ON SOME TYPES OF ALMOST-PERIODIC POINT IN BI-TOPOLOGICAL DYNAMICS

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ABSTRACT
In this paper We introduce some new types of almost bi-periodic points in topological bitransformation groups and their effects on some types of minimality in topological dynamics.

INTRODUCTION
Let X be a topological space, A be any subset of X, the sets $\overline{A}$, $\overline{A}^c$ denote the interior, closure and complement of A respectively. A is called semi-open in X if $A \subseteq \overline{A}$ [10] then A is called semi-closed [6] such that every open set in X is semi-open set and every closed set in X is semi-closed set. A subset $N_x$ of X is called a semi-neighborhood (SNbd) of a point $x \in X$ if there exists a semi-open set A in X s.t $x \in A \subseteq N_x$ [2], every nbhd is semi-nbhd. The smallest semi-closed set containing A is called the semi-closure of A (scl A) [2] s.t A is semi-closed set iff $A = \text{scl} A$ and scl $A \subseteq \overline{A}$. p $\in X$ is called a semi-limit point of A if $A \cap (U \setminus \{p\}) \neq \emptyset$ for each semi-open set $U \subseteq X$ which is containing p [2]. A family $@$ of semi-open subsets of X is called a semi-open cover of X if X is a subset of the union of elements of $@$ and X is called a semi-compact space if every semi-open cover of X containing a finite sub cover [3]. every semi-compact set is compact, X is called locally semi-compact space if for every $x \in X$ there exist a s.nbd Ux is compact set. If G is a topological group and $A \subseteq G$ then A is called a left syndatic set if there exists a compact set $K \subseteq G$ s.t. $G = AK$ and $B \subseteq G$ is called a right syndatic set if $G = KB$ [4], is called left semi-syndatic set if there exists a semi-compact set $K \subseteq G$ s.t $G = AK$, similar B is called right semi-syndatic set if $G = KB$ [1], every semi-syndatic set is syndatic. If $(X, G, \pi)$ is a right topological transformation group $(0, H, X)$ is a left topological transformation group then $(H, X, G)$ is called right-left topological bitransformation group $(T.B.G)$ if satisfy the condition: $(hx)_{x}(h, x, g)_{x}(h, (x, g))_{x} = (h, (x, g))_{x} = (h, x)_{x}$ for every $x \in X$, $h \in H, g \in G$.

Definition 1: Let $(H, X, G)$ be a T.B.G and $A \subseteq X$ then A is called a bi-invariant set under G and H if $HAG \subseteq A$.

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Definition 2: Let \( (I, X, G) \) be a T.B.G. and \( x \in X \) then the set \( HxG=\{hxg|\ \ g \in G, h \in H\} \) is called the bi-orbit of \( x \) under \( G \) and \( H \), the set \( HxG \) is called the bi-orbit closure of \( x \) under \( G \) and \( H \) and \( \text{sel} \ (HxG) \) is called an bi-orbit semi-closure of \( x \) under \( G \) and \( H \).

Lemma 1: Let \( (I, X, G) \) be a T.B.G., \( A \subseteq X \) and \( g \in G, h \in H \) then \( hAg = hAg \).

Proof: It is obvious.

Lemma 2: Let \( (I, X, G) \) be a T.B.G., \( A \subseteq X \) and \( A \) is bi-invariant set then \( A \) is bi-invariant set.

Proof: By using lemma 1.

Proposition 1: Let \( (I, X, G) \) be a T.B.G. then the following statements are true:
1) If \( x \in X \) then the bi-orbit of \( x \) is a smallest bi-invariant subset of \( X \) contain \( x \).
2) If \( x \in X \) and \( y \in IxG, \) then \( HxG=HyG \).
3) If \( x \in X \) then the bi-orbit closure of \( x \) is a smallest bi-invariant subset of \( X \) contain \( x \).
4) If \( x \in X \) and \( y \in IxG \) then \( IxG=HyG \).
5) The class of all bi-orbit closure under \( G \) and \( H \) covered \( X \).

Proof:
(1) Let \( IxG \) be the bi-orbit of \( x \) then \( HIXG=HXG \) i.e. \( HxG \) is bi-invariant set. Let \( A \subseteq HxG \) and \( \text{A is bi-invariant set} \). s.t. \( x \in A \), let \( a \in HxG - A \) then there exist a \( g \in G \) s.t. \( a=thxg \) but this contradiction.
(2) Since \( y \in IxG \) then there exist a \( g \in G \) s.t. \( y=hxg \) i.e. \( x^{-1}g^{-1}x \) this means \( x \in IxG \) and \( HxG \subseteq IxG \) then \( HxG=HyG \).
(3) Since \( HxG \subseteq IxG \) then \( HxG \) is a smallest closed set contain \( x \) and the closed of bi-invariant set is bi-invariant set then \( HxG \) is smallest closed bi-invariant set contain \( x \).
(4) It is obvious.

(5) Let \( a=[HxG|\ x \in X] \) then \( X=\cup HxG \subseteq IxG=Ua \).

Definition 3: Let \( (I, X, G) \) be a T.B.G. and \( M \subseteq X \) then \( M \) is called a \( b_1 \)-minimal set \( (b_1 \)-minimal set) in \( T.B.G \) if satisfy the following conditions:
1) \( M \neq \emptyset \), \( M \) is closed \{semi-closed\} bi-invariant set.
2) \( M \) is not containing closed \{semi-closed\} bi-invariant subset.

Proposition 2: Let \( (I, X, G) \) be a T.B.G. and \( M \subseteq X \) then the following statements are equivalent:
(1) \( M \) is \( b_1 \)-minimal set.
(2) \( M \neq \emptyset \), \( M \) is \( IxG \) for every \( x \in M \).
(3) \( M \neq \emptyset \), \( M \) is closed and \( M \subseteq IxG \) for every nonempty open subset \( U \) of \( M \).

Proof:
(1)\( \Rightarrow \) (2) Since \( M \) is \( b_1 \)-minimal set then \( M \) is \( b_1 \)-invariant set and closed, this means \( HxG \subseteq M \) and \( IxG \subseteq M \), if \( HxG \neq M \) means \( M \) is not \( b_1 \)-minimal set then \( HxG=M \).\( \Rightarrow \) (1) Since \( M \neq \emptyset \) and \( M \neq IxG \) then \( M \) is \( b_1 \)-invariant set if \( A \subseteq M \) closed \( b_1 \)-invariant then there exist a \( e \in A \) s.t. \( M \subseteq HxG \), this lead to \( M \subseteq A \) i.e. \( M \subseteq A \).
(4)\( \Rightarrow \) (1) \( M \) is \( b_1 \)-invariant set \( (\exists M G=HxG \subseteq IxG) \). Let \( A \subseteq M \) and \( A \neq \emptyset \). closed and \( b_1 \)-invariant set, then \( M \subseteq A \) is open set in \( M \) and \( M \subseteq A \).
(4)\( \Rightarrow \) (1) \( M \) is \( b_1 \)-minimal set and \( U \subseteq M \) nonempty open set, then \( HUG \subseteq IMG=IxG \) and \( M \subseteq HUG \subseteq IxG \) closed set. If \( x \in M-IxG \) and \( g \in G \), \( h \in H \) s.t. \( hxg \subseteq IxG \) then \( HxG \subseteq IxG \) this means \( M \) is contain \( IxG \) \( b_1 \)-minimal set but this contradiction then \( M \subseteq IxG \).

Lemma 3: Let \( (I, X, G) \) be a T.B.G., \( A, B, C \subseteq X \) are compact sets then \( ABC \) is compact subset of \( X \).
Proof: Since 0:1x(χxG)p →X is continuous map and since A, B, C are compact sets and 0(χxAB) = CAB is compact set under continuous map.

Remark: We can use the above proposition to b2-minimal set with take difference of b1-minimal.

Definition 4: Let (I, X, G) be a T, B, G and x ∈X then x is called b1-almost periodic point (b1) if for every nbh U of x there exist a left syndatic set A ⊆G and right syndatic set B ⊆H s.t. BxAG ⊆U.

Proposition 3: Let (I, X, G) be a T, B, G space and x ∈X then x is b1 ill HxG is b1-minimal compact subset of X.

Proof: → Let x be a b1 point. Ux be a nbh of x then there exist a left syndatic set A ⊆G and right syndatic set B ⊆G s.t. BxAG ⊆U. Since A, B are syndatic set this means there exist two compact set K ⊆G, J ⊆H s.t. G = AK, H = JB and HxG = BxAK ⊆K, J. By Lemma (3) for any K, J U, K is compact subset of T2-space then it is closed set. Hence, HxG ⊆K, J and HxG is compact.

If y ∈HxG then HxG ⊆HxG i.e. y ∈Ux, K this means there exists k ∈K and j ∈J s.t. y = jy, k ∈Ux then HxG ∩Ux = ∅, hence, x is a limit point of HxG and x ∈HxG then HxG is b1-minimal compact subset of X.

←Let HxG be a b1-minimal compact subset of X. Let Ux be a nbh of x in X. Since HxG ⊆HxG ⊆U then there exist a finite sets F ⊆G, F ⊆H s.t. HxGF ⊆F. Hence, HGF is an open cover HxG then it contains a finite subcover F.

For every g ∈G and h ∈H there exist a u ∈U, f ∈F and e ∈F s.t. hgx = Fue.

Let A = {g | hgx ∈U}, B = {h | hgx ∈U} we prove A and B are right, left syndatic sets respectively. Since hgx = Fue this leads to Fhgx = F−1 then Fhgx = F−1 ∈U and g ∈A, and h ∈FB then G = AB, H = FB. Since E and F are finite set then A is left syndatic set and B is right syndatic set.

Then x is b1-almost periodic point.

Definition 5: Let (I, X, G) be a T, B, G and x ∈X then x is called b2-almost periodic point (b2) if for every nbh U of x there exist a left semi-syndatic set A ⊆G and right semi-syndatic set B ⊆H s.t. BxA ⊆U.

Proposition 4: Let (I, X, G) be a T, B, G space and x ∈X is b2 point then sel(HxG) is b2-minimal semi-compact subset of X.

Proof: Similarly of first part of proposition 3 by supposing x ∈X is a b2 point and using the statement “every nbh is semi-nbh and change compact set by semi-compact”.

Proposition 5: Let (I, X, G) be a T, B, G, and sel(HxG) is b2-minimal semi-compact subset of X, then x ∈X is b2.

Proof: Similarly of second part of proposition 3.

Definition 6: Let (I, X, G) be a T, B, G and x ∈X then x is called b3-almost periodic point (b3) if for every nbh U of x there exist a left semi-syndatic set A ⊆G and right syndatic set B ⊆H s.t. BxA ⊆U.

Proposition 6: Let (I, X, G) be a T, B, G space and x ∈X is b3 point then HxG is b3-minimal compact subset of X.

Proof: Similarly of first parts of proposition 3 by use the statement in introduction “every semi-syndatic set is syndatic”.

Proposition 7: Let (I, X, G) be a T, B, G, and sel(HxG) is b2-minimal semi-compact subset of X, then x ∈X is b2.
Proof: Similarly of second part of proposition 3 by use the statement "every open set is semi-open". The definition of $b_1$ point ($b_1$-almost periodic point) is similar of $b_1$ point by suppose $A$ is left sydadic set and $B$ is right semi-syndadic set and we can satisfy propositions 6.7 by use it.

**Definition 7:** Let $(H, X, G)$ be a T.B.G and $x \in X$ then $x$ is called $b_2$-almost periodic point ($b_2$) if for every semi-nbhb $U$ of $x$ there exist a left semi-syndadic set $A \subseteq G$ and right semi-syndadic set $B \subseteq H$ s.t. $B \times A \subseteq U$.

**Proposition 8:** Let $(H, X, G)$ be a T.B.G. $X$ is $T_2$ locally semi-compact space and $x \in X$ is $b_3$ point then $sc(HxG)$ is $b_3$-minimal semi-compact subset of $X$.

**Proof:** Similarly of first parts of proposition 3.

**Proposition 9:** Let $(H, X, G)$ be a T.B.G. and $sc(HxG)$ is $b_2$-minimal semi-compact subset of $X$ then $x \in X$ is $b_2$.

**Proof:** Similarly of second part of proposition 3 by suppose $sc(HxG)$ is a $b_2$-minimal semi-compact subset of $X$ and let $U$ be a semi-nbhb of $x$ in $X$.

**Definition 8:** Let $(H, X, G)$ be a T.B.G and $x \in X$ then $x$ is called $b_3$-almost periodic point ($b_3$) if for every semi-nbhb $U$ of $x$ there exist a left syndadic set $A \subseteq G$ and right syndadic set $B \subseteq H$ s.t. $B \times A \subseteq U$.

**Proposition 10:** Let $(H, X, G)$ be a T.B.G. $X$ is $T_2$ locally semi-compact space and $x \in X$ is $b_3$ point then $sc(HxG)$ is $b_3$-minimal semi-compact subset of $X$.

**Proof:** Let $x$ be a $b_3$ point, $U$ be a semi-nbhb semi-compact of $x$ then there exist a left syndadic set $A \subseteq G$ and right syndadic set $B \subseteq H$ s.t. $B \times A \subseteq U$. Since $A$, $B$ are syndadic set this means there exist two compact set $K \subseteq G$, $J \subseteq H$ s.t. $G = AK$, $H = JB$ and $HxG = JB \times AK \subseteq UxK$. By Lemma (3) $UxK$ is compact subset of $T_2$-space then it is closed set. Hence, $sc(HxG) \subseteq U \times K$ and $sc(HxG)$ is semi-compact.

If $y \in sc(HxG)$ then $sc(HyG) \subseteq sc(HxG)$ i.e. $y \in UxK$ this means there exist $u \in U$ and $j \in J$ s.t. $y = j \times UxK$ and $u = j \times y \in UxK$, then $HyG \subseteq U \times J$ hence, $x$ is a semi-limit points of $HyG$ and $x \in sc(HyG)$ then $sc(HxG)$ is $b_2$-minimal semi-compact subset of $X$.

**Proposition 11:** Let $(H, X, G)$ be a T.B.G. and $sc(HxG)$ is $b_2$-minimal semi-compact subset of $X$ then $x \in X$ is $b_2$.

**Proof:** Let $sc(HxG)$ be a $b_2$-minimal semi-compact subset of $X$. Let $U$ be a semi-nbhb of $x$ in $X$.

Since $HxG \subseteq sc(HxG) \subseteq HUG$ then there exist a finite sets $E \subseteq G$, $F \subseteq H$ s.t. $HxG \subseteq FU$E. Hence, $HUG$ is an semi-open cover of $sc(HxG)$ then its contain a finite subcover $FU$E.

For every $g \in G$ and $h \in H$ there exist a $u \in U$, $f \in F$ and $e \in E$ s.t. $hxg = fu$. Let $A = \{g \times x \in G\}$, $B = \{hx \times e \in U\}$ we prove $A$ and $B$ are right, left semi-syndadic sets respectively. Since $hxg = fu$ this leads to $f \times hxg^{-1} = u$ then $f \times hxg^{-1} \in U'$ and $g \in AE$ and $h \in FB$ then $G = AE$. $H = FB$. Since $E$ and $F$ are finite set then $A$ is left semi-syndadic set and $B$ is right semi syndadic set.

Then $x$ is $b_3$-almost periodic point.

**Definition 9:** Let $(H, X, G)$ be a T.B.G and $x \in X$ then $x$ is called $b_4$-almost periodic point ($b_4$) if for every semi-nbhb $U$ of $x$ there exist a left semi-syndadic set $A \subseteq G$ and right syndadic set $B \subseteq H$ s.t. $B \times A \subseteq U$.

**Proposition 12:** Let $(H, X, G)$ be a T.B.G. $X$ is $T_2$ locally semi-compact space and $x \in X$ is $b_3$ point then $sc(HxG)$ is $b_3$-minimal semi-compact subset of $X$.
REFERENCES

Proof. Similarly, if the first part of the definition of semi compact set is not satisfied by any semi-syndetic set, any semi compact set must contain a minimal semi compact subset. Then if we suppose A is a semi-compact set and B is a semi-syndetic set, then by proposition 11, we can find a semi-compact subset of X which contains a minimal semi-compact subset of B. Then if we suppose A is a semi-compact set and B is a semi-syndetic set, then by proposition 11, we can find a semi-compact subset of X which contains a minimal semi-compact subset of B.

Proposition 13: Let \( (X, \mathcal{A}) \) be a semi compact set and \( \mathcal{A} \) a minimal semi compact subset of \( X \). Then \( \mathcal{A} \) is semi-compact.

Proof. Similarly, if the first part of the definition of semi-compact set is not satisfied by any semi-syndetic set, any semi-compact set must contain a minimal semi-compact subset. Then if we suppose A is a semi-compact set and B is a semi-syndetic set, then by proposition 11, we can find a semi-compact subset of X which contains a minimal semi-compact subset of B. Then if we suppose A is a semi-compact set and B is a semi-syndetic set, then by proposition 11, we can find a semi-compact subset of X which contains a minimal semi-compact subset of B.