Detour Polynomials of Generalized Vertex Identified of Graphs

Haveen J. Ahmed1 Ahmed M. Ali2 * Gashaw A. Mohammed Saleh3

1Department of Mathematics, College of Science, Duhok University, Duhok, Iraq.
2Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq.
3Department of Mathematics, College of Science, Salahaldin University, Arbil, Iraq.
*Corresponding author: ahmedgraph@uomosul.edu.iq
E-mail addresses: haveen.ahmed@uod.ac, gashaw.mohammed@su.edu.krd

Received 22/5/2021, Accepted 10/4/2022, Published Online First 20/9/2022

This work is licensed under a Creative Commons Attribution 4.0 International License.

Abstract:
The Detour distance is one of the most common distance types used in chemistry and computer networks today. Therefore, in this paper, the detour polynomials and detour indices of vertices identified of n-graphs which are connected to themselves and separated from each other with respect to the vertices for n≥3 will be obtained. Also, polynomials detour and detour indices will be found for another graphs which have important applications in Chemistry.

Keywords: Detour distance, Detour polynomial, Detour index, Special graphs, Vertices identified.

Introduction:
Let \( \Phi = \Phi(V(\Phi), E(\Phi)) \) be a connected graph without muliedges and loops (which write short \( \Phi \) be a conn.graph) such that order of \( \Phi \) is \( |V(\Phi)| = p(\Phi) \) and size of \( \Phi \) is \( |E(\Phi)| = q(\Phi) \). The distance function \( \sigma \) from \( V(\Phi) \times V(\Phi) \) in to non-negative integer is the number of edges between \( u \) and \( v \) in the shortest \( u - v \) path in \( \Phi \). It is denoted by \( \sigma(u, v) \). But, there are other types of distance between any two distinct vertices \( u \) and \( v \) in a conn.graph. \( \Phi \) is called detour distance and it has been defined by the length of a longest \( u - v \) path in \( \Phi \); it is denoted by \( \mathcal{D}(u, v) \).

We note that \( \mathcal{D}(u, v) = 1 \) iff the edge \( uv \) which removal resulted disconnected graph of \( \Phi \), and \( \mathcal{D}(u, v) = \sigma(u, v) \) iff the graph \( \Phi \) is a connected without cycles. From clearly that \( \mathcal{D}(u, v) = p(\Phi) \) minus one iff \( \Phi \) contains a “Hamiltonian \( u - v \) path”. The eccentricity distance of this vertex \( v \) which is denoted \( e_v(\Phi) \) and defined by \( e_v(\Phi) = \max_{u \in V(\Phi)} \mathcal{D}(u, v) \) in a conn.graph \( \Phi \). The detour diameter \( \text{diam}_2(\Phi) = \delta_2(\Phi) \) of a conn.graph \( \Phi \) is defined by \( \delta_2(\Phi) = \max_{v \in V(\Phi)} \{ e_v(\Phi) \} \) and the detour radius \( r_2(\Phi) = \min_{v \in V(\Phi)} \{ e_v(\Phi) \} \). We see that \( e(v) \leq e_v(\Phi), \forall v \in V(\Phi) \), since \( d(u, v) \leq \mathcal{D}(u, v), \forall u, v \in V(\Phi) \). This requires us to be \( \text{diam}(\Phi) = \text{diam}_2(\Phi) \) and \( \text{rad}(\Phi) = \text{rad}_2(\Phi) \).

There is another concept related to the detour distance which is called the connected detour numbers. Also, there is another concept of numbers in graph theory which is called domination and chromatic number.

“The detour polynomial \( \mathcal{D}[\Phi, x] \) of a conn.graph \( \Phi \), is defined” by
\[
\mathcal{D}[\Phi, x] = \sum_{(u,v) \in E(G)} x^{\mathcal{D}(u,v)} \text{ or } \mathcal{D}[\Phi, x] = \sum_{k \geq 1} C_2(\Phi, k) x^k,
\]
where \( C_2(\Phi, k) \) be the number of unordered pairs \( u, v \) such that \( \mathcal{D}(u, v) = k \).

In 1995, the authors introduced definition the detour index of a conn.graph \( \mathcal{D}(\Phi) \) which defined as \( \mathcal{D}(\Phi) = \sum_{i=1}^{p(\Phi)} \sum_{j=i+1}^{p(\Phi)} d_{ij} \), where \( d_{ij} \) is a matrix of detour distance. The detour index is horribly important in applied sciences. After that Lukovits tested the detour index on the correlation between some chemical compositions. Also, the detour index can be obtained from another method by:
\[
\mathcal{D}(\Phi) = \sum_{(u,v) \in E(\Phi)} \mathcal{D}(u, v) = \mathcal{D}(\Phi; x) |_{x=1} = \sum_{k \geq 1} k C_2(\Phi, k).
\]
Let \( C_2(\Phi, k) \) be the number of vertices \( u, \ (u \neq v) \) such that \( \mathcal{D}(u, v) = k \).
The detour polynomial of a vertex $v$ in $\Phi$ is defined as:
\[
\mathcal{D}[v, \Phi, x] = \sum_{k \geq m_2(v)} c_2(v, \Phi, k)x^k,
\]
where $m_2(v) = \min_{u \in V(\Phi)} \{\mathcal{D}(v, u)\}$. We note that:
\[
\mathcal{D}([v, \Phi, x]) = \frac{1}{2} \sum_{u \in V(\Phi)} \mathcal{D}[v, \Phi, x].
\]
The detour index of a vertex $v$ which denoted by $\mathcal{D}(v, \Phi)$ and defined as:
\[
\mathcal{D}(v, \Phi) = \sum_{u \in V(\Phi)} \mathcal{D}(v, u).
\]

**Theorem 1:**
1. $\mathcal{D}[K_p, x] = \binom{p}{2}x^{p-1}$.
2. $\mathcal{D}[W_p, x] = \binom{p}{2}x^{p-1}$.
3. $\mathcal{D}[G_p, x] = px^2(p + 1)$, where $p$ is even.
4. $\mathcal{D}[G_p, x] = px^2(p - 1)$, where $p$ is odd.

A number of writers had obtained detour polynomials, restricted detour polynomials and indices of detour for many graphs, applications in graph theory and graphs resulting from the synthesis of other graphs under operations defined in graph theory. In 1993, Gutman constructed new graph from two graphs not connected to each other. The two vertices $u$ and $v$ of any two conn. graphs $\Phi_1$ and $\Phi_2$ with disjoint vertex sets, respectively. Then, the vertex identified graph $\Phi_1 \cdot \Phi_2$ is obtain from $\Phi_1$ and $\Phi_2$ by two vertices identically $u$ and $v$. Mohammed Saleh obtained $\mathcal{D}(\Phi_1 \cdot \Phi_2; x)$, as specified in the following theorem.

**Theorem 2:** Let $u, (v)$ be any vertex of $\Phi_1, (\Phi_2)$, then
\[
\mathcal{D}[\Phi_1 \cdot \Phi_2, x] = \mathcal{D}[\Phi_1, x] + \mathcal{D}[\Phi_2, x] + \mathcal{D}[u, \Phi_1, x] \mathcal{D}[v, \Phi_2, x].
\]

See the books, there are certain classes of graphs that occur for their definitions. The detour polynomials and detour indices will be determined of a conn. graphs resulting from others graphs using the idea of vertex identification.

**Main Results:**
In the following sections, polynomials of detour distance are found from the graphs by identified their vertices with examples of the important special graphs.

**Detour Polynomial of Vertex Identified Graphs**
Let $(\Phi_1, \Phi_2, ..., \Phi_n)$ be the set of pairwise disjoint graphs to vertices $u_i, v_i \in V(\Phi_i), i = 1, 2, ..., n, n \geq 3$, then the vertex identified graph $\prod_{i=1}^n V_{\psi}(\Phi_i) = (\Phi_1 \cdot \Phi_2 \cdot ... \cdot \Phi_n)$ is obtained from the graphs $\Phi_1, \Phi_2, ..., \Phi_n$ of $\{\Phi_i\}_{i=1}^n$ with respect to the vertices $\{v_1, u_2, v_2, u_3; ...; v_{n-1}, u_n\}$ is the graph obtained from the graphs $\Phi_1, \Phi_2, ..., \Phi_n$ by identifying the vertex $v_i$ with the vertex $u_{i+1}$ for all $i = 1, 2, ..., n - 1$. (See Fig. 1).

**Figure 1.** Graph $\prod_{i=1}^n V_{\psi}(\Phi_i), \Phi_i = G_i$.

From clearly that:
\[
p(\prod_{i=1}^n V_{\psi}(\Phi_i)) = \sum_{i=1}^n p(\Phi_i) - n + 1,
\]
\[
q(\prod_{i=1}^n V_{\psi}(\Phi_i)) = \sum_{i=1}^n q(\Phi_i).
\]

**Proposition:** $\text{diam}_2(\prod_{i=1}^n V_{\psi}(\Phi_i)) = \max_{i<l}(e_2(v_l) + e_2(u_l) + \sum_{r=i+1}^{l-1} \mathcal{D}(v_r, u_r))$.

**Proof:** Obviously.

Now, the detour polynomial will be given of $\prod_{i=1}^n V_{\psi}(\Phi_i)$ for all $n \geq 2$ in the following theorem.

**Theorem 3:** $\forall n \in \mathbb{N} - \{0, 1\}$, there will be:
\[
\mathcal{D}[\prod_{i=1}^n V_{\psi}(\Phi_i), x] = \sum_{i=1}^n \mathcal{D}[\Phi_i, x] + \sum_{i=1}^{n-1} \mathcal{D}[v_1, \prod_{i=1}^n V_{\psi}(\Phi_i), x] \mathcal{D}[u_{n+i}, \Phi_h+1, x].
\]

**Proof:** The theorem is proved using mathematical induction on, $n \geq 2$.
For $n = 2$, then the formula in this theorem is true.
The theorem is assumed that is true for each value less than $n$, $n \geq 3$ and prove it for all $n$. Then, using Theorem 2, will be get the following:

$$
\mathcal{D}[\prod_{i=1}^{n} V_I(\Phi_i), x] = \mathcal{D}[\prod_{i=1}^{n-1} V_I(\Phi_i)] \Phi_n, x
$$

Using the mathematical induction hypothesis, the following is obtained:

$$
\mathcal{D}[\prod_{i=1}^{n} V_I(\Phi_i), x] = \sum_{i=1}^{n-1} \mathcal{D}[\Phi_i, x]
$$

Lemma 4: For all $u_i, v_i \in V(\Phi_i), i = 1, 2, ..., t$, $t \geq 2$, will be:

$$
\mathcal{D}[v_t, \prod_{i=1}^{t-1} V_I(\Phi_i), x] = \mathcal{D}[v_t, \Phi_t, x]
$$

Proof: The mathematical induction will be used on $t, t \geq 2$. From Fig.1, the following is noticed that, when $t = 2$,

$$
\mathcal{D}[v_2, \prod_{i=1}^{1} V_I(\Phi_i), x] = \mathcal{D}[v_2, \Phi_2, x] + \mathcal{D}[v_1, \Phi_1, x] x^{\xi_{h+1}^t}(v_2, v_4).
$$

Therefore, the theorem is true for $t = 2$.

Now, the theorem assumed that is true for all value less than $t$, $t \geq 3$. For all $t$, will be

$$
\mathcal{D}[v_t, \prod_{i=1}^{t-1} V_I(\Phi_i), x] = \mathcal{D}[v_t, \Phi_t, x]
$$

Proof: From Theorem 5, the following will be obtained:

$$
\mathcal{D}[\prod_{i=1}^{n} V_I(\Phi_i), x] = \sum_{i=1}^{n} \mathcal{D}[\Phi_i, x]
$$

Corollary 6: For any conn.graph. $\Phi$ and for all $n \in \mathbb{N}, n \geq 3$, there will be:

$$
\mathcal{D}[V_{\Phi}^0(\Phi), x] = \sum_{i=1}^{n} \mathcal{D}[\Phi_i, x]
$$

Proof: From Theorem 5, the following will be obtained:

$$
\mathcal{D}[\prod_{i=1}^{n} V_I(\Phi_i), x] = \sum_{i=1}^{n} \mathcal{D}[\Phi_i, x]
$$

Corollary 7: For a conn.graph. $\Phi$ and for all $n \in \mathbb{N}, n \geq 3$, there will be:

$$
\mathcal{D}[V_{\Phi}^0(\Phi)] = \mathcal{D}[V_{\Phi}^0(\Phi), x]_{x=1} = \mathcal{D}(\Phi) + (p-1)^2 \mathcal{D}(u, v, \Phi)_{x=1}
$$

Proof: From Corollary 6, the following will be obtained:

$$
\mathcal{D}[\prod_{i=1}^{n} V_I(\Phi_i), x] = \sum_{i=1}^{n} \mathcal{D}[\Phi_i, x]
$$
\[ = n\mathcal{D}(\Phi) + (p - 1)^2\mathcal{D}(u, v) \binom{n}{3} \]
\[ + (p - 1)\mathcal{D}(u, v, G) + \mathcal{D}(u, \Phi) \binom{n}{2}, \]
where \([v, \Phi, 1] = p - 1\), for any vertex \(v \in \Phi\).

Examples:
In this section, detour polynomials and detour indices of generalized vertex identified of special graphs will be given a general formula such as: complete graph \(K_p\), wheel graph \(W_p\), cycle graph \(G_p\), quadruple circle with horizontal chord \(G_4^+\) and quadruple circle with vertical chord \(G_4^-\).

Example 1: If \(K_p\) is a complete graph of order \(p\), \(p \geq 3\), then
\[ \mathcal{D}(V_1^p(K_p); x) = n \binom{p}{2} x^{p-1} \]
\[ + (p - 1)^2 x^{2p-2} \sum_{i=0}^{n-2}(n - 1 - i)x^{i(p-1)}. \]

Special case, if 4 , see Fig. 2. Then

**Figure 2. Graph \(V_1^p(K_4)\)**

\[ \mathcal{D}(V_1^p(K_4); x) = 6nx^3 + 9x^6 \sum_{i=0}^{n-2}(n - 1 - i)x^{3i}. \]

Example 2: If \(\Phi \equiv W_p\), \(p \geq 3\), then
\[ \mathcal{D}(V_1^p(W_p); x) = n \binom{p}{2} x^{p-1} \]
\[ + (p - 1)^2 x^{2p-2} \sum_{i=0}^{n-2}(n - 1 - i)x^{i(p-1)}. \]

Example 3: If \(\Phi \equiv G_p\), \(p \geq 4\), and \(\mathcal{D}(u, v) = \left[ \begin{array}{c} p \\ 2 \end{array} \right]\), where \(u \) and \(v \) are identical vertices, then

(1) If \(p \) is odd ,
\[ \mathcal{D}(V_1^p(G_p); x) = npx^{\frac{p+1}{2} \left[ \frac{p-1}{2} \right]} \]
\[ + 4x^{p+1}\left[ \frac{p-1}{2} \right]^2 \sum_{i=0}^{n-2}(n - 1 - i)x^{i\left( \frac{p+1}{2} \right)}. \]

**Figure 5a. Graph \(V_1^p(G_4^+)\)**

(2) If \(\Phi \equiv G_4^+\), see Fig. 5.a, then
\[ \mathcal{D}(V_1^p(G_4^+); x) = 5nx^3(1 + x) + 4x^6(1 + x^2)\sum_{i=0}^{n-2}(n - 1 - i)x^{3i}. \]

**Example 4:**

(1) If \(\Phi \equiv G_4^+\), see Fig. 5.a, then
\[ \mathcal{D}(V_1^p(G_4^+); x) = n^2x^2(1 + 2x) + 9x^6\sum_{i=0}^{n-2}(n - 1 - i)x^{2i}. \]

**Corollary 8:** For \(n \geq 3\), there will be:

(1) \[ \mathcal{D}(V_1^p(K_p)) = n.(p - 1).\binom{p}{2} \]
\[ + (p - 1)^3.(\binom{n}{3} + 2 \binom{n}{2}). \]

(2) \[ \mathcal{D}(V_1^p(G_p)) = \frac{1}{6}np^2.(3p - 4) \]
\[ + \binom{p}{2}.((p - 1).\binom{n}{3} + (3p - 4).\binom{n}{2}). \]

when \(p\) is an odd.

(3) \[ \mathcal{D}(V_1^p(G_p)) = \frac{1}{8}np.(p - 1).(3p - 1) \]
\[ + \frac{1}{2}(p - 1)^2.((p + 1).\binom{n}{3} + (3p - 1).\binom{n}{2}). \]

where \(p\) is even

(4) \[ \mathcal{D}(V_1^p(G_4^+)) = n(3n^2 + 15n - 1). \]

(5) \[ \mathcal{D}(V_1^p(G_4^-)) = \frac{1}{2}n(63n^2 - 81n + 52). \]
The Graphs with Equal Detour Polynomials

The two polynomials will be equal if and only if the coefficients of \( x^i \) are equal for all \( i \), that is, \( \sum_{i=0}^{\beta} a_i x^i = \sum_{i=0}^{\beta} b_i x^i \) iff \( a_i = b_i \), for all \( i \). It is clear that if \( \Phi_1 \cong \Phi_2 \), then \( \mathcal{D}[\Phi_1, x] = \mathcal{D}[\Phi_2, x] \), but the converse is not true, for example, the detour polynomials of complete graph \( \mathcal{K}_p \) and wheel graph \( \mathcal{W}_p \), \( p \geq 4 \) are equal, but \( \mathcal{K}_p \) and \( \mathcal{W}_p \) are not isomorphic, \( \mathcal{K}_p \not\cong \mathcal{W}_p \).

Corollary 9: Let \( \Phi, \Phi^* \) and \( \Phi^{**} \) be conn. graphs. and the vertices identification \( v^* \in V(\Phi), v^{**} \in V(\Phi^*) \) if they satisfy the conditions:

- \( \mathcal{D}[\Phi^*, x] = \mathcal{D}[v^{**}, x] \).
- \( \mathcal{D}[v^{**}, \Phi^*, x] = \mathcal{D}[v^{***}, \Phi^{**}, x] \).

Then \( \mathcal{D}[\Phi, \Phi^*, x] = \mathcal{D}[\Phi, \Phi^{**}, x] \).

Proof: Obviously.

Corollary 10: Let, \( \Phi^*_i \) and \( \Phi^{**}_i \) be conn. graphs. and the vertices identification \( v^*_i, u^*_i \in V(\Phi^*_i) \) and \( v^{**}_i, u^{**}_i \in V(\Phi^{**}_i) \), for all \( i = 1, 2, \ldots, n \), \( n \geq 2 \). If they satisfy the conditions for all \( i = 1, 2, \ldots, n \), \( n \geq 2 \):

1. \( \mathcal{D}[\Phi^*_i, \Phi^{**}_i, x] = \mathcal{D}[v^{**}_i, \Phi^{**}_i, x] \).
2. \( \mathcal{D}[v^{**}_i, \Phi^*_i, x] = \mathcal{D}[v^{***}_i, \Phi^{**}_i, x] \) and \( \mathcal{D}[u^{**}_i, \Phi^*_i, x] = \mathcal{D}[u^{***}_i, \Phi^{**}_i, x] \).
3. \( \mathcal{D}[v^{**}_i, u^{**}_i] = \mathcal{D}[v^{***}_i, u^{**}_i] \).

Then \( \prod_{i=1}^{n} V_{\mathcal{D}}(\Phi^*_i) = \prod_{i=1}^{n} V_{\mathcal{D}}(\Phi^{**}_i) \), see Examples 1 and 2.

Proof: Obviously.

Conclusions:

The construct operation generalized by vertices identified of \( n \) – graphs are found from any two graphs from a sequence of \( n \) pairwise vertex disjoint connected graphs \( \Phi_1, \Phi_2, \ldots, \Phi_n \), \( n \geq 3 \). Moreover, the detour polynomial and detour index will be obtained for generalized vertex identified graphs when every \( \Phi_i, i = 1, 2, \ldots, n \), \( n \geq 3 \) is an isomorphic to a special graph which has important applications in chemistry.

Acknowledgment:

This paper is supported from College of Science/University of Duhok, Kurdistan-Iraq, College Computer Sciences & Math./University of Mosul, Iraq and College of Science/Salahaladdin University, Kurdistan - Iraq.

Authors’ declaration:

- Conflicts of Interest: None.

- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for republication attached with the manuscript.

Ethical Clearance: The project was approved by the local ethical committee in University of Mosul.

Authors’ contributions statement:

H. J. A., A. M. A. and G. A. M. S. uses their initials to explain his/her contribution (The definition: Generalized Vertex Identified of Graphs was presented by A. M. to PhD student H., with some problems which unsolved G. A. also made suggestions Something and gave other unresolved problems for some special graphs.

References:


متعددات حدود الالتفاف لتعزيز تطابق الرأس للبيانات

هامين جلال أحمد
أحمد محمد علي
كه شاو محمد صالح

الخلاصة:

تعد مسافة الالتفاف من أهم أنواع المسافات التي لها تطبيقات حديثة في الكيمياء وشبكات الكمبيوتر، لذلك حصلنا في هذا البحث على متعددات حدود الالتفاف وأدلتها لـ n من البيانات المنفصلة عن بعضها البعض بالنسبة للرؤوس ، n≥3. أيضنا وجدنا متعددات حدود الالتفاف وأدلتها لبعض البيانات الخاصة والتي لها تطبيقات مهمة في الكيمياء.

الكلمات المفتاحية: مسافة الالتفاف، متعدد الحدود الالتفاف، دليل الالتفاف، بيانات خاصة، تطابق الرؤوس.