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Subgroups and Orbits by Companion Matrix in Three Dimensional Projective Space

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Abstract:

The aim of this paper is to construct cyclic subgroups of the projective general linear group over F_{23} from the companion matrix, and then form caps of various degrees in $PG(3,23)$. Geometric properties of these caps as secant distributions and index distributions are given and determined if they are complete. Also, partitioned of $PG(3,23)$ into disjoint lines is discussed.

Keywords: Cap, Companion matrix, Group action, Projective geometry, Secant.

Introduction:

Let $PG(n, q)$ be the finite projective space of dimension n over the finite field F_q (Galois field, $GF(q)$). The projective space of dimension 0, 1, and 2 is called point, projective line, and projective plane, respectively. From the Gaussian coefficient formula, the number of m -subspaces of $PG(n, q)$ is

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^{n+1} - 1)(q^{n+1} - q) \dots (q^{n+1} - q^m)}{(q^{m+1} - 1)(q^{m+1} - q)(q^{m+1} - q^m)}.$$

As special cases, the number of points (hyperplanes, by duality), lines and planes in $PG(n, q)$ are as follows:

$$\begin{aligned} \theta(n, q) &= \frac{q^{n+1} - 1}{q - 1}, \varphi(n, q) \\ &= \frac{(q^{n+1} - 1)(q^n - 1)}{(q^2 - 1)(q - 1)}, \phi \\ &= \frac{(q^{n+1} - 1)(q^{n+1} - q)(q^{n+1} - q^2)}{(q^3 - 1)(q^3 - q)(q^3 - q^2)}. \end{aligned}$$

Definition 1: $(k; r)$ -cap, K of degree r is a set of k points in $PG(n \geq 3, q)$ such that no $r + 1$ points are collinear, but some r of them are collinear. The set K is called a complete $(k; r)$ -cap if K is maximal with respect to set-theoretical inclusion¹.

The maximum size of a cap of degree r will be denoted by $m_r(n, q)$. The size of the smallest

complete cap of degree r will be denoted by $t_r(n, q)$.

Definition 2: Let K be a cap of degree r . An s -secant of a K in $PG(n, q)$ is a line l such that $|K \cap l| = s$. The number of i -secants of K will be denoted by τ_i ¹.

Let Q be a point not on the $(k; r)$ -cap, K . The number of i -secants of K through Q will be denoted by $\sigma_i(Q)$. The number $\sigma_r(Q)$ of r -secants is called the index of Q with respect to K . The set of all points of index i will be denoted by C_i and the cardinality of C_i denoted by c_i . The sequence (τ_0, \dots, τ_r) will refer to secant distribution and the sequence (c_0, \dots, c_d) , $d \leq \lfloor \frac{k}{r} \rfloor$ to the index distribution.

Definition 3: The group of projectivities of $PG(n, q)$ is called the projective general linear group, $PGL(n + 1, q)$ ¹.

The elements of $PGL(n + 1, q)$ are non-singular matrices of dimension $n + 1$, and its cardinality is

$$\frac{(q^{n+1} - 1)(q^{n+1} - q) \dots (q^{n+1} - q^n)}{(q - 1)}.$$

Definition 4: Let $f(x) = x^{n+1} - s_n x^n - \dots - s_1 x - s_0$ be a primitive polynomial over F_q of

degree $n + 1$. A companion matrix for f is a $(n + 1) \times (n + 1)$ matrix²

$$C_f = \begin{pmatrix} 0 & & & \\ 0 & I_n & & \\ \vdots & \vdots & \vdots & \\ s_0 & \dots & s_n & \end{pmatrix}.$$

The companion matrix C_f has the property that it is cyclic of order $\theta(n, q)$ so, all points (hyperplanes) of $PG(n, q)$ can be formed using the formula: $P(i) = U_0 C_f^i$, where $U_0 = [1, 0, \dots, 0]$ and i from 0 to $\theta(n, q) - 1$. Also, C_f formed a cyclic group of $PGL(n, q)$ which is called the Singer group. According to Lagrange's theorem (group theory), any natural number m divided the order of the Singer group generated by C_f will give a cyclic subgroup of order m . Therefore, the points of $PG(n, q)$ will be partitioned into a number of classes from the act of Singer group on $PG(n, q)$; that is, for a fixed natural number r divided $\theta(n, q)$, the point X related to point Y iff there is a natural number t such that $t \cdot r \leq \theta(n, q)$ and $Y = X C_f^{tr}$.

To know more about the three-dimensional projective space, which is of interest in this research, see source³.

The paper aims to study the geometric structure of classes formed from the Singer group of $PGL(4, 23)$ like cap, secant distributions and index distributions. The topic of caps has been studied extensively by many researchers in finite projective spaces. For three dimensional projective space, some of them attempted to find the smallest complete caps or the largest complete caps as in^{4, 5, 6}. A spectrum size of complete caps has been given for certain fields^{7, 8}. As for studying the caps in the three dimensional projective space, there are many researchers who have studied this concept⁹, and the link between it and the concepts of linear coding¹⁰ and with cubic curves¹¹. The idea of action of groups on the points of projective space to construct geometrical objects that appear in¹² for line and with respect to projective plane in^{13, 14}.

Caps from Companion Matrix:

Let $\gamma = 2$ be the primitive element of F_{23} . The polynomial $f(x) = x^4 - \gamma^{11}x^3 - \gamma^9x^2 - x - \gamma^{16}$ is primitive polynomial over F_{23} . The companion matrix for f is

$$C_f = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma^{16} & \gamma^9 & \gamma^{11} & \end{pmatrix}.$$

In $PG(3, 23)$, $\theta(3, 23) = 12720 = \phi(3, 23)$, and $\varphi(3, 23) = 293090$. The number 12720 has 2, 2, 2, 2, 3, 5 as prime factor integers so, there are 38 non-trivial divisors of $\theta(3, 23)$. These 38 factors are $\chi = \{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 40, 48, 53, 60, 80, 106, 120, 159, 212, 240, 265, 318, 424, 530, 636, 795, 848, 1060, 1272, 1590, 2120, 2544, 3180, 4240, 6360\}$. Therefore, subgroups of $PGL(4, 23)$ of these orders are existing.

Lemma 5: The space $PG(3, 23)$ is partition into 530 disjoint lines.

Proof: This space has a 1-space of cycle 530 which is less than $\theta(3, 23)$ as shown below:

Let $\ell_1 = \{0, 530, 1060, \dots, 530 \cdot 23 \text{ mod } \theta(3, 23)\}$ be the first line of the cycle in numeral form; that is, $i := [x^i, y^i, z^i]$. Then the points of others 529 lines have the form $\ell_i[j] := \ell_1[j] + i \text{ mod } \theta(3, 23)$, $i = 1, \dots, 529$.

Lemma 6: There are only 38 cyclic subgroups of $PLG(4, 23)$ of order t divided $\theta(3, 23)$.

Proof. The integers in χ are all non-trivial factors of $\theta(3, 23)$. The order of the companion matrix C_f is also $\theta(2, 23)$ which is give cyclic subgroup, $\langle C_f \rangle$ of $PGL(4, 23)$ such that $PG(3, 23)$ invariant with respect to it. So, all elements of χ divided the order of C_f and give cyclic subgroups of $\langle C_f \rangle$. Let denote these subgroups by S_i , i in χ . Any other cyclic subgroup of $PGL(4, 23)$ of order divided $\theta(3, 23)$ will be a copy isomorphic to S_i for some i in χ .

Theorem 7: There are unique 38 equivalence classes up to projectivity of $PG(3, 23)$ of order t in χ such that $t \cdot i = \theta(3, 23)$.

Proof. The action of the groups S_i , $i \in \chi$ in Lemma 6, on the projective space $PG(3, 23)$, will divide the space points into i equivalence classes (orbits) of order $t \in \chi$; that is, $t = \frac{\theta(3, 23)}{i}$. All these i equivalence classes will be projectively equivalent by C_f .

Let denote the classes in the Theorem 7, by \mathcal{D}_i . The intersection of these classes has been tested with planes to find out the degrees of the caps that it formed. Also, its non-zero i -secant distributions and c_0 -distributions have computed.

Outline of Algorithm:

The algorithm that has been used to construct the caps by the action of the subgroups on the projective space $PG(3, 23)$ is summarized as below, and it has been executed by GAP programming¹⁵.

Step 1: For fixed i in χ , construct the subgroup S_i .

Step 2: Construct the orbit \mathcal{D}_i .

Step 3: Compute the number of intersection points of \mathcal{D}_i with all lines of $PG(3,23)$.

Step 4: The maximum point of intersection will be determined as a degree of the cap \mathcal{D}_i .

Step 5: Compute the number of points out of \mathcal{D}_i that are not on the 3-secants to determine if it is complete or not.

Theorem 8: The 38 equivalence classes \mathcal{D}_i in Theorem 7 are divided into 6 complete caps of degree 2,4,6,8,13 and 32 incomplete caps of degree 2,3,4,6,8,12,24.

Proof.

Let $(k; r)$ denote the $(k; r)$ -cap.

Table 1. Caps in $PG(3, 23)$.

\mathcal{D}_i	$(k; r)$	$(0,1, \dots, r)$ (τ_0, \dots, τ_r)	$(0,1, \dots, d)$ (c_0, \dots, c_d)
\mathcal{D}_2	(6360;24)	(0, 8, 9, 10, 11, 12, 13, 14, 15, 16, 24) (265,12720,50880,38160,25440, 38160,25440,38160, 50880,12720, 265)	(0) (6360)
\mathcal{D}_3	(4240;13)	(3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13) (8480, 16960, 16960, 33920, 8480, 123490, 8480, 33920, 16960, 16960, 8480)	(11) (8480)
\mathcal{D}_4	(3180;12)	(0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12) (265, 19080, 12720, 50880, 25440, 76320, 25440, 50880, 12720, 19080, 265)	(0,1) (6360, 3180)
\mathcal{D}_5	(2544;24)	(0, 2, 3, 4, 5, 6, 7, 8, 10, 24) (8056, 10176, 61056, 66144, 30528, 71232, 30528, 5088, 10176, 106)	(0) (10176)
\mathcal{D}_6	(2120;8)	(0, 1, 2, 3, 4, 5, 6, 7, 8) (8745, 8480, 63600, 16960, 97520, 16960, 63600, 8480, 8745)	(10, 12, 16) (2120, 4240, 4240)
\mathcal{D}_8	(1590;6)	(0, 1, 2, 3, 4, 5, 6) (28885, 6360, 98580, 25440, 98580, 6360, 28885)	(44, 47, 49) (3180, 4770, 3180)
\mathcal{D}_{10}	(1272; 24)	(0, 1, 2, 3, 4, 5, 6, 7, 24) (20829, 71232, 61056, 81408, 40704, 7632, 7632, 2544, 53)	(0) (11448)
\mathcal{D}_{12}	(1060;4)	(0, 1, 2, 3, 4) (63865,12720, 139920, 12720, 63865)	(107, 109, 111) (3180, 2120, 6360)
\mathcal{D}_{15}	(848;8)	(0, 1, 2, 3, 4, 5, 6, 8) (60632, 79712, 94976, 40704, 11872, 1696, 3392, 106)	(0,1) (10176, 1696)
\mathcal{D}_{16}	(795;4)	(0, 1, 2, 3, 4) (79765,57240,104940,32065,19080)	(12, 18, 29, 32, 36, 37, 39, 43) (795, 1590, 1590, 1590, 1590, 1590, 1590, 1590)
\mathcal{D}_{20}	(636;12)	(0, 1, 2, 3, 4, 5, 12) (86973, 103032, 71232, 22896, 7632, 1272,53)	(0,1) (11448, 636)
\mathcal{D}_{24}	(530;2)	(0,1,2) (140185, 12720, 140185)	(253) (12190)
\mathcal{D}_{30}	(424;8)	(0, 1, 2, 3, 4, 6, 8) (124285, 119568, 37312, 9328, 1696, 848, 53)	(0,1) (11448, 848)
\mathcal{D}_{40}	(318;6)	(0, 1, 2, 3, 4, 6) (158205, 101124, 28620, 3180, 1908, 53)	(0,1) (1448, 954)
\mathcal{D}_{48}	(265;2)	(0,1,2) (181525, 76585, 34980)	(0, 56, 60, 63, 65, 66) (265, 530, 2650, 3710, 2650, 2650)
\mathcal{D}_{53}	(240;24)	(0, 1, 2, 3, 4, 24) (184120, 87360, 20160, 960, 480, 10)	(0) (12480)
\mathcal{D}_{60}	(212;4)	(0, 1, 2, 4) (195517, 79712, 16960, 901)	(0, 1, 2, 3, 4, 5) (2544, 4664, 3392, 1272, 424, 212)
\mathcal{D}_{80}	(159;3)	(0,1,2,3) (217353, 63918, 11448, 371)	(0, 1, 2, 3) (6996, 3975, 954, 636)
\mathcal{D}_{106}	(120;24)	(0, 1, 2, 3, 24) (232365, 55440, 5040, 240, 5)	(0) (12600)
\mathcal{D}_{120}	(106;2)	(0,1,2) (240037, 47488, 5565)	(0, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 21) (424, 424, 212, 1484, 1060, 3180, 1484,

			1484, 1272, 424, 212, 424, 318, 212)
\mathcal{D}_{159}	(80;8)	(0,1,2,8) (251800, 38400, 2880, 10)	(0,1) (12480, 160)
\mathcal{D}_{212}	(60;12)	(0, 1, 2, 12) (261405, 30240, 1440, 5)	(0,1) (12600, 60)
\mathcal{D}_{240}	(53;2)	(0,1,2) (265159, 26553, 1378)	(0, 1, 2, 3, 4, 5, 6, 7) (1113, 2544, 3710, 2438, 1908, 424, 424, 106)
\mathcal{D}_{265}	(48;24)	(0, 1, 2, 24) (267168, 25344, 576, 2)	(0) (12672)
\mathcal{D}_{318}	(40;8)	(0, 1, 2, 8) (271645, 20800, 640, 5)	(0,1) (12600, 80)
\mathcal{D}_{424}	(30;6)	(0,1,2,6) (276885, 15840, 360, 5)	(0,1) (12600, 90)
\mathcal{D}_{530}	(24;24)	(0,1,24) (279841, 13248, 1)	(0) (12696)
\mathcal{D}_{636}	(20;4)	(0,1,2,4) (282205, 10720, 160, 5)	(0,1) (12600, 100)
\mathcal{D}_{795}	(16;8)	(0,1,2,8) (284320, 8704, 64, 2)	(0,1) (12672, 32)
\mathcal{D}_{848}	(15;3)	(0,1,2,3) (284895, 8100, 90, 5)	(0,1) (12600, 105)
\mathcal{D}_{1060}	(12;12)	(0,1,12) (286465, 6624, 1)	(0,1) (12696, 12)
\mathcal{D}_{1272}	(10;2)	(0,1,2) (287605, 5440, 45)	(0,1,2) (11780, 870, 60)
\mathcal{D}_{1590}	(8;8)	(0,1,8) (288673, 4416, 1)	(0,1) (12696, 16)
\mathcal{D}_{2120}	(6;6)	(0,1,6) (289777, 3312, 1)	(0,1) (12696, 18)
\mathcal{D}_{2544}	(5;2)	(0,1,2) (290335, 2745, 10)	(0,1) (12495, 220)
\mathcal{D}_{3180}	(4;4)	(0,1,4) (290881, 2208, 1)	(0,1) 12696, 20
\mathcal{D}_{4240}	(3;3)	(0,1,3) (291433, 1656, 1)	(0,1) 12696, 21
\mathcal{D}_{6360}	(2;2)	(0,1,2) (291985, 1104, 1)	(0,1) 12696, 22

Note that, all the results in Theorem 8 can be discussed with respect to linear codes. From this point of view, there are many authors who discussed this case, for instance, Al-Zangana in^{16, 17}, and also in^{5, 10}.

Corollary 9.

- i. $m_2(3,23) = 530$, (see (3),(9)),
- ii. $m_4(3,23) \geq 1060$, $t_4(3,23) \leq 95$,
- iii. $m_6(3,23) \geq 1590$,
- iv. $m_8(3,23) \geq 2120$,
- v. $m_{13}(3,23) = 4240$.
- vi. \mathcal{D}_2 is the union of 265 disjoint lines.
- vii. \mathcal{D}_{265} is the union of two disjoint lines.
- viii. \mathcal{D}_{106} is the union of five disjoint lines.
- ix. \mathcal{D}_{53} is the union of ten disjoint lines.
- x. \mathcal{D}_{10} is the union of 53 disjoint lines.
- xi. \mathcal{D}_5 is the union of 106 disjoint lines.
- xii. \mathcal{D}_{530} is just a line.

Proof.

- i. \mathcal{D}_{24} is a complete cap of size 530 and degree 2 which achieved the largest size of cap $q^2 + 1$ of degree 2.
- ii. \mathcal{D}_{12} is a complete cap of size 1060 and degree 4, and \mathcal{D}_{16} is complete of size 795 and degree 4.
- iii. \mathcal{D}_8 is a complete cap of size 1590 and degree 6.
- iv. \mathcal{D}_6 is a complete cap of size 2120 and degree 8.
- v. \mathcal{D}_3 is a complete cap of size 4240 and degree 13.
- vi. The class \mathcal{D}_2 is the union of the 265 lines ℓ_{2i} , $i = 1, \dots, 265$.
- vii. The class \mathcal{D}_{265} is the union of the two lines ℓ_2 and ℓ_{267} .
- viii. The class \mathcal{D}_{106} is the union of the five lines ℓ_{2+106i} , $i = 0, \dots, 4$.

- ix. The class \mathcal{D}_{53} is the union of the ten lines $\ell_{2+53i}, i = 0, \dots, 9$.
- x. The class \mathcal{D}_{10} is the union of the 53 lines $\ell_{2+10i}, i = 0, \dots, 52$.
- xi. The class \mathcal{D}_5 is the union of the 106 lines $\ell_{2+5i}, i = 0, \dots, 105$.
- xii. The class \mathcal{D}_{530} is just the line ℓ_2 .

Remark 10: All complete caps in Theorem 8 have no points in common with lines in the 1-space cycle of length 530.

Conclusion:

From the study of group action in $PG(3,23)$, new caps of different sizes and degrees have been found, and show that the maximum size of cap of degree 2 has many copies; that is, it is not unique. Also, it has been shown that the six complete caps in Corollary 9, have no point-contacts with lines in the 1-space cycle.

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Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Mustansiriyah University.

Authors' contributions:

Al-Zangana E.B and Kasm Yahya N.Y. contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

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زمر جزئية ومدارات بواسطة المصفوفة المترافقة في الفضاء الإسقاطي من البعد الثالث

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الخلاصة:

الهدف من هذا البحث هو انشاء زمر جزئية دورية من الزمرة الخطية العامة الإسقاطية على الحقل F_{23} من المصفوفة المصاحبة ، ثم تكوين أغطية بدرجات مختلفة في $PG(3,23)$. تم إعطاء الخصائص الهندسية لهذه الأغطية كنوزيعة القطع و توزيعات الدليل ، وتحديد فيما إذا كانت كاملة. كذلك، تجزئة $PG(3, 23)$ الى خطوط غير متقاطعة تم دراسته.

الكلمات المفتاحية: الغطاء، مصفوفة الترافق، فعل الزمر، الهندسة الإسقاطية، القاطع.