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# An Analytic Solution for Riccati Matrix Delay Differential Equation using Coupled Homotopy-Adomian Approach

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## Abstract:

An efficient modification and a novel technique combining the homotopy concept with Adomian decomposition method (ADM) to obtain an accurate analytical solution for Riccati matrix delay differential equation (RMDDE) is introduced in this paper . Both methods are very efficient and effective. The whole integral part of ADM is used instead of the integral part of homotopy technique. The major feature in current technique gives us a large convergence region of iterative approximate solutions .The results acquired by this technique give better approximations for a larger region as well as previously. Finally, the results conducted via suggesting an efficient and easy technique, and may be addressed to other non-linear problems.

**Keywords:** ADM, Homotopy analysis method, RMDDE.

## Introduction:

In this study, a homotopy –Adomian approach is presented for addressing the following RMDDE:

$$\dot{G}(t)+G(t-\tau)D+D^{T}G(t)-G(t)PG(t)+S=0,\quad t\in[c,T], \qquad 1$$

by history condition:

$$G(t)=Z_0(t),\ c-\tau\le t\le c\ ,c \text{ is}$$

non negative integer number. 2

For many years, Liao implemented the initial thoughts of homotopy in topology to suggest a semi analytic approach for non-linear equations, called Homotopy Analysis Method (HAM); [1], [2]. According to homotopy of topology, the evidence of this approach exists with or without small physical parameters in the considered problem. It is an analytic approach for finding series solution of various kinds of non-linear problems [3]. This technique is a strong technique used to find the solution of non-linear phenomena [3, 4].

ADM since 1980s, is an example of a semi analytic approach which can be implemented to many kinds of equations including non-linear PDEs [5, 6, 7].

It gives an effective computational manner for equations and can get results as an infinite series converging to exact solution [6, 8, 9]. Authors in [8], implemented a new Adomian's technique for finding type of Riccati and compared with original ADM.

Also, Abazari [10], found the approximate analytical approach for RMDE using DTM. Khalid et al. in [11], implemented VIM to find the analytic solution to RMDE. A modification of DTM has been made by Shaher et al. [12], to find the approximate results to FHN neurons mode. Mohammed et al. [13] found that the solution methodology lies in generating an infinite conformable chain with reliable wave pattern by minimizing the residual error functions. Mohammed et al. [14], applied the DTM to obtain results of the fractional order Lu types. Approximate results for a SEIR epidemic type by homotopy method have been conducted [15]. In this study, a new implementation of homotopy-Adomian approach is implemented to obtain solve RMDDE.

A motivation in these methods provides the solution of DEs in infinite series. The most important advantages of these methods is its ability in providing us a continuous representation of the approximate solution, which allows better information of the results with whole interval. In addition it gives a highly accurate solution.

A feature for this technique as compared with the other methods for obtaining the solution, is that it is efficient not only for a small value of $x$ but also for a larger value. Absolute residual error for each component is conducted. A suggested approach showed that it is so efficient, accurately and gives more extend of the concourse region.

**2. Homotopy-Adomian Approach for RMDDE**

Firstly, the standard HAM for RMDDE is reviewed. For this reason, consider the nonlinear $m \times m$ of equation 1:

$$\dot{G}(t) + G(t-\tau)D + D^T G(t) - G(t)PG(t) + S = 0, \quad t \in [c,T], \qquad 3$$

where $\tau > 0$ is a constant and $c$ an arbitrary non-negative in $R$ .
where $G$, $P$, $S$ and $D$ are matrices $m \times m$;
$P = P^T, S = S^T, \tau = 1,$ and $G(t)$ is a matrix, with history condition:

$$G(t) = Z_0(t), \quad t \in [c-\tau, c]. \qquad 4$$

Now, by the method of successive for delay, generally to every time step $[c + i\tau, c + (i+1)\tau]$:

$$\dot{G}(t) + D^T G(t) - G(t)PG(t) + (S(t) + Z_i(t-\tau)D) = 0, \qquad 5$$

$$\dot{G}(t) + D^T G(t) - G(t)PG(t) + S^*(t) = 0, \qquad 6$$

where $S^*(t) = S(t) + Z_i(t-\tau)D$. Suppose $\mu(t;s)$ is a matrix function of homotopy, so:

$$V_m(\phi) = \frac{1}{m!} \frac{\partial^m \mu}{\partial s^m}\bigg|_{s=0},$$

is $m$th- order derivative $\mu(t;s)$, $m \geq 0$. The zero order formulation of matrix equation is:

$$(1-s)\mathrm{L}[\mu(t;s) - G_0(t)] = s\hbar N[\mu(t;s)], \qquad 7$$

and $m$th-order deformation matrix equation related to (3) can see:

$$\mathrm{L}[G_m(t) - \chi_m G_{m-1}(t)] = \hbar V_{m-1}(N[G]), \quad m \geq 1, \qquad 8$$

$$N[G] = \dot{G}(t) + D^T G(t) - G(t)PG(t) + S^*(t), \qquad 9$$

where $\mathrm{L}[G] = \dot{G}(t),$

$$\chi_m = \begin{cases} 0, m \leq 1, \\ 1, m > 1. \end{cases},$$

$$G(t) = \sum_{i=0}^{\infty} G_i(t) s^i, \qquad 10$$

$$R_m G_{m-1} = \frac{\partial G_{m-1}}{\partial t} + D^T G_{m-1} - \sum_{i=0}^{m-1} G_i P G_{m-1-i} + (1-\chi_m)(S^*). \qquad 11$$

Now, the $m$th-order formulation matrix equation 11 after addressing the inverse operator with given history condition, for $m \geq 1$ will transform to:

$$\begin{aligned} G_m(t) &= \chi_m G_{m-1} + \hbar \int_0^t \left( \dot{G}_{m-1}(u) + D^T \dot{G}_{m-1}(u) - \sum_{i=0}^{m-1} G_i(s) P G_{m-1-i}(u) + (1-\chi_m)(S^*) \right) du \\ &= \chi_m G_{m-1} + \hbar \left[ G_{m-1}(t) - (1-\chi_m) G_{m-1}(0) \right] + \hbar \int_0^t \left( D^T G_{m-1}(u) - \sum_{i=0}^{m-1} G_i(u) P G_{m-1-i}(u) + (1-\chi_m)(S^*) \right) du, \\ &= (\chi_m + \hbar) G_{m-1}(t) - \hbar(1-\chi_m) G_{m-1}(0) + \hbar \int_0^t \left( D^T G_{m-1}(u) - \sum_{i=0}^{m-1} G_i(u) P G_{m-1-i}(u) + (1-\chi_m)(S^*) \right) du. \end{aligned} \qquad 12$$

Now, ADM for solving RMDDE is addressed. For this reason, hence the nonlinear RMDDE $m \times m$ of the form:

$$\dot{G}(t) + G(t-\tau)D + D^T G(t) - G(t)PG(t) + S(t) = 0, \quad t \in [c,T], \qquad 13$$

with history condition: $G(t) = Z_0(t),$ $t \in [c-\tau, c].$

Consider with the method of successive for delay, generally for every time step $[c + i\tau, c + (i+1)\tau], \; i = 0,1,\ldots,N; \; N \in \mathrm{R}$

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$\dot{G}(t)+D^T G(t)-G(t)PG(t)+(S(t)+Z_i(t-\tau)D)=0,$

$\dot{G}(t)+D^T G(t)-G(t)BG(t)+S^*(t)=0,$ 14

where we set $S^*(t)=S(t)+Z_i(t-\tau)D,$

with condition history:

$G(c)=Q.$ 15

when $Q$ is a matrix $m\times m$ and $G(t)$ is a matrix $m\times m$ assumed to be bounded, $\forall t\in[c,T]$, $\left|g_{ij}(t)\right|\le M,\ c\le t\le T,\ G(t)=(g_{ij}(t))_{n\times n},$ the noise term $N(G)=GPG,$ has polynomial matrices:

$$N(G)=GPG=\sum_{n=0}^{\infty}F_n\ ,$$

and $F_n$ can be express:

$F_0(t)=G_0(t)PG_0(t)$

$F_1(t)=G_0(t)PG_1(t)+G_1(t)PG_0(t)$

$\vdots$

$$F_n(t)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i G_i(t)\right)P\left(\sum_{i=0}^{\infty}\lambda^i G_i(t)\right)\right]\Bigg|_{\lambda=0},$$

16

$F_n(t)$ is polynomial Adomian matrices. Where:

$$W_n=\sum_{i=0}^{n}G_i, \qquad 17$$

The implementation of ADM to the matrix equation 13 is:

$$G(t)=\sum_{i=0}^{\infty}G_i(t)\ ,$$

and

$$G_0(t)=Q+L^{-1}(-S^*(t)),$$

$$G_i(t)=L^{-1}(-F^T G_{i-1})+L^{-1}F_{i-1},\ i\ge 1, \qquad 18$$

Currently, the whole integral for HAM in equation 12 is replaced by integral of ADM in equation 18. In this case, more accurate solution, efficient, and more extend of the concourse area is obtained compare with the obtained by equation 12, means that:

$$G_{m-1}(t)=(\chi_m+\hbar)G_{m-1}(t)-\hbar(1-\chi_m)G_{m-1}(0)+\hbar\int_0^t\left((-F^T G_{i-1})+F_{i-1}\right)ds$$

, $i,m=1,2,\ldots$ . 19

### 3. Numerical Example

Now, the homotopy-Adomian approach presented above to RMDDE is applied. In the following example, the term of delay in the other part. It can be seen that this technique is very strong, effect and reliable. Hence the non-linear $2\times 2$ RMDDE:

$$\dot{G}(t)+G(t)D+D^T G(t)-G(t)PG(t-\tau)+S=0,$$

$\tau=1,\ t_0=0,$ 20

$$D=\begin{pmatrix}2&1\\1&2\end{pmatrix},\ P=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ S=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

and stationary $G_0(t)=\begin{pmatrix}t&0\\0&1\end{pmatrix},\ -1\le t\le 0.$

Now, by using method of successive:

$$\dot{G}(t)+G(t)D+D^T G(t)-G(t)PG_0(t-1)+S=0,$$

21

$$\begin{pmatrix}\dot{x}(t)&\dot{y}(t)\\\dot{z}(t)&\dot{w}(t)\end{pmatrix}+\begin{pmatrix}x(t)&y(t)\\z(t)&w(t)\end{pmatrix}\begin{pmatrix}2&1\\1&2\end{pmatrix}+\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}x(t)&y(t)\\z(t)&w(t)\end{pmatrix}+\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}x(t)&y(t)\\z(t)&w(t)\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}t-1&0\\0&1\end{pmatrix}\right)=\begin{pmatrix}0&0\\0&0\end{pmatrix},$$

$$\begin{pmatrix}4x(t)+\dot{x}(t)+y(t)+z(t)+x(t)(t-1)+1 & w(t)+x(t)+3y(t)+\dot{y}(t)\\ w(t)+x(t)+4z(t)+\dot{z}(t)+z(t)(t-1) & 3w(t)+\dot{w}(t)+y(t)+z(t)+1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}. \qquad 22$$

In this example, it can be noted that the delay is in the noise part [equation 21], it can also be observed that the delay part disappears after addressing the method of successive [system 22], which can be considered as RMDE. Then apply the HAM-ADM approach to system 22 for each equation of with initial condition $G_0(t)=\begin{pmatrix}0&0\\0&1\end{pmatrix},$ with Maple program, it obtained first twentieth iterate.

**Table 1. Numerical results of HAM-ADM for RMDDE of twenty iterate**

| *t* | *Absolute residual error of HAM-ADM for the first component* $x^{20}{}_{11}(t)$ | *Absolute residual error of HAM-ADM for the second component* $x^{20}{}_{12}(t)$ | *Absolute residual error of HAM-ADM for the third component* $x^{20}{}_{21}(t)$ | *Absolute residual error of HAM-ADM for the fourth component* $x^{20}{}_{22}(t)$ |
|---|---|---|---|---|
| 0.1 | $6.93889\times10^{-18}$ | 0 | $5.55112\times10^{-17}$ | 0 |
| 0.2 | $1.38778\times10^{-17}$ | 0 | $1.11022\times10^{-16}$ | $6.66134\times10^{-16}$ |
| 0.3 | $3.4177\times10^{-14}$ | $2.93099\times10^{-14}$ | $3.33067\times10^{-14}$ | $2.95319\times10^{-14}$ |
| 0.4 | $9.0021\times10^{-12}$ | $7.49312\times10^{-12}$ | $9.0028\times10^{-12}$ | $7.49467\times10^{-12}$ |
| 0.5 | $7.12908\times10^{-10}$ | $5.67208\times10^{-10}$ | $7.12984\times10^{-10}$ | $5.67268\times10^{-10}$ |
| 0.6 | $2.60291\times10^{-8}$ | $1.98031\times10^{-8}$ | $2.60321\times10^{-8}$ | $1.98053\times10^{-8}$ |
| 0.7 | $5.57142\times10^{-7}$ | $4.05514\times10^{-7}$ | $5.57212\times10^{-7}$ | $4.05564\times10^{-7}$ |
| 0.8 | $8.0691\times10^{-6}$ | $5.62152\times10^{-6}$ | $8.07019\times10^{-6}$ | $5.62227\times10^{-6}$ |
| 0.9 | $8.67334\times10^{-5}$ | $5.78689\times10^{-5}$ | $8.67461\times10^{-5}$ | $5.78772\times10^{-5}$ |
| 1.0 | $7.36974\times10^{-4}$ | $4.71195\times10^{-4}$ | $7.37091\times10^{-4}$ | $4.71268\times10^{-4}$ |

## Conclusion:

A homoptopy-Adomian approach is discussed to obtain RMDDE. All results from the table above indicate that the suggested approach award highly precise results in extended area. Results conducted by this approach award better approximations for whole interval time as well as the neighborhood of the initial condition.

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## Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

## Authors' contributions:

Khalid Hammood AL-Jizani. and *Jawad Kadhim K. Al-Delfi* contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript

## References:

1. Belal B, Mohamed S, Ishak H. Approximate solutions of singular two-point BVPs by modified homotopy analysis method. Phys. Lett. A. 2008; 372 (22): 4062-4066.
2. Sami Al, Mohd Salmi Md, Alomari A. Numerical Solution of Fractional-Order HIV Model Using Homotopy Method. DDNS. 2020; 2020: 1-13.
3. Alao S, Oderinu R, Akinpelu F, Akinola E. Homotopy Analysis Decomposition Method for the Solution of Viscous Boundary Layer Flow Due to a Moving Sheet. J. Adv. Math. Comp. Sci.. 2019; 32(5): 1-7.
4. AL-Jawary M, Rhahdi G, Ravnik J. Boundary-domain Integral Method and Homotopy Analysis Method for Systems of Nonlinear Boundary Value Problems in Environmental Engineering. Arab. J. Basic Appl. Sci. 2020; 27(1): 121-133.
5. Chi-Min L, Ray-Yeng Y. Application of Adomian Decomposition Method to Bounded and Unbounded Stokes' Problems. Math. Probl. Eng. 2018; 2018: 1-11.
6. Mohammad A. Adomian Decomposition Method with Modified Bernstein Polynomials for Solving Nonlinear and Volterra Integral Equations. Math. Stat. J. 2020; 8(3): 278-285.
7. Al-Mazmumy M, Almuhalbedi S. Modified Decomposition Method by Adomian and Rach for Solving Nonlinear Volterra Integro Differential Equations. Nonlinear Analy. Differ. Equ. 2017; 5(4): 157-170.
8. Rasool S, Hassan K, Muhammad A, Poom K. Application of Laplace-Adomian Decomposition Method for the Analytical Solution of Third-Order Dispersive Fractional Partial Differential Equations. Entropy. 2019; 21(4): 1-17.
9. Béyi B, Justin L, Joseph B, Gabriel B. Application of the Adomian Decomposition Method (ADM) for Solving the Singular Fourth-Order Parabolic Partial Differential Equation. J. Appl. Math. Phys. 2018; 6(7): 1476-1480.
10. Abazari, R. Solution of Riccati types matrix differential equations using matrix differential transform method, J. Appl. Math. Informatics,2009; 27(5-6): 1133 -1143.
11. Khalid H, Noor A, Fadhel S. Variational Iteration Method for Solving Riccati Matrix Differential Equations. Indones. J. Electr Eng. Comput. Sci. 2017; 5(3): 673-683.
12. Shaher M, Asad F, Mohammed A. Analytical Study of Fractional-Order Multiple Chaotic FitzHugh-Nagumo Neurons Model Using Multistep

Generalized Differential Transform Method. Abs. Appl. Analysis. 2014;2014(27): 1-10.
13. Mohammed A, Omar A, Shaher M. Numerical computations of coupled fractional resonant Schrödinger equations arising in quantum mechanics under conformable fractional derivative sense. Phys. Scripta. 2020; 95(7): 1-9.
14. Mohammed A, Asad F, Omar A, Nabil S. A Novel Multistep Generalized Di§erential Transform Method for Solving Fractional-order L¸ Chaotic and Hyperchaotic Systems. 2015; 19(4): 713-724.
15. Mohammed H, Ghaleb N. On the Homotopy Analysis Method for Fractional SEIR Epidemic Model.Res. J. Appl. Sci. Eng. Technol. 2014; 7(18): 3809-3820.

# الحل التقريبي لحل معادلة المصفوفات التفاضلية الاعتيادية التباطئية لمعادلة ريكاتي باستخدام طريقتي الهوموتوبي و الادوميان

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**الخلاصة:**

في هذا البحث تناولنا طريقة فعالة وجديدة وهي الدمج بين طريقتي الهوموتوبي و الادوميان مع إستخدام مفهوم المعادلات التفاضلية الاعتيادية التباطئية لحل معادلة المصفوفات التباطئية لمعادلة ريكاتي والحصول على حل تقريبي دقيق جدا قليل الخطأ ويقترب من الحل المظبوط او الحل الحقيقي. في هذه الطريقة تم الكشف على نتائج ادق خلال فترة التأخير التي مرت بها المعادلة. أيضا في هذه الطريقة ان الاقتراب للحل الحقيقي يأخذ منطقة او فترة اوسع وكلما استمرينا ب التكرارا حيث نحصل على نتائج دقيقة جدا ويكون الخطأ جدا صغير.

**الكلمات المفتاحية:** طريقة الادوميان، طريقة الهوموتوبي، معادلة ريكاتي التفاضلية الاعتيادية التباطئية للمصفوفات