Oscillation of The Per Capita Growth Rate

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Abstract

In this paper, equations of the per capita growth rate are considered. Sufficient conditions for Oscillation of all solutions are obtained. The asymptotic behavior of the nonoscillatory solutions is investigated.

Keywords. The per capita growth rate, oscillation, nonoscillatory, neutral differential equations.

Introduction.

In this paper we discussed the " *Oscillation of the per capita growth rate*" which is a delay differential equation transformed to a neutral delay differential equation of first order. Consider the classical delay logistic equation [10].

$$N'(t) = rN(t)[1 - \frac{N(\tau(t))}{K}],$$

which investigated in this paper, the per capita growth rate

$$h(t) = \frac{N'(t)}{N(t)} = r[1 - \frac{N(\tau(t))}{K}],$$

where $N(t)$ denotes the density of population at time t, r is the growth rate and K is the carrying capacity of the environment. $\tau(t)$ denotes the feedback mechanism, which responds to changes in the size of the population. In [10] assumed that the per capita growth rate $h(t)$ is given by $h(t) = h_\sigma(t) - h_\tau(t)$, where

$h_\sigma(t) = r[1 - \frac{N(\sigma(t))}{K}]$, is the per capita growth rate associated with density dependence, and $h_\tau(t) = c\frac{N'(\tau(t))}{N(\tau(t))}$, is the per capita growth rate associated with per capita growth rate at time $\tau(t)$. This leads to the neutral delay differential equation

$$N'(t) = N(t)\{r[1 - \frac{N(\sigma(t))}{K}] - c\frac{N'(\tau(t))}{N(\tau(t))}\}, \quad t \geq 0 \qquad \ldots\ldots(1)$$

Where

$r, K \in (0,\infty)$, $\sigma(t), \tau(t) \in C[[0,\infty);[0,\infty)]$, and $c \in (-\infty,\infty)$. An alternative model may be obtained by considering the growth rates rather than the per capita growth rates, this leads to the neutral delay differential equation

$$N'(t) = N(t)\{r[1 - \frac{N(\sigma(t))}{K}] - c\frac{N'(\tau(t))}{K}\}, \quad t \geq 0 \qquad \ldots\ldots(2)$$

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with Eq. (1) and (2) one associates an initial condition of the form $N(t) = \phi(t), \quad -\gamma(t) \le t \le 0$ with $\gamma(t) = \max\{\sigma(t), \tau(t)\}$, where the initial function $\phi(t)$ satisfy the following condition :

$\phi(t) \in C[[-\gamma(t), 0]; R^{+}], \quad \phi(t) > 0$ for $-\gamma(t) \le t \le 0$,

$$\qquad (3)$$

and $\phi(t)$ is absolutely continuous with locally bounded derivative on $-\tau(t) \le t \le 0$. By uniqueness theorem the initial value problem (1) and (3) has unique solution which exists and remains positive on $[0, \infty)$. The same is true for the initial value problem (2) and (3). By introducing the change of variable $x(t) = \ln \frac{N(t)}{K}$ Eq. (1) and (2) transformed to

$$\frac{d}{dt}[x(t) + c\,x(\tau(t))] + r[e^{x(\sigma(t))} - 1] = 0, \quad t \ge 0 \qquad (4)$$

And

$$\frac{d}{dt}[x(t) + c[e^{x(\tau(t))} - 1]] + r[e^{x(\sigma(t))} - 1] = 0, \quad t \ge 0 \qquad (5)$$

respectively. In this paper we considered the oscillatory properties of the positive solutions of equations (1) and (2) about the unique positive steady state K. Clearly,

every positive solution of (1) and (2) oscillatory about K if and only if every solution of (4) and (5) oscillates about zero. We can extended Eq.(4) and (5) by making r is a function of t and so Eq. (4) and (5) can be written in the form

$$\frac{d}{dt}[x(t) + c\,x(\tau(t))] + r(t)\,f(x(\sigma(t))) = 0, \quad t \ge 0 \qquad (4')$$

and

$$\frac{d}{dt}[x(t) + c\,f(x(\tau(t)))] + r(t)\,f(x(\sigma(t))) = 0, \quad t \ge 0 \qquad (5')$$

respectively, where $r(t) \in C([t_0, \infty); (0, \infty))$. We will establish necessary and sufficient conditions for the oscillation of every solution of Eq. (4') and sufficient conditions for the nonoscillatory solutions of Eq. (4') to converge to zero as $t \to \infty$.

Main Results.

In this section we will obtain the sufficient conditions for the oscillation of *every* solution of Eq.(4'), where the function f satisfies the condition $x f(x) > 0$, where $x > 0$, and sufficient conditions for the nonoscillatory solution of equation (4') to converge to 0. We define the function $z(t) = x(t) + c\,x(\tau(t))$. Then equation (4') will be as follows:

$$z'(t) = -r(t)\,f(x(\sigma(t))) \qquad (6)$$

Now we prove the following theorems.

Theorem 1. Suppose that, f is an increasing function, $0 \le c < 1$, $r(t) > 0$, $\tau(t) > t$ for $t \ge t_0$ and

$$\int^{\infty} r(s)\,ds = \infty \qquad (7)$$

Then all solutions of equation (4') are oscillatory.

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (4') and $x(t) > 0$ for large t. for $t_1 \ge t_0$ we have $x(\sigma(t))$ and $x(\tau(t))$ are positive for $t \ge t_1 \ge t_0$, and f is an increasing function. Then by equation (6) we have only the case $z'(t) \le 0, \quad z(t) > 0$ to discuss.

We can see that $z(t) \ge x(t)$ and $z(t)$ is decreasing, that leads to

$z(\tau(t)) \le z(t) \le x(t) + c\,z(\tau(t))$

From the last inequality we obtain

$z(\tau(t))(1 - c) \le x(t)$.

that is $z(\tau(\sigma(t)))(1-c) \le x(\sigma(t))$ so, equation (6) will be

$$z'(t)+r(t)f(z(\tau(\sigma(t))))(1-c) \le 0$$

Integrating the last inequality from t_1 to t

$$z(t)-z(t_1)+\int_{t_1}^{t} r(s)f(z(\tau(\sigma(s))))(1-c)\,ds \le 0$$

But f increasing so

$$z(t)-z(t_1)+f(z(\tau(\sigma(t_1))))(1-c)\int_{t_1}^{t} r(s)\,ds \le 0$$

As $t \to \infty$ we get a contradiction. Then $x(t)$ can not be eventually positive and in the same way we can proof that $x(t)$ can not be eventually negative, and so $x(t)$ can not be nonoscillatory.

Theorem 2. Suppose that, f is an increasing function, $c \ge 0$, $r(t) > 0$ for $t \ge t_0$, and (7) holds. Then all solutions of equation (4') are oscillatory.

Proof. Let $x(t)$ be an eventually positive solution of equation (4'), so we can choose t_1 such large that $x(\sigma(t))$ and $x(\tau(t))$ are positive for $t \ge t_1$, we have f is an increasing function. By equation (6) we can see that $z'(t) < 0$ then we have only the possible $z(t) > 0$, $t \ge t_1 \ge t_0$, to discuss.

Integrating equation (6) from t_1 to t

$$z(t)-z(t_1)=$$

$$-\int_{t_1}^{t} r(s)f(z(\sigma(s)))\,ds \le -f(z(\sigma(t_1)))\int_{t_1}^{t} r(s)\,ds$$

as $t \to \infty$ we get $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction .

Remark 1. According to the two Theorems (1) and (2) we can conclude that all solutions of equation (1) are oscillatory about a unique positive steady state K under the conditions of these theorems.

Theorem 3. Suppose that $c \ge 0$, $r(t) > r > 0$, and

$$|f(u)| \ge a|u|, \quad a > 0 \quad \text{...............(8)}$$

For $t \ge t_0$. Then every nonoscillatory solution of equation (4') tends to zero as $t \to \infty$.

Proof. Assume that $x(t)$ be an eventually positive solution of equation (4'), and $x(\sigma(t)) > 0$, $x(\tau(t)) > 0$, for $t \ge t_1 \ge t_0$. Then we have only the case $z'(t) < 0$, $z(t)>0$, to discuss, which means $z(t)$ is bounded, then there exists $M > 0$ such that $z(t) \le M$, this implies that $x(t) \le M$, let $\lim_{t\to\infty} z(t) = L \ge 0$,

suppose that $L \ne 0$ then $L > 0$. Integrating equation (6) from t_1 to ∞ we get

$$z(t_1)-L=\int_{t_1}^{\infty} r(s)f(z(\sigma(s)))\,ds$$

by (8) we get

$$z(t_1)-L \ge r\,a\int_{t_1}^{\infty} x(\sigma(s))\,ds$$

that means $x(t) \in L_1[t_1,\infty)$, since $z(t)$ is bounded, we conclude that $z(t) \in L_1[t_1,\infty)$

by integrating the inequality $z(t) \ge L > 0$, from t_1 to ∞ we get

$$\int_{t_1}^{\infty} z(s)\,ds \ge L\int_{t_1}^{\infty} ds,$$

which leads to

$$\int_{t_1}^{\infty} z(s)\,ds = \infty,$$

and this is impossible since $z(t) \in L_1[t_1,\infty)$ then $\lim_{t\to\infty} z(t) = 0$ but $z(t) \ge x(t)$, and so $\limsup_{t\to\infty} x(t) = 0$ which implies that $\lim_{t\to\infty} x(t) = 0$. The proof is complete.

Remark 2. According to Theorem (3) we can conclude that every nonoscillatory solution of equation (1) tends to a unique positive steady state K as $t \to \infty$, under the conditions of this theorem .

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التذبذب في سعة معدل النموالسكاني

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المستخلص

أن المعادلة التباطؤيةاللوجستيةمعرفة بالشكل

$N'(t) = r\,N(t)[1 - \frac{N(\tau(t))}{K}]$، أنظر [1]، في هذاالبحث تمت دراسة معادلة سعة معدل النموالسكاني $h(t) = \frac{N'(t)}{N(t)} = r[1 - \frac{N(\tau(t))}{K}]$، حيث تم تحويلها الى معادلة محايدة من الرتبة الأولى، حيث اعطيت بعض الشروط الكافية لتذبذب حلول هذه المعادلةو التي تمثل الكثافةالسكانية $N(t)$ في الزمن t، أما r يمثل معدل النمو،كذلك أعطيت بعض الشروط الكافية لأستقرارية هذه الحلول أو تباعدها.