First Order Nonlinear Neutral Delay Differential Equations

Hussain Ali Mohamad *

Date of acceptance 9/11/2004

Abstract

The author obtain results on the asymptotic behavior of the nonoscillatory solutions of first order nonlinear neutral differential equations.

Keywords. Neutral differential equations, Oscillatory and Nonoscillatory solutions.

INTRODUCTION

The oscillation theory for delay differential equations has been developed over twenty years, only in the last few years has much effort been devoted to the study of neutral differential equations, i.e. equations in which the highest order derivative of the unknown function appears both with and without delay which can be found in [1],[2],...In this paper we study the behavior of the solutions of the first order neutral delay differential equations of the type

$$\frac{d}{dt}[\,x(t)+P(t)\,x(\sigma(t))\,]+Q(t)\,f(x(\tau(t)))=0 \qquad (1)$$

where $P,Q:[t_0,\infty)\to R$ are continuous with neither P nor Q identically zero on any half line $[t_x,\infty)$, σ,τ are continuous and strictly increasing and $\lim_{t\to\infty}\sigma(t)=\infty$, $\lim_{t\to\infty}\tau(t)=\infty$,

$f:R\to R$ is continuous.we will assume that every solution $x(t)$ of (1) is continuous and nontrivial, such solution is said to be oscillatory if it has unbounded sequence of zeros, and is said to be nonoscillatory otherwise.

Properties of nonoscillatory solutions

In all results f will satisfy $u\,f(u)>0$ for $u\neq 0$. Throughout this paper we will assume that $Q(t)\geq 0$, and

$\sigma(t)<t,\ \tau(t)<t,\quad \sigma^2(t)=\sigma(\sigma(t))$.

Theorem 1.

Suppose that $P(t)\geq 0$, f is increasing, $\tau(t)<t$

$$\int^{\infty} Q(s)\,ds<\infty \qquad (2)$$

$$\int_{\alpha}^{\infty}\frac{1}{f(z)}dz=\infty,\quad \int_{-\alpha}^{-\infty}\frac{1}{f(z)}dz=\infty \qquad (3)$$

where $\alpha>0$, then all nonoscillatory solutions of (1) are bounded.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), and suppose that $x(t)>0$, $t\geq t_0$ (the case for $x(t)<0$

*Dr.-Teacher-College of Science for women-Department of Mathematics-University of Baghdad

is similar) and $x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$ and let $u(t) = x(t) + P(t)x(\sigma(t))$ so for t large enough we have only the case $u'(t) \ge 0$, $u(t) > 0$ for $t \ge t_2 \ge t_1$ to discuss:

$u(t) \ge x(t)$ then

$f(u) \ge f(u(\tau(t))) \ge f(x(\tau(t)))$ hence from (1) we get

$$\frac{u'(t)}{f(u(t))} \le Q(t)$$

Integrating the last inequality we obtain

$$\int_{t_2}^{t} \frac{u'(s)}{f(u(s))} ds \le \int_{t_2}^{t} Q(s)\, ds$$

$$\int_{u(t_2)}^{u(t)} \frac{1}{f(z)} dz \le \int_{t_2}^{t} Q(s)\, ds$$

According to (2) and (3) the last inequality implies that $u(t)$ must be bounded, but $x(t) \le u(t)$, so $x(t)$ is also must be bounded.

Corollary.

Suppose that all conditions of theorem 1 hold, then every unbounded solution of (1) are oscillatory.

Theorem 2.

Suppose that $-1 < -\lambda \le P(t) \le 0$

$$\int^{\infty} Q(t)\, dt = \infty$$

(4)

$f(u)$ is bounded away from zero if u is bounded away from zero, if $x(t)$ is nonoscillatory solution of (1) then either $|x(t)| \to \infty$ or $x(t) \to 0$ as $t \to \infty$.

Proof. Let $x(t)$ be nonoscillatory solution of (1), and assume that $x(t) > 0$, $t \ge t_0$ then $u'(t) \ge 0$, $t \ge t_1$

1. Suppose that $\liminf_{t\to\infty} x(t) \ne 0$ then $\exists\, c > 0$ such that

$x(\tau(t)) \ge c$, $t \ge t_2 \ge t_1$ and so $f(x(t)) \ge k > 0$, integrating eq.(1) we get

$$u(t) - u(t_2) = \int_{t_2}^{t} Q(s) f(x(\tau(s)))\, ds \ge k \int_{t_2}^{t} Q(s)\, ds$$

this implies that $u(t) \to \infty$ as $t \to \infty$ but $u(t) < x(t)$ then $x(t) \to \infty$.

2. Suppose that $\liminf_{t\to\infty} x(t) = 0$ since $u(t) < x(t)$ and $u'(t) \ge 0$ we have $u(t) \le 0$ and so

$x(t) \le -P(t)x(\sigma(t)) \le \lambda x(\sigma(t))$ then

$x(\sigma^{-1}(t)) \le \lambda x(t)$, $x(\sigma^{-2}(t)) \le \lambda^2 x(t)$ and by induction we get

$x(\sigma^n(t)) \le \lambda^n x(t$ and as $n \to \infty$ this implies that $x(t) \to 0$ as $t \to \infty$.

Example 1. Consider the neutral differential equation

$$\frac{d}{dt}\left[x(t) + \left(2 + e^{-\frac{t}{2}}\right)x\left(\frac{t}{2}\right)\right] - \left(2e^{-t} - e^{-\frac{3}{2}t}\right)\frac{1}{x\left(\frac{t}{2}\right)} = 0 \quad , \qquad (E\_1)$$

all conditions of theorem 1 are satisfy so all nonoscillatory solution of eq.(E_1) are bounded, for instance $x(t) = 2 - e^{-t}$ which is such bounded solution.

Example 2. Consider the neutral differential equation

$$\frac{d}{dt}\left[x(t) - \frac{1}{2}x\left(\frac{t}{2}\right)\right] - e^{t}\left(e^{-\frac{t}{2}} - \frac{1}{4}e^{-t}\right)x\left(\frac{t}{2}\right) = 0, \quad t \ge 0 \qquad (E\_2)$$

we can see that all conditions of theorem 2 are satisfied so all nonoscillatory solutions of eq.(E_2) are either tends to ∞ or 0, for instance $x(t) = e^{t}$ is such unbounded solution.

Note : One can established some other results when $Q(t) \le 0$, or $\tau(t) > t$, $\sigma(t) > t$.

References

1. Hussain A.Mohamad, Rudolf Olach: 1998,Oscillation of Second order neutral differential equation, international scientific conference of mathematics, ȝilina, 195-201.
2. Hussain A. Mohamad; 1999 ,Oscillation of n-th Order Linear Differential quation of Neutral Type, Iscm Herlany International Scientific Conference on Mathematics Proceedings, Ko□ice, 130-135.
3. Hussain A. Mohamad, Rudolf Olach: 2000,Asymptotic Behavior of Second Order Linear Neutral Differential quation, roceedings of international conference in EQDF for post graduate and researcher, Bratislava, Scam 86-92.
4. Jianhua, Shen 1994,: Lineariazed Oscillations for Differential Equations of Neutral Type, Math. Nachr. 176, 265-275.
5. S. R. Grace, B. S. Lalli: 1987, An Oscillation Criterion for certain Second rder trongly ub inear Differential quation, ournal o Math. Anal. And Appl.123,584-588.
6. Wan-Tong Li: 1999, Asymptotic Behavior of Nonoscillatory Solutions of n-th Order Neutral Nonlinear Equations with Oscillating Coefficient, Nonlinear Analysis 36,891-898.

المعادلات التفاضلية المحايدة غير الخطية من الرتبة الأولى

حسين علي محمد

دكتوراه–قسم الرياضيات–كلية العلوم للبنات–جامعة بغداد

المستخلص

في هذا البحث تم أيجاد نتائج جديدة لسلوك الحلول غير المتذبذبة للمعادلات التفاضلية المحايدة غير الخطية من النوع

$$\frac{d}{dt}[\,x(t)+P(t)\,x(\sigma(t))\,]+Q(t)\,f(x(\tau(t)))=0 \qquad (1)$$

حيث أن $P,Q:[t_0,\infty)\rightarrow R$ دوال مستمرة غير مكافئة للصفر في أي فـــترة σ,τ *دوال مســـتمرة ومـــتزايدة* بحيث $\lim_{t\to\infty}\tau(t)=\infty,\ \lim_{t\to\infty}\sigma(t)=\infty$ وان $f:R\rightarrow R$ مستمرة. أعطيت في هذا البحث بعض الشـــروط الكافية لضمان كون الحلول غير المتذبذبة للمعادلة (١) مقيدة وشروط كافية أخرى لتقارب الحلول الى الصفر أو تباعدها. تضمن البحث أيضاً بعض الأمثلة التوضيحية للنتائج.