J. of Um-Salama for Science Vol 1 (2) 2004

B-splines Algorithms for Solving Fredholm Linear Integro-Differential Equations

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Date of acceptance 14/1/2004

Abstract

Algorithms using the second order of B-splines $\left[B_i^2(x)\right]$ and the third order of B-splines $\left[B_i^3(x)\right]$ are derived to solve $1^{st}, 2^{nd}$ and 3^{rd} linear Fredholm integro-differential equations (FIDEs). These new procedures have all the useful properties of B-spline function and can be used comparatively greater computational ease and efficiency.The results of these algorithms are compared with the cubic spline function.Two numerical examples are given for conciliated the results of this method.

1. Introduction

B-splines were introduced around $1940's$ in the context of approximation theory. They have been applied for geometric modeling since $1970's$. According schoenberg, B-splines basis and the letter B in B-spline stands for basis [1]. The general form of linear Fredholm integro-differential equation of order n is:

$$\left[D^n + \sum_{i=0}^{n-1} P_i(x) D^i\right] u(x) = f(x) + \int_a^b k\ (x,t)\ u(t)\ dt\ ,$$

$x \in [a,b]$

with two point boundary conditions:

$$\sum_{i=0}^{n-1} r_{j,i} \frac{d_i}{dx^i}\ u(a) + r_{j,i} \frac{d_i}{dx^i}\ u(b) = c_j\ ,$$

$j = 0,1,\ldots,n-1$

where $u(x)$ is the unknown function and $f(x), k(x,t)$ and $P_i(x), (i = 0,1,\ldots,n-1)$ are known continuous functions on $[a,b]$. This equation was treateded using collocation ,least square and Galerkin methods as well as linear programming method [2]. To facilitate the presentation of the material that followed, a brief review of some background on the B-splines and their relevant properties are given in the following section.

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2. Definitions and Properties of B-splines

Definition (1)

Let $t=(t_i)$ be a non decreasing squence " which may be finite , infinite or binfinite", the $i-th$ B-splines of order k for the knot sequence t is denoted by $B_i^k(t)$ are defined by the rule :

$$B_i^k(x)=(t_{i+k+1}-t_i)[t_i,\dots,t_{i+k+1}](t-x)_+^{k-1} \quad for \quad all \quad x\in\Re \tag{1}$$

It is used here to indicate that the $k-th$ divided difference of the function $(t-x)^{k-1}$ of the two variables t and x is to be taken by fixing x and considering $(t-x)^{k-1}$ as a function of t alone. It is used here to indicate that the k-th divided difference of the function $(t-x)^{k-1}$ of the two variables t and x is to be taken by fixing x and considering $(t-x)^{k-1}$ as a function of t alone.

Definition (2)

The B-splines of order 0 are defined by:

$$B_i^0(x)=\begin{cases}1 & t_i\le x<t_{i+1}\\ 0 & otherwise\end{cases}$$

obviously B_i^0 is discontinuous. However, it is continuous from the right at all points. Thus

$\lim_{x\to t_i^+} B_i^0(x)=1=B_i^0(t_i)$ and $\lim_{x\to t_{i+1}^-} B_i^0(x)=0=B_i^0(t_{i+1})$

Property (1)

Let $k\ge1$, and suppose $t_i<t_{i+k+1}$, then for all $x\in\Re$ we have

$$B_i^k(x)=\left(\frac{x-t_i}{t_{i+k}-t_i}\right)B_i^{k-1}(x)+\left(\frac{t_{i+k+1}-x}{t_{i+k+1}-t_{i+1}}\right)B_{i+1}^{k-1}(x) \tag{2}$$

with $k=1,2,\dots$; and $i=0,\pm1,\pm2,\dots$. This equation is called "***recurrence relation***"

Property(2)

We have

$B_i^k(x)>0$ *for* $x\in(t_i,t_{i+k+1})$, $k\ge0$

and

$B_i^k(x)=0$ *for* $x\notin[t_i,t_{i+k+1})$, $k\ge0$

these two properties are easily shown by induction.

Property(3)

If we take x in the open interval (t_r,t_s), then

$\sum_i B_i(x)=\sum_{i=r-k}^{s-1} B_i(x)=1$ for all $t_r<x<t_s$

3. The Second Order of B-splines [$B_i^2(x)$]

In this section, we find only the second order of B-spline which is called quadratic B-splines, because the first order is straight and like linear spline $L(x)$[3].

Take $k=2$ in eq. (2) , we get

$$B_i^2(x)=\left(\frac{x-t_i}{t_{i+2}-t_i}\right)B_i^1(x)+\left(\frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}\right)B_{i+1}^1(x)$$

$$=\left(\frac{x-t_i}{t_{i+2}-t_i}\right)\left[\left(\frac{x-t_i}{t_{i+1}-t_i}\right)B_i^0(x)+\left(\frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}\right)B_{i+1}^0(x)\right]+$$

$$\left(\frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}\right)\left[\left(\frac{x-t_{i+1}}{t_{i+2}-t_{i+1}}\right)B_{i+1}^0(x)+\left(\frac{t_{i+3}-x}{t_{i+3}-t_{i+2}}\right)B_{i+2}^0(x)\right]$$

since the distance between the knots t_i, $i=0,1,\dots,n$ is equal space , then $t_{i+1}-t_i=h$, $t_{i+2}-t_i=2h$. Therefore

$$B_i^2(x)=\frac{(x-t_i)^2}{2h^2}B_i^0(x)+\frac{(x-t_i)(t_{i+2}-x)+(t_{i+3}-x)(x-t_{i+1})}{2h^2}$$

$$B_{i+1}^0(x)+\frac{(t_{i+3}-x)^2}{2h^2}B_{i+2}^0(x)$$

That is:

$$B_i^2(x)=\begin{cases}\dfrac{(x-t_i)^2}{2h^2} & t_i<x\le t_{i+1}\\ \dfrac{(x-t_i)(t_{i+2}-x)+(t_{i+3}-x)(x-t_{i+1})}{2h^2} & t_{i+1}<x<t_{i+2}\\ \dfrac{(t_{i+3}-x)^2}{2h^2} & t_{i+2}\le x<t_{i+3}\\ 0 & t_{i+3}\le x \quad or \quad x\le t_i\end{cases}$$

since $s(x)=\sum_{i=0}^{n}u_iB_{i-2}^2(x)$, then for $n=3$ (see figure 1) we have

$$s(x)=u_0B_{-2}^2(x)+u_1B_{-1}^2(x)+u_2B_0^2(x)+u_3B_1^2(x)$$

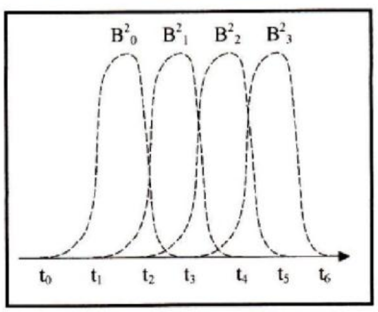

Figure (1)

where

$$B_{-2}^2(x)=\begin{cases}\dfrac{(x-t_{-2})^2}{2h^2} & t_{-2}<x\le t_{-1}\\ \dfrac{(x-t_{-2})(t_0-x)+-(t_1-x)(x-t_{-1})}{2h^2} & t_{-1}<x<t_0\\ \dfrac{(t_1-x)^2}{2h^2} & t_0\le x<t_1\end{cases}$$

$$B_{-1}^2(x)=\begin{cases}\dfrac{(x-t_{-1})^2}{2h^2} & t_{-1}<x\le t_0\\ \dfrac{(x-t_{-1})(t_1-x)+(t_2-x)(x-t_0)}{2h^2} & t_0<x<t_1\\ \dfrac{(t_2-x)^2}{2h^2} & t_1\le x<t_2\end{cases}$$

$$B_0^2(x)=\begin{cases}\dfrac{(x-t_0)^2}{2h^2} & t_0<x\le t_1\\ \dfrac{(x-t_0)(t_2-x)+(t_3-x)(x-t_1)}{2h^2} & t_1<x<t_2\\ \dfrac{(t_3-x)^2}{2h^2} & t_2\le x<t_3\end{cases}$$

and

$$B_1^2(x)=\begin{cases}\dfrac{(x-t_1)^2}{2h^2} & t_1<x\le t_2\\ \dfrac{(x-t_1)(t_3-x)+(t_4-x)(x-t_2)}{2h^2} & t_2<x<t_3\\ \dfrac{(t_4-x)^2}{2h^2} & t_3\le x<t_4\end{cases}$$

For second order $B-$splines, we shall see that $B_{i-2}^2(x)$ is zero at every knot except at:

$$B_{i-2}^2(t_{i-1})=\frac{t_{i-1}-t_{i-2}}{t_i-t_{i-2}} \qquad \text{and} \qquad B_{i-2}^2(t_i)=\frac{t_{i+1}-t_i}{t_{i+1}-t_{i-1}}$$

Therefore if $x\in[t_i\ ,t_{i+1}]$, we have

$$s(x)=A_i(x)u_i+B_i(x)u_{i+1}+C_i(x)u_{i+2} \tag{3}$$

where

$$A_i(x)=\frac{(t_{i+1}-x)^2}{2h^2},$$
$$B_i(x)=\frac{(x-t_{i-1})(t_{i+1}-x)+(t_{i+2}-x)(x-t_i)}{2h^2}$$
$$C_i(x)=\frac{(x-t_i)^2}{2h^2}$$

with $i=0,1,\ldots,n-1$.

4. The Third-order of B-splines $[\mathbf{B}_i^3(\mathbf{x})]$

The third order of B-splines is called cubic $B-$splines, where $s(x)$ denotes the cubic polynomial over each subinterval $[t_i,t_{i+1}]$.

The formula of $s(x)$ is obtained from eq.(2) :

$$B_i^3(x)=\left(\frac{x-t_i}{t_{i+3}-t_i}\right)B_i^2(x)+\left(\frac{t_{i+4}-x}{t_{i+4}-t_{i+1}}\right)B_{i+1}^2(x)$$
$$=\left(\frac{x-t_i}{t_{i+3}-t_i}\right)\left[\left(\frac{x-t_i}{t_{i+2}-t_i}\right)B_i^1(x)+\left(\frac{t_{i+3}-x}{t_{i+3}-t_{i+1}}\right)B_{i+1}^1(x)\right]+$$
$$\left(\frac{t_{i+4}-x}{t_{i+4}-t_{i+1}}\right)\left[\left(\frac{x-t_{i+1}}{t_{i+3}-t_{i+1}}\right)B_{i+1}^1(x)+\left(\frac{t_{i+4}-x}{t_{i+4}-t_{i+2}}\right)B_{i+2}^1(x)\right]$$

After some manipulations and by the same way in $B_i^2(x)$-splines , the following important equation is concluded:

$$s(x)=A_i(x)u_i+B_i(x)u_{i+1}+C_i(x)u_{i+2}+D_i(x)u_{i+3}$$
(4)

where

$$A_i(x)=\left(\frac{t_{i+1}-x}{6h}\right)^3$$

$$B_i(x)=\frac{(x-t_{i-2})(t_{i+1}-x)^2+(t_{i+2}-x)(x-t_i)+(t_{i+2}-x)(x-t_{i-1})(t_{i+1}-x)}{6h^3}$$

$$C_i(x)=\frac{(x-t_{i-1})^2(t_{i+1}-x)+(x-t_{i-1})(t_{i+2}-x)(x-t_i)+(t_{i+3}-x)(x-t_i)^2}{6h^3}$$

$$D_i(x)=\left(\frac{x-t_i}{6h}\right)^3$$

with $i=0,1,\ldots,n-1$

5. Solution of 1st and 2nd order linear FIDEs using B_i^2(x)-splines

First we consider the 1^{st} order linear FIDE of the form:

$$\frac{du(x)}{dx}+p(x)u(x)=f(x)+\int_a^b k(x,t)u(t)dt$$
(5)

with the boundary conditions $u(a)=u_0$ and $u(b)=u_n$.

Substituting eq.(3) into eq.(5) for $u(x)$ and with $x=t_i$ gives the following formula

$$(2+hp_i)u_{i+1}+(hp_i-2)u_i=2hf_i$$
$$+2h\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_i,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$
(6)

for $i=0,1,\ldots,n-1$.

the iterated integrals in eq.(6) are approximateded using trapezoidal rule. Now, the following formula of the 2^{nd} order linear FIDE is considerd:

$$\frac{d^2u(x)}{dx^2}+q(x)\frac{du(x)}{dx}+p(x)u(x)=f(x)+\int_a^b k(x,t)u(t)dt$$
(7)

where q,p,f and k are known continuous functions with the boundary conditions

$u(a)=u_0$ and $u(b)=u_n$.

After substituting eq.(3) into eq.(7) with $i=0,1,\ldots,n-1$ and $x=t_{i+1}$, yields:

$$u_i+\left(-2-hq_{i+1}+\frac{h^2}{2}p_{i+1}\right)u_{i+1}+\left(1+hq_{i+1}+\frac{h^2}{2}p_{i+1}\right)u_{i+2}$$
$$=h^2f_{i+1}+h^2\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_{i+1},t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$
(8)

6. Solution of 3rd order linear FIDEs using B_i^3(x)-spline

Consider the 3^{rd} order linear FIDE of the form:

$$u'''(x)+w(x)u''(x)+q(x)u'(x)+p(x)u(x)$$
$$=f(x)+\int_a^b k(x,t)u(t)dt$$
(9)

where w,q,p,f and k are known continuous functions with the boundary conditions $u(a),u(b)$ and $u'(b)$.

Put eq.(4) into eq.(9) to get:

$$(-6+6hw_i-3h^2q_i+h^3p_i)u_i+(18-12hw_i$$
$$+4h^3p_i)u_{i+1}+(-18+6hw_i+3h^2q_i+h^3p_i)u_{i+2}+6u_{i+3}$$
$$=6h^3f_i+6h^3\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_i,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}+D_j(t)u_{j+3}\right)dt$$
(10)

where $i=0,1,\ldots,n-1$ and $x=t_{i+1}$.

Note that the involved integrals in eq.(10) are approximated using trapezoidal rule with n subintervals. The result of eq.(10) is the ($n\times n+1$) matrix. Therefore with the aid of the central difference formula :

$$u'_n=\frac{u_{n+1}-u_{n-1}}{2h}$$

we can use $(2hu'_n+u_{n-1})$ instead of u_{n+1} in order to obtain $(n\times n)$ matrix .

7. The Algorithms

B_{21}-Spline Algorithm:

The numerical solution of 1^{st} order FIDEs using quadratic $B-$splines function is summerized as follows:

Step 1:
Input the boundary conditions u_0 and u_n then put $h=(b-a)/n$, $n \in \mathrm{N}$

Step 2:
Substitute $i=0,1,\ldots,n-1$ into eq.(6), yields

$$(2+hp_0)u_1+(hp_0-2)u_0=2hf_0$$
$$+2h\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_0,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$
$$\vdots$$
$$(2+hp_{n-1})u_{n1}+(hp_{n-1}-2)u_{n-1}=2hf_{n-1}$$
$$+2h\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_{n-1},t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$

Step 3:
Use Gauss elimination procedure to solve the above system and find $u_1,u_2,\ldots,u_{n-1.}$

B_{22}-Spline Algorithm

The numerical solution of 2^{nd} order FIDEs using quadratic $B-$splines function is summerized as follows:

Step 1:
Input the boundary conditions u_0 and u_n then put $h=(b-a)/n$, $n \in \mathrm{N}$

Step 2:
Substitute $i=0,1,\ldots,n-1$ into eq.(8), yields

$$u_0+\left(-2-hq_1+\frac{h^2}{2}p_1\right)u_1+\left(1+hq_1+\frac{h^2}{2}p_1\right)u_2$$
$$=h^2f_1+h^2\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_1,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$
$$\vdots$$
$$u_{n-1}+\left(-2-hq_n+\frac{h^2}{2}p_n\right)u_n+\left(1+hq_n+\frac{h^2}{2}p_n\right)u_{n+1}$$
$$=h^2f_n+h^2\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_n,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}\right)dt$$

Step 3:
Use Gauss elimination procedure to solve the above system and find $u_1,u_2,\ldots,u_{n-1.}$

B_{33}-Spline Algorithm:

The numerical solution of 3^{rd} order FIDEs using $B^3(x)$ splines function is summerized as follows:

Step 1:
Input the boundary conditions u_0, u_n and u'_n then put $h=(b-a)/n$, $n \in \mathrm{N}$

Step 2:
Put $i=0,1,\ldots,n-1$ into eq.(10), yields

$$\left(-6+6hw_0-3h^2q_0+h^3p_0\right)u_0+\left(18-12hw_0+4h^3p_0\right)u_1$$
$$+\left(-18+6hw_0+3h^2q_0+h^3p_0\right)u_2+6u_3$$
$$=6h^3f_0+6h^3\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_0,t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}+D_j(t)u_{j+3}\right)dt$$
$$\vdots$$
$$\left(-6+6hw_{n-1}-3h^2q_{n-1}+h^3p_{n-1}\right)u_{n-1}+\left(18-12hw_{n-1}+4h^3p_{n-1}\right)u_n$$
$$+\left(-18+6hw_{n-1}+3h^2q_{n-1}+h^3p_{n-1}\right)u_{n+1}+6u_{n+2}$$
$$=6h^3f_{n-1}+6h^3\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}k(x_{n-1},t)\left(A_j(t)u_j+B_j(t)u_{j+1}+C_j(t)u_{j+2}+D_j(t)u_{j+3}\right)dt$$

Step 3:
Solve this system by Gauss elimination procedure to find u_j's for $i=0,1,\ldots,n-1$.

8. Illustrative Exampes

Example (1):

Consider the following 2^{nd} order linear FIDE :

$$u''(x)=-\sin(x)-0.45969x+0.30116+\int_0^1(x-t)u(t)dt$$

with the boundary conditions $u(0)=0$ and $u(1)=\sin(1)$, while the exact solution is $u(x)=\sin(x)$. This problem is solved using the method derived in section 5, the solution of $u(x)$ is obtained as shown in Table 1 for $n=10,\quad h=0.1$ and $x_i = a + ih,\quad i = 0,1,\ldots,n$. For the comparison of computing accuracy (depend on the least square error L.S.E.), the solution obtained by using cubic spline [8] is also tabulated.

x	B_{33} − spline	cubic spline [8]	Exact
0.0	0.0	0.0	0.0
0.1	0.10050	0.103428	0.099833
0.2	0.198949	0.205341	0.198669
0.3	0.295749	0.304288	0.295520
0.4	0.389521	0.399335	0.389418
0.5	0.479367	0.489587	0.479426
0.6	0.564428	0.574200	0.564642
0.7	0.643891	0.653830	0.644218
0.8	0.717001	0.723412	0.717356
0.9	0.783068	0.786633	0.783327
1.0	0.841471	0.841471	0.841471
L.S.E	0.0000005	0.0005415	

Table (1) solution of example (1)

Example (2):

Consider the 3^{rd} order linear FIDE :

$$u'''(x) + (x^2+2)u'(x) - u(x) = f(x) + \int_0^1 (x+2t)u(t)dt$$

with the boundry conditions $u(0)=0$ and $u(1)=2.71828$,where $f(x) = e^x(x^2+2) - 1.718281x - 2$.

The solution of $u(x)$ for $0 \le x \le 1$ with $n=10,\quad h=0.1$ and $x_i = a + ih,\quad i = 0,1,\ldots,n$ is required. Using the method derived in section 6 , is obtained as shown in Table 2. Again , the results obtained by other method is also listed in Table 2 for comparison with the exact solution $u(x)=e^x$.

x	B_{33} − spline	cubic spline [8]	Exact
0.0	1.0	1.0	1.0
0.1	1.100822	1.093367	1.100822
0.2	1.214284	1.201841	1.214294
0.3	1.341251	1.334840	1.341251
0.4	1.482913	1.493106	1.482913
0.5	1.640566	1.671793	1.640566
0.6	1.815555	1.863659	1.815555
0.7	2.009296	2.062340	2.009296
0.8	2.223273	2.265665	2.223273
0.9	2.459052	2.478961	2.459052
1.0	2.718281	2.718282	2.718281
L.S.E	0.0003579	0.0073524	

Table (2) solution of example (2)

9. Conclusion

A method of using B-splines functions has been presented for solving FIDEs.It has been shown that the propersed method is comparable in accuracy with other method . The results show a marked improvement in the least square errors from which we conclde that:

- The B-splines functions give better accuracy and more stable than spline functions.
- As n "the number of knotes" is increased, the error term is decreased.

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خوارزميات B-splines
لحل معادلات فريدهولم الخطية التكاملية- التفاضلية

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الجامعة التكنولوجية-قسم العلوم التطبيقية

الخلاصة

تم اشتقاق خوارزميات لحل معادلات فريدهولم الخطية التكاملية- التفاضلية ذات الرتبة الأولى,الثانية والثالثة باستخدام $\left[B_i^2(x)\right]$ و $\left[B_i^3(x)\right]$. من الممكن ملاحظة كفاءة هذه الخوارزميات و سسهولة الحسابات فيها ,حيث تمت مقارنة نتائج هذه الخوارزميات مع نتائج دوال الثلمة التكعيبية من خلال بعض الأمثلة العددية.