} \right| \leq \kappa \quad w \in U, \zeta \in \partial U/E(\rho U) \quad \kappa \geq 2 \] In case of \( \Omega = C \) this means it is a connected domain \( U \) then \( \Omega = \mu(U) \) for some conformal mapping in open unit disk \( U \) onto \( \Omega \). In this case, the class \( \Phi_n[\mu(U), q] \), it can be formed as \( \Phi_n[\mu, q] \).
This leads to the following immediate consequence of Theorem 1.

**Theorem 2.** Let \( \varnothing \in \Phi_n[\mu, q] \). If the function \( f \) belongs to \( \mathcal{A} \) and \( q \in Q_0 \) satisfies the following condition :
\[\text{Re} \left( \frac{q''(\zeta)}{q'(\zeta)} \right) \geq 0 \quad \text{and} \quad \left| \frac{NT^m_k \cdot f(w)}{q'(\zeta)} \right| \leq \kappa \quad \text{,} \]

(26) and
\[\varnothing (NT^m_k \cdot f(w), NT^m_k \cdot f(w), NT^m_k \cdot f(w), \Omega) \]

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\( \mathcal{N}_{T_{s+1,0}} f(w), \mathcal{N}_{T_{s+3,0}} f(w); w < \mu(w), \)

(27)

then \( \mathcal{N}_{T_{s,n}} f(w) < q(w) \) \( w \in U \).

**Corollary 2.** Let \( q(w) \) be an univalent function in open unit disk \( U \) with \( q(0) = 0 \). Also, let \( \Omega \subset \subset \mathbb{C} \) be a subset of the complex plane and \( \emptyset \in \Phi_n[\mu, q \rho] \) for some \( \rho \in (0, 1) \), where \( q \rho(w) = \emptyset (\mathcal{N}_{T_{s,n}} f(w), \mathcal{N}_{T_{s+1,n}} f(w), \mathcal{N}_{T_{s+2,n}} f(w); w) < \mu, \)

then \( \mathcal{N}_{T_{s,n}} f(w) < q(w) \) \( w \in U \). 

Proof: 

Case 1. To prove this Corollary by using Theorem 1, and have \( \mathcal{N}_{T_{s,n}} f(w) < q \rho(w) \) and since \( q \rho(w) < q(w) \) implies that \( \mathcal{N}_{T_{s,n}} f(w) < q(w) \).

**Theorem 3.** Let \( \mu(w) \) be a univalent function in the open unit disk. Also let \( \emptyset \subset \subset \mathbb{C} \times U \rightarrow \Gamma \) and \( \Psi \) be given by (24). Suppose that

\( \psi(q(w), wq'(w), w^2q''(w), w^3q'''(w); w) = \mu(w), \)

(28)

the differential equation has a solution \( q(w) \in Q_0 \) which satisfies (15). If \( f(w) \) belongs to \( A \) satisfies condition (27) and if

\( \emptyset (\mathcal{N}_{T_{s,n}} f(w), \mathcal{N}_{T_{s+1,n}} f(w), \mathcal{N}_{T_{s+2,n}} f(w), \mathcal{N}_{T_{s+3,n}} f(w); w) \)

is analytic in open unit disk \( U \), then

\( \emptyset \left( M e^{i \theta}, (\kappa + s) M e^{i \theta}, \frac{L + [(2s + 1)\kappa + s^2] M e^{i \theta}}{(s + 1)^2}, \frac{N + (3s + 3) M + [(3s + 2s + 3)\kappa + s^3] M e^{i \theta}}{(s + 1)^3}, w \right) \in \mathbb{C} \)

(29)

where \( \kappa(\lambda(w) = \rho w \) is any mapping from \( U \) into itself, \( g \rho(w) < q \rho(w) \) for \( \rho \in (0, 1) \). By \( \rho \rightarrow 1^- \), get \( q(w) > q \rho(w) \). Hence \( \mathcal{N}_{T_{s,n}} f(w) < q(w) \).

**Corollary 3.** Let \( \emptyset \in \Phi_n[\Omega, M] \) be in this class. If the function \( f(w) \) belongs to \( A \), then it satisfies the following conditions:

\( |\mathcal{N}_{T_{s+1,n}} f(w) | \leq \kappa M \) \( (w \in U, \kappa \geq 2; M > 0) \) and

\( \emptyset (\mathcal{N}_{T_{s,n}} f(w), \mathcal{N}_{T_{s+1,n}} f(w), \mathcal{N}_{T_{s+2,n}} f(w), \mathcal{N}_{T_{s+3,n}} f(w); w) \in \Omega \), then

\( |\mathcal{N}_{T_{s,n}} f(w) | < M \).

The special case of the above Corollary 3 when \( \Omega = q(U) = \{ y | y < M \} \), the class \( \Phi_n[\Omega, M] \) is simply symbolized by its \( \Phi_n[M] \). Corollary 4 can be rewritten as follows.

**Corollary 4.** Let \( \emptyset \in \Phi_n[M] \) be in this class. If \( f \in A \) is a function, it would satisfy the following conditions:

\( |\mathcal{N}_{T_{s+1,n}} f(w) | \leq \kappa M \) \( (w \in U, \kappa \geq 2; M > 0) \) and

\( \emptyset (\mathcal{N}_{T_{s,n}} f(w), \mathcal{N}_{T_{s+1,n}} f(w), \mathcal{N}_{T_{s+2,n}} f(w), \mathcal{N}_{T_{s+3,n}} f(w); w) \in \Omega \), then

\( |\mathcal{N}_{T_{s,n}} f(w) | < M \).
Corollary 5. A non-zero $s$ belongs to the complex plane $\mathbb{C}$, let $k \geq 2$ and $M > 0$. If $f \in \mathcal{A}$ it would satisfy the conditions:

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| \leq \kappa M$$

and

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) - N_{T_{s,\eta,\zeta}}^{m,k} f(w) \right| \leq \frac{M}{|s+1|}, \text{ then}$$

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| < M.$$  

Proof: Let $\Phi(a, b, d, e; w) = N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) - N_{T_{s,\eta,\zeta}}^{m,k} f(w) = b - a$ be equal and $\Omega = \mu(U)$, where

$$\mu(z) = Mw., \text{ (} M > 0 \text{).}$$

Use Corollary 3 that can be shown as $\Phi \in \Phi_n[\Omega, M]$. This means that the admissibility condition (29), is satisfied. This follows easily, because

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| \leq \kappa M,$$

and

$$\text{Re}\left\{ (s+1)^3e - (3s+6)(s+1)^2d - (s+1)^3a - (s+1)^3a \right\} + 3s^2 + 12s + 11 \geq \kappa^2 \text{Re}\left\{ \Re\left( \frac{s^2q''(\zeta)}{q'(\zeta)} \right) \right\}$$

where $w \in U$, $\zeta \in \partial U / E(q)$ and $\kappa \geq 2$.

Theorem 4. Let $\Phi \in \Phi_{n,1}[\Omega, q]$. If a function $f$ belongs to $\mathcal{A}$ and $q$ belongs to $Q$ satisfying the following condition:

$$\text{Re}\left\{ \left| \frac{s^2q'(\zeta)}{q'(\zeta)} \right| \right\} \geq 0 \text{ and}$$

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| \leq \kappa,$$  

(30)

and

$$\left\{ \left( N_{T_{s+z,\eta,\zeta}}^{m,k} f(w), N_{T_{s,\eta,\zeta}}^{m,k} f(w), N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right) : w \in U \right\} \subseteq \Omega.$$  

(31)

then

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| < q(w).$$

Proof: From the relation between (10) and (11), gives

$$N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) = \frac{w N_{T_{s,\eta,\zeta}}^{m,k} f(w)}{(s+1)}.$$  

(32)

Let $g(w)$ be analytic in open unit disk defined by:

$$g(w) = \frac{N_{T_{s+z,\eta,\zeta}}^{m,k} f(w)}{(s+1)}.$$  

(33)

Then

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| = \frac{wg'(w) + (s+1)g(w)}{(s+1)}.$$  

(34)

Based on that

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| = \frac{wg'(w) + (s+1)g(w)}{(s+1)}.$$  

(35)

and

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| \leq \frac{w^2g'(w) + (s+1)^2g(w)}{(s+1)}.$$  

(36)

Now, the transformation from $\mathbb{C}^+ \to \mathbb{C}$ defined by

$$a(v, u, t, r) = v, b(v, u, t) = \frac{u+(s+1)v}{s+1},$$

$$d(v, u, t, r) = \frac{t+(2s+3)u+(s+1)^2v}{(s+1)^2},$$

and

$$e(v, u, t, r) = \frac{r+(3s+2)t+(3s^2+9s+7)u+(s+1)^2v}{(s+1)^3}.$$  

(37)

Let $\Psi(v, u, t, r) = \Phi(a, b, d, e)$

$$= \Phi \left( \frac{u+(s+1)v}{s+1}, \frac{u+(s+1)v}{s+1}, \frac{t+(2s+3)u+(s+1)^2v}{(s+1)^2}, \frac{r+(3s+2)t+(3s^2+9s+7)u+(s+1)^2v}{(s+1)^3} \right) : w.$$  

(38)

The proof will make use of Lemma 1. Using (33) to (36), and from equation (39), get

$$\left| N_{T_{s+z,\eta,\zeta}}^{m,k} f(w) \right| = \frac{w^2g'(w) + (s+1)^2g(w)}{(s+1)}.$$  

(39)
\[ \psi(\lambda(w), w \lambda'(w), w^2 \lambda''(w), w^3 \lambda'''(w); w) = \left( \left( N_{\mathcal{T}_{s+2,n,c}}^m f(w), N_{\mathcal{T}_{s+3,n,c}}^m f(w), N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right) \right) \leq \kappa \] 
\[ \text{such that } \]
\[ 1 + \frac{t}{u} = (s+1)^d - (s+1)^2 - 2(s+1) \text{ and } \]
\[ \frac{r}{u} = (s+1)^2 e - (s+6+3(s+1)^2) - (s+1)^4 a + 6s^2 + 12s + 11 \]

And since the admissibility condition for \( \varnothing \in \Phi_{n,1}[\Omega, q] \equiv \Psi \in \Psi_s[\Omega, q] \), given in Definition 11 with \( n = 2 \).

Thus, using (30) and Lemma 1, it implies that
\[ N_{\mathcal{T}_{s+1,n,c}}^m f(w) < q(w). \]

In case \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = \mu(U) \) for some conformal mapping in open unit disk onto \( \Omega \). In this case, the class \( \bar{\Phi}_{n,1}[\mu, q] \) is rewritten as \( \bar{\Phi}_{n,1}[\mu, q] \). Thus can show below the result of Theorem 4.

**Theorem 5.** Let \( \varnothing \in \Phi_{n,1}[\mu, q] \) be in this class. If the function \( f \in \mathcal{A} \) and \( q \) belong to \( \mathcal{Q}_1 \), satisfying the following condition:
\[ \text{Re} \left( \frac{q'(\zeta)}{q(\zeta)} \right) > 0 \text{ and } \left| N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right| \leq \kappa \]
(42)
and
\[ \left\{ \begin{array}{l}
N_{\mathcal{T}_{s+1,n,c}}^m f(w), N_{\mathcal{T}_{s+2,n,c}}^m f(w), N_{\mathcal{T}_{s+3,n,c}}^m f(w) < \mu(w), \quad (w \in U)
\end{array} \right. \]
(43)

then
\[ N_{\mathcal{T}_{s+1,n,c}}^m f(w) < q(w), \quad w \in U. \]

In a particular case where \( q(w) = Mw \), where ( \( M > 0 \)), and in saw of Definition 11, the class \( \bar{\Phi}_{n,1}[\Omega, q] \) of admissibility functions and symbol by \( \bar{\Phi}_{n,1}[\Omega, M] \) is:

**Definition 12.** Let \( \Omega \) be a set in complex plane \( \mathbb{C} \) and \( M > 0 \). The class \( \bar{\Phi}_{n,1}[\Omega, M] \) of the admissible function consists of the function \( \varnothing : \mathbb{C} \times U \to \mathbb{C} \) such that
\[ \left( \begin{array}{l}
Me^{i\theta}, (k+1+s)Me^{i\theta}, L + [(2s+3)k+(s+1)^2]Me^{i\theta},
N + [(3s+6)L + [(2s+3)^2](s+1)^2]Me^{i\theta}, Z
\end{array} \right) \in \Omega \]
(44)

where
\[ w \in U, \quad \text{Re}(Le^{-i\theta}) \geq (k-1)kMm \text{ and } \text{Re}(Ne^{-i\theta}) \geq 0, \quad (\forall \theta \in \mathbb{R}, k \geq 2) \]

**Corollary 6.** If \( f \in \mathcal{A} \) and let \( \varnothing \in \Phi_{n,1}[\Omega, M] \) satisfies the following conditions:
\[ \left| N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right| \leq kM \quad (w \in U, k \geq 2; M > 0) \]
and
\[ \varnothing \left( N_{\mathcal{T}_{s+1,n,c}}^m f(w), N_{\mathcal{T}_{s+2,n,c}}^m f(w), N_{\mathcal{T}_{s+3,n,c}}^m f(w) ; w \right) \in \Omega, \]

then
\[ \left| N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right| < M. \]

The particular case of above Corollary 6 when \( \Omega \) = \( q(U) = \{ y : |y| < M \} \), the class \( \bar{\Phi}_{n,1}[\Omega, M] \) is simply denoted by \( \bar{\Phi}_{n,1}[M] \). Corollary 6 can be rewritten as shape.

**Corollary 7.** Let \( \varnothing \in \Phi_{n,1}[M] \) be in this class. If the function \( f \) belong to \( \mathcal{A} \) satisfying the following conditions:
\[ \left| N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right| \leq kM \quad (w \in U, k \geq 2; M > 0) \]
and
\[ \varnothing \left( N_{\mathcal{T}_{s+1,n,c}}^m f(w), N_{\mathcal{T}_{s+2,n,c}}^m f(w), N_{\mathcal{T}_{s+3,n,c}}^m f(w) ; w \right) \in \Omega, \]

then
\[ \left| N_{\mathcal{T}_{s+1,n,c}}^m f(w) \right| < M. \]

**Some Properties of the Third-Order Differential Subordination Results Involving the Operator: \( N_{\mathcal{T}_{s+1,n,c}}^m f(w) \)**

Note that, making use of the recurrence relation (12), certain differential subordination consequences associated with the operator \( N_{\mathcal{T}_{s+1,n,c}}^m f(w) \) are obtained. Because the proofs of the consequences contained in this part are similar to those from the prior part, they will be omitted.

**Definition 13.** Let \( q \in Q_0 \cap \mathbb{Y}[0,1] \) and \( \Omega \) be a set in complex plane \( \mathbb{C} \). The class \( \varnothing_{n}[\Omega, q] \) of admissible function consists of the function \( \varnothing : \mathbb{C} \times U \to \mathbb{C} \) that satisfies the following conditions:
\[ \varnothing(a, b, d, e; w) \notin \Omega, \quad \text{whenever} \]
$a = q(\zeta), \quad b = \frac{\kappa \tilde{q}(w) + \left(\frac{1 - \eta}{\eta}\right) q(w)}{\left(\frac{1}{\eta}\right)},$
\[
\frac{1}{\eta^2} d - \frac{1}{\eta^2} a \left(\frac{1}{\eta} - \frac{1}{\eta}\right)
\frac{1}{(b - a)} - 2 \left(\frac{1 - \eta}{\eta}\right) \geq \kappa \text{ Re} \left(\frac{\tilde{q}'(\zeta)}{q'(\zeta)}\right)
\]

\[
\frac{1}{\eta^2} \left(\frac{1}{\eta} - 3 \left(\frac{1}{\eta}\right) d + 2 \left(\frac{1 - \eta}{\eta}\right) a + 3 \left(\frac{1 - \eta}{\eta}\right)|b - a| + b\right)
\frac{1}{\eta^2} \left(\frac{1}{\eta} - 3 \left(\frac{1}{\eta}\right) d + 2 \left(\frac{1 - \eta}{\eta}\right) a + 3 \left(\frac{1 - \eta}{\eta}\right)|b - a| + b\right) \geq \kappa^2 \text{ Re} \left(\frac{\tilde{q}'^2(\zeta)}{q'(\zeta)}\right),
\]

where $w \in U, \zeta \in \partial U / E(q)$ and $\kappa \geq 2$.

**Theorem 6.** If the function $f \in A$. Let $\emptyset \in \Phi_n[\Omega, q]$ and $q$ that belongs to $Q_0$ satisfy the next condition:

\[
\text{Re} \left(\tilde{q}'(\zeta)q'(\zeta)\right) \geq 0 \quad \text{and} \quad \left|\frac{N_{n+1}^{m+1}f(w)}{q'(\zeta)}\right| \leq \kappa,
\]

(45)

and

\[
\emptyset(NT_{n+1}^m f(w), NT_{n+1}^{m+1} f(w), NT_{n+1}^{m+2} f(w), NT_{n+1}^{m+3} f(w); w) \in \Omega,
\]

(46)

then

\[
NT_{n+1}^{m+1} f(w) < q(w).
\]

In a particular case $\Omega \neq \mathbb{C}$. This means it is a simply connected domain, then $\Omega = \mu(U)$ for some conformal mapping in open unit disk $U$ onto $\Omega$. In this status, the class $\Phi_n[\mu(U), q]$ is written as $\Phi_n[\mu, q]$ . The next consequence is immediate of Theorem 6.

\[
\emptyset \left(\mu \left(\frac{1}{\eta}, \beta \right) f(w), \mu \left(\frac{1}{\eta}, \beta + 1\right) f(w), \mu \left(\frac{1}{\eta}, \beta + 2\right) f(w), \mu \left(\frac{1}{\eta}, \beta + 3\right) f(w); w \right) \in \Omega,
\]

(49)

where

\[
\mu(\kappa - 1) \kappa M \text{ and } \mu(\kappa - 1) \kappa M \geq 0, \quad (\forall \theta \in \mathbb{R}, \kappa \geq 2).
\]

**Corollary 8.** Let $\emptyset \in \Phi_n[\Omega, M]$ be in this class. If the function $f \in A$ that satisfies the following conditions:

\[
\left|N_{n+1}^{m+1} f(w)\right| \leq \kappa M \quad (w \in U), \quad \kappa \geq 2; M > 0 \quad \text{and}
\]

\[
\emptyset(NT_{n+1}^m f(w), NT_{n+1}^{m+1} f(w), NT_{n+1}^{m+2} f(w), NT_{n+1}^{m+3} f(w); w) \in \Omega
\]

then

\[
\left|N_{n+1}^{m+1} f(w)\right| \leq \kappa M
\]

The particular case of the above Corollary 8 when $\Omega = q(U) = \{y : |y| < M\}$ the class
Φ_{n}[Ω,M] simply symbolizes itΦ_{n}[M], Corollary 9 can be written as the following form.

**Corollary 9.** Let Φ ∈ Φ_{n}[M]. If ∈ A , satisfies the following conditions:

\[ |N^{T_{m+1}}_{S_{n,c}} f(w)| \leq κM \quad (w \in U, κ \geq 2; M > 0) \]

and

\[ Φ \left( N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) \right) ∈ Ω \]

then

\[ |N^{T_{m}}_{S_{n,c}} f(w)| < M . \]

**Definition 1.** Let q ∈ Q \cap Y_{1}[1,1]and Ω be a set in complex plane C. The class \( φ_{n,1}[Ω,q] \) of admissible function consists of those function \( φ : C \rightarrow U \rightarrow C \) that satisfies the next conditions:

\[ \phi (a,b,c,d,e; w) \in Ω, \]

whenever

\[ a = q(ζ), b = \frac{κq'(w) + \left( \frac{1 - η}{η} + 1 \right)q(w)}{\frac{1}{η}}, \quad \text{Re} \left( \left( \frac{2}{η} \right)^{2} - \left( \frac{1 - η}{η} + 1 \right)^{2} a \right) - 2(s + 1) \geq κ \text{Re} \left( \frac{q'(ζ)}{q'(ζ)} \right) \]

and

\[ \frac{1}{η} = \frac{3}{η}, \frac{2}{η} = \frac{1 - η}{η} + 1, \frac{3}{η} = \frac{1 - η}{η} + 1 \]

for

\[ \text{Re} \left( \frac{q'(ζ)}{q'(ζ)} \right) ≥ 0 \quad \text{and} \quad |N^{T_{m}}_{S_{n,c}} f(w)| \leq κ \]

(50)

and

\[ \left\{ \begin{array}{l}
N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) ; w \in U \end{array} \right\} \subset \Omega , \]

then

\[ \frac{N^{T_{m}}_{S_{n,c}} f(w)}{q(w)} < q(w) . \]

**Theorem 8.** Let \( φ \in φ_{n,1}[Ω,q] \) be in this class. If \( f \in \mathcal{A} \) and q belong to Q \_1 satisfying the following condition :

\[ \text{Re} \left( \frac{wq'(w) + msq(w)}{m(s + 1)} \right) \geq 0 \]

(51)

and

\[ \left\{ \begin{array}{l}
φ \left( N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) ; w \in U \right) \end{array} \right\} \subset \Omega , \]

\[ \left\{ \begin{array}{l}
N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) ; w \in U \end{array} \right\} \subset \Omega , \]

where \( w \in U, ζ \in \partial U / E(q) \) and \( κ ≥ 2 \).

**Main Results of the Third-Order Differential Superposition Involving the Operator:** \( N^{T_{m}}_{S_{n,c}} f(w) \)

In this part, the third order differential superposition for the operator \( N^{T_{m}}_{S_{n,c}} f(z) \) defined in (10) is obtained and proving many Theorems, for the purpose, consider the next class of admissible functions.

**Definition 16.** Let q ∈ Y[0,1] with \( q'(w) \neq 0 \) and Ω be a set in complex plane C. The class \( φ_{n,1}[Ω,q] \) of admissible function consists of those function \( φ : C \rightarrow U \rightarrow C \) that satisfies the next admissibility conditions:

\[ \left\{ \begin{array}{l}
φ (a,b,c,d,e; w) ∈ Ω, \end{array} \right\} \]

where

\[ a = q(ζ), b = \frac{wq'(w) + msq(w)}{m(s + 1)}, \]

\[ \text{Re} \left( \frac{wq'(w)}{q'(w)} \right) \geq 0 \quad \text{and} \quad |N^{T_{m}}_{S_{n,c}} f(w)| \leq j \]

(52)

and

\[ \left\{ \begin{array}{l}
φ \left( N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) ; w \in U \right) \end{array} \right\} \subset \Omega , \]

\[ \left\{ \begin{array}{l}
N^{T_{m}}_{S_{n,c}} f(w), N^{T_{m+1}}_{S_{n,c}} f(w), N^{T_{m+2}}_{S_{n,c}} f(w), N^{T_{m+3}}_{S_{n,c}} f(w) ; w \in U \end{array} \right\} \subset \Omega . \]

(53)
It means that \( q(w) < NT_{5;3c} f(w) \).

Proof. Let \( g(w) \) be a function defined by (18), and \( \Psi \) defined by (24). Since \( \phi \in \Phi_{n}^{1}[\Omega, q] \), from (25) and (53) yield,

\[
\Omega \subseteq \left\{ \phi(NT_{5;3c} f(w), NT_{5;3+1;3c} f(w)), \right. \\
\left. NT_{5;3+2;3c} f(w), NT_{5;3+3;3c} f(w); \right) w \in U \}
\]

From (22) and (23) deduce that the admissibility condition for \( \Psi \in \Psi_{2}[\Omega, q] \) given in Definition 8 with \( n = 2 \). Subsequently \( \psi \in \psi_{2}[\Omega, q] \) and, by using (53) and Lemma 2, get

\[
q(w) < g(w) \text{this equivalently,}
\]

\[
q(w) < NT_{5;3c} f(w) \quad w \in U
\]

In case of \( \Omega \neq \emptyset \) this means it is a connected domain, then \( \Omega = \mu(U) \) for some conformal mapping \( \mu(w) \) of open unit disk \( U \) on to \( \Omega \). In this case, the class \( \phi \in \Phi_{n}^{1}[\mu(U), q] \) is formed as \( \phi \in \Phi_{n}^{1}[\mu(U), q] \). The next is instant consequence of Theorem 10.

**Theorem 11.** Let \( \phi \in \Phi_{n}^{1}[\mu(U), q] \) and \( \mu \) be analytic in unit disk \( U \). If the function \( f \in \mathcal{A} \) and \( NT_{5;3c} f(w) \in \mathcal{Q}_0 \), and if \( q \in \psi_{2}[0, 1] \) with \( q(\cdot) \neq 0 \) satisfying the condition (52) and the function \( \phi(NT_{5;3c} f(w), NT_{5;3+1;3c} f(w), \right. \\
\left. NT_{5;3+2;3c} f(w), NT_{5;3+3;3c} f(w); w) \right) \) is univalent in unit disk \( U \), then

\[
M(w) < \phi(NT_{5;3c} f(w), NT_{5;3+1;3c} f(w), \\
NT_{5;3+2;3c} f(w), NT_{5;3+3;3c} f(w); w) \right)
\]

It means that \( q(w) < NT_{5;3c} f(w) \) \( w \in U \).

The next Theorem proves the presence of the best subordination of (54) for appropriate \( \phi \).

\[
\text{Re}\left( (s+1)^3 - 3(s+6)((s+1)^2 - (s+1)^2 a) - (s+1)^3 a \\
(s+1)(b - (s+1)a) \right) \leq \text{Re}\left( wq^{"}(w) \right) q^{"}(w)
\]

where \( w \in U \), \( \zeta \in \partial U \) and \( j \geq 2 \).

**Theorem 13:** Let \( \phi \in \Phi_{n}^{1}[\Omega, q] \). If \( f \in \mathcal{A} \) with \( NT_{5;3c} f(w) \) belongs to \( \mathcal{Q}_0 \), and \( q \) belongs to \( \psi_{2}[0, 1] \) with \( q'(\cdot) \neq 0 \) satisfying the condition:

\[
\text{Re}\left( \frac{wq'(w)}{q'(w)} \right) \geq 0 \quad \text{and} \quad \frac{NT_{5;3+1;3c} f(w)}{NT_{5;3c} f(w)} \leq j
\]

(56)

and the function \( \phi(NT_{5;3+1;3c} f(w), NT_{5;3+1;3c} f(w), \right. \\
\left. NT_{5;3+2;3c} f(w), NT_{5;3+3;3c} f(w); w) \right) \) is univalent in open unit disk \( U \), then

\[
\Omega \subseteq \left\{ \phi(NT_{5;3c} f(w), NT_{5;3+1;3c} f(w), \\
NT_{5;3+2;3c} f(w), NT_{5;3+3;3c} f(w); w) ; w \in U \right\}
\]

(57)

It means that \( q(w) < NT_{5;3c} f(w) \).

Proof: Let \( g(w) \) be a function defined by (33) and \( \Psi \) defined by (39). Since \( \phi \in \Phi_{n}^{1}[\Omega, q] \), from (40) and (57) yield,
\[ \Omega \subset \left\{ \left( N_T^{m,k}_{s,\eta,c} f(w), N_T^{m,k}_{s+1,\eta,c} f(w), N_T^{m,k}_{s+2,\eta,c} f(w), N_T^{m,k}_{s+3,\eta,c} f(w) ; w \right) ; w \in U \right\} \]

From (37) and (38), deduce that the admissibility condition for \( \Psi \) belongs to \( \Psi_2(\Omega, q) \) given in Definition 8 with \( n=2 \). Subsequently \( \psi \in \psi'_1(\Omega, q) \) and, by using (56) and Lemma 2, it implies that
\[ q(w) < \frac{N_T^{m,k}_{s,\eta,c} f(w)}{w}, w \in U. \]
This is the complete proof of Theorem 1.

In case of \( \Omega \neq \emptyset \) this means that it is a simply connected domain, then \( \Omega = \mu(U) \) for some conformal mapping \( \mu(w) \) of open unit disk \( U \) onto \( \Omega \). In this case, the class \( \phi \in \Phi_{n,1}[\mu(U), q] \) is formed as \( \phi \in \Phi'_{n,1}[\mu, q] \). The next is an instant consequence of Theorem 13.

**Theorem 14.** Let \( \phi \in \Phi'_{n,1}[\mu, q] \), and let \( \mu \) be analytic function in unit disk \( U \). If \( f \) belong to \( \mathcal{A} \) and \( q \) belong to \( Y_1[0,1] \) with \( q'(w) \) not equal to zero satisfying the condition:
\[ \Re \left( \frac{\mu(q')(w)}{q'(w)} \right) \geq 0 \] and
\[ \frac{N_T^{m,k}_{s,\eta,c} f(w)}{q'(w)} \leq j(58) \]
the function \( \Omega \subset \left\{ \left( N_T^{m,k}_{s,\eta,c} f(w), N_T^{m,k}_{s+1,\eta,c} f(w), N_T^{m,k}_{s+2,\eta,c} f(w), N_T^{m,k}_{s+3,\eta,c} f(w) ; w \right) ; w \in U \right\} \).

It means that
\[ q(w) < \frac{N_T^{m,k}_{s,\eta,c} f(w)}{w}. \]

**A Set of Sandwich – Type Results**

Collect Theorem 2 and Theorem 11. The next sandwich-type Theorem is obtained.

**Theorem 15:** Let \( \mu_1 \) and \( q_1 \) be analytic functions in \( U \), \( \mu_2 \) is an univalent function in open unit disk \( U \), \( q_2 \in Q_0 \) with \( q_1(0) = q_2(0) = 0 \) and \( \phi \in \Phi_{n,1}[\mu_2, q_2] \cap \Phi'_{n,1}[\mu_1, q_1] \). If the function \( f \in \mathcal{A} \) with

\[ N_T^{m,k}_{s,\eta,c} f(w) \in Q_0 \cap Y_0[0,1] \] and the function
\[ \Theta \left( N_T^{m,k}_{s,\eta,c} f(w), N_T^{m,k}_{s+1,\eta,c} f(w) \right) \] is univalent in open unit disk, and if the conditions (15) and (52) are satisfied
\[ \mu_1(w) < \Theta \left( N_T^{m,k}_{s,\eta,c} f(w), N_T^{m,k}_{s+1,\eta,c} f(w), N_T^{m,k}_{s+2,\eta,c} f(w), N_T^{m,k}_{s+3,\eta,c} f(w) ; w \right) < \mu_2(w), \]
it means that \( q_1(w) < N_T^{m,k}_{s,\eta,c} f(w) < q_2(w) \).

**Conclusion:**
In this work, in the case of applying the new operator to the geometric functions, it was concluded that they remain preserving their geometric features. If are put some restrictions on the functions that belong to these subclasses characteristics, can get new results inside the unit disk.

**Authors’ declaration:**
- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Anbar.

**Authors’ contributions statement:**
All authors contributed equally to the writing of this paper.

All authors read and approved the final manuscript.

**References**

التابعية التفاضلية من الرتبة الثالثة للدوال التحليلية التي تتضمن مؤثر الالتفاف

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الخلاصة:
في هذا البحث، من خلال استخدام المؤثر المعمم الجديد تم الحصول بعض نتائج التابعية التفاضلية من الدرجة الثالثة ونتائج التابعية التفاضلية العليا للدوال التحليلية، كذلك تم تقديم بعض نظريات من نوع الساندوج.

الكلمات المفتاحية: الدوال التحليلية، التابعية التفاضلية، ضرب هادمرد، الدوال احادية التكافؤ.