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# A Numerical scheme to Solve Boundary Value Problems Involving Singular Perturbation

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## Abstract

The Wang-Ball polynomials operational matrices of the derivatives are used in this study to solve singular perturbed second-order differential equations (SPSODEs) with boundary conditions. Using the matrix of Wang-Ball polynomials, the main singular perturbation problem is converted into linear algebraic equation systems. The coefficients of the required approximate solution are obtained from the solution of this system. The residual correction approach was also used to improve an error, and the results were compared to other reported numerical methods. Several examples are used to illustrate both the reliability and usefulness of the Wang-Ball operational matrices. The Wang Ball approach has the ability to improve the outcomes by minimizing the degree of error between approximate and exact solutions. The Wang-Ball series has shown its usefulness in solving any real-life scenario model as first- or second-order differential equations (DEs).

**Keywords:** Boundary conditions, Integral equations, Numerical solutions, Singularly perturbed, Wang-Ball polynomials.

## Introduction

The perturbation problems involve differential equations with a higher-order derivative. The derivative order of these problems increases by a small positive parameter known as the perturbation parameter [1-4]. Modeling a phenomenon in science and engineering often requires looking at differential equations with very small (or very large) parameters. When these parameters get close to zero (or infinity), the solutions of these differential equations behave very differently, which makes it harder to get close numerical solutions that are accurate. This concept is called "singular perturbation" [5]. In several disciplines, such as magneto-hydrodynamics, fluid dynamics and mechanics, aerodynamics, plasma dynamics, elasticity, rarefied gas dynamics, oceanography, and other domains of the fantastic world of fluid motion, singularly perturbed differential equations with epsilon as a small parameter are used in the mathematical simulation of procedures.

Steady and unsteady viscous flow problems, especially with the large Reynolds numbers and boundary layers, usually are difficult to model on the basis of a tiny positive parameter. Big challenges were known to be remarkable cases. There are boundaries in this issue class, which are places where the solution quickly alters close to one of the boundary points. The solution to this equation fluctuates quickly in some parts of the domain and slowly in others.

Recently, a large number of techniques have been presented to solve singularly perturbed boundary value problems. For example, Abdullah[6] has recently introduced a number of strategies for solving ordinary differential equations and Volterra integral equations (VIEs) utilizing the operation matrix of differentiation and integration. To solve VIEs using Touchard Polynomials, Al-Saif & Ameen[7] employ the collection approach. They apply the collocation method for solving mixed Volterra – Fredholm integral equations (MVFIEs). Also, for the singularly perturbed boundary value problems, a novel exponentially fitted integration approach on a uniform mesh is developed by Alam et al.[8] . And a Hermite approximation is developed for solving the singular perturbed delay differential equations under the boundary conditions [9]. Furthermore, a numerical method was presented based on quintic B-spline functions to find the solution of the singular Emden–Fowler Equation [10]. In addition, a new fractional-order derivative operational matrix was suggested by Ghomanjani in which the matrix depends on Genocchi polynomials [11]. For the computational solution of singularly perturbed boundary-value problems, Farajeyan et al. designed a class of new approaches focused on changing the polynomial spline equation [3].

While there are various disciplines that deal with singularly perturbed boundary problems and have used a variety of asymptotic expansion approaches to solve them, more effective and simplified computational methodologies are needed to handle uniquely disrupted boundary value problems [12].

The equation is as follows,

$$\varepsilon x''(r) + p_1(r)x'(r) + p_2(r)x(r) = f(r), \quad 1$$

With boundary conditions (BCs)

$$x(0) = \alpha_1, x(1) = \alpha_2. \quad 2$$

Wang-Ball polynomial has various applications, such as surface interpolation in geometric modeling [13,14]**.** However, to the best of our knowledge, the first application of the Wang-Ball and DP-Ball polynomials in a numerical approach were by Kherd et al. [14, 15], who obtained surprising results when compared to existing methods. This paper is an extension of the Wang- Ball series to solve problems involving singular perturbation.

The following outline constitutes this paper's structure:

At the beginning of the article, there is a concise explanation of the Wang-Ball polynomial, as well as its conventional derivation and its operational matrix differentiation. In addition, the applications of the operational matrix of the derivative are explained. We provide a brief explanation of a method for estimating the error of a solution that has already been found. This makes it effective to bring improvement to the solution itself. Afterward, we will proceed to explain our findings by going through four numerical examples. Finally, we arrive at some conclusions about the existing approach.

## Review on Ball polynomial

The Ball polynomial was introduced by A. A. Ball in his well-known aircraft design system CONSURF [16]. It is described as a cubic polynomial and defined mathematically as [14].

$$(1-r)^2, 2r(1-r)^2, 2r^2(1-r), r^2, \quad 0 \le r \le 1 \quad 3$$

Previous studies have investigated the Ball polynomial's high generality and qualities. For instance, in the 1980s, two distinct Ball polynomials of an arbitrary degree, namely Said-Ball and Wang-Ball [14,15] were introduced.

## Wang-Ball Polynomial Representation

Wang-Ball polynomial $W_i^m(r)$ of degree, $m$ can be defined by[13-15, 17].

$$W_i^m(r) = \begin{cases} (1-r)^{2+i}\ (2r)^i & ,0 \le i \le \frac{m-3}{2} \\ (1-r)^{\frac{1+m}{2}}\ (2r)^{\frac{1-m}{2}} & ,i = \frac{m-1}{2} \\ \left(2(1-r)\right)^{\frac{m-1}{2}}\ r^{\frac{m+1}{2}} & ,i = \frac{m+1}{2} \\ (2(1-r)r)^{m-i}\ r^{m+2-i} & ,\frac{m+3}{2} \le i \le m \end{cases} \quad 4$$

when $m$ is odd, and

$$W_i^m(r) = \begin{cases} (1-r)^{2+i}\ (2r)^i & ,0 \le i \le \frac{m}{2} - 1 \\ \left(2(1-r)\right)^{\frac{m}{2}} & ,i = \frac{m}{2} \\ \left(2(1-r)\right)^{m-i}\ t^{m+2-i} & ,\frac{m+3}{2} \le i \le m \end{cases} \quad 5$$

## Wang-Ball Monomial Form

Given a Wang-Ball curve of degree $m$ represented by $A_m(r)$ together with $m+1$ control points, represented by$\{w_i\}_{i=0}^m$. The degree m Wang-Ball

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$W_i^m(r)$ is shown in the form of power basis as given below[17]

$$W_i^m(r) = \sum_{i=0}^{m}\sum_{l=0}^{m} w_{k,l}\, r^l \,, 0 \le r \le 1 \qquad 6$$

where

$$w_{lk} = \begin{cases} (-1)^{(k-l)}2^l\binom{l+2}{k-l}, & \text{for } 0 \le l \le \left\lfloor\frac{m}{2}\right\rfloor - 1, \\ (-1)^{(k-l)}2^l\binom{n-l}{k-l}, & \text{for } l = \left\lfloor\frac{m}{2}\right\rfloor, \\ (-1)^{(k-l)}2^{n-l}\binom{n-l}{k-l}, & \text{for } l = \left\lceil\frac{m}{2}\right\rceil, \\ (-1)^{(k-n+l)}2^{n-l}\binom{n-l}{k-n+l-2}, & \text{for } \left\lceil\frac{m}{2}\right\rceil + 1 \le l \le n \end{cases}$$

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where $\lfloor x \rfloor$ represents GI $\le$ x and $\lceil x \rceil$ represents $LI \ge x$ where $GI$ and $LI$ are the greatest integer and least integer, respectively. The Wang-Ball monomial matrix is

$$\mathcal{A} = \begin{bmatrix} w_{00} & w_{01} & \cdots & \cdots & w_{0m} \\ w_{10} & w_{11} & \cdots & \cdots & w_{1m} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & \cdots & w_{mm} \end{bmatrix}_{(m+1)(m+1)} \qquad 8$$

where $w_{lk}$ is given as in (7).
The Wang-Ball basis function satisfies the following properties.

i. The Wang-Ball basis function is non-negative, that is,
$W_i^m(r) \ge 0, \forall i = 0,1,\dots,m$ 9

ii. The partition of unity that is,

$$\sum_{i=0}^{m} W_i^m(r) = 1 \ . \qquad 10$$

In general, any function $x(r)$ can be written with the first $(m+1)$ Wang-Ball polynomials and get approximated as

$$x(r) \approx \sum_{i=0}^{m} c_i' W_i^m(r) = \Omega(r)C'$$
$$= H_m(r)\mathcal{A}^T C' \qquad 11$$

where $C' = [c_0', c_1', \dots, c_m']^T$, $H_m(r) = [1\ r\ r^2\ \dots\ r^m]$ and $\mathcal{A}$ is the monomial matrix form given in Eq 8 . The $m+1$ by $m+1$an operational matrix of derivative of the Wang-Ball polynomials set $\Omega(r)$ is given by:

$$\frac{d\Omega(r)}{dr} = \frac{d}{dr}H_m(r)\mathcal{A}^T$$
$$= [0\ 1\ 2r\ \dots\ mr^{m-1}]\mathcal{A}^T$$
$$= [1\ r\ r^2\ \dots\ r^m]\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathcal{A}^T$$

$$= H_m(r)\Lambda\mathcal{A}^T \qquad 12$$

Where

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & m \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$x'(r) = H_m(r)(\Lambda)^n\mathcal{A}^T C' \qquad 13$$

Eq 13 can be generalised as

$$x^{(n)}(r) = \frac{d^n}{dr^n}H_m(r)(\Lambda)^n\mathcal{A}^T C' \,, n = 1,2, \dots$$

## Applications of the Operational Matrix of Derivative

The following is the derivation of the Wang-Ball Polynomials method for solving differential equations of the form Eq 1

$$\varepsilon H_m(r)(\Lambda)^2\mathcal{A}C' + p(r)H_m(r)\Lambda\mathcal{A}C' + q(r)H_m(r)\Lambda\mathcal{A}C' = f(r) \qquad 14$$

First Eq 14 is collocated at $(m-1)$ points. For suitable points, the following $r_i = \frac{1}{2}\left(\cos\left(\frac{i\pi}{N}\right) + 1\right), i = 1,2,\dots,N$ is used. Then Eq 14 can be written as a system of equation

$$(\varepsilon H_m(r_i)(\Lambda)^2\mathcal{A} + p(r_i)H_m(r_i)\Lambda\mathcal{A} + q(r_i)H_m(r_i)\Lambda\mathcal{A})C' = f(r_i)$$

or in matrix form

$$(\varepsilon H(\Lambda)^2\mathcal{A} + PH\Lambda\mathcal{A} + QH\mathcal{A})C' = F \qquad 15$$

Where

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$$\mathcal{E} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \varepsilon \end{bmatrix}, P = \begin{bmatrix} p(r_0) & 0 & 0 & 0 \\ 0 & p(r_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & p(r_N) \end{bmatrix}, F = \begin{bmatrix} f(r_1) \\ f(r_2) \\ \vdots \\ f(r_N) \end{bmatrix}$$

$$Q = \begin{bmatrix} q(r_0) & 0 & 0 & 0 \\ 0 & q(r_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & q(r_N) \end{bmatrix} \text{ and } H = \begin{bmatrix} H(r_1) \\ H(r_2) \\ \vdots \\ H(r_N) \end{bmatrix} = \begin{bmatrix} 1 & r_0 & \cdots & r_0^N \\ 1 & r_1 & \cdots & r_1^N \\ \vdots & \vdots & \cdots & \vdots \\ 1 & r_N & \cdots & r_N^N \end{bmatrix}$$

Eq.15 can be written as

$$SC' = F \quad \text{or} \quad [S; F], \qquad 16$$

where $S = [S_{i,j}] = \mathcal{E}H(\Lambda)^2\mathcal{A} + PH\Lambda\mathcal{A} + QH\mathcal{A}$, $i = 0,1, \ldots, N-2$ and $j = 0,1, \ldots, N$.
The boundary conditions in Eq 1 in matrix form as $H_m(0)\Lambda\mathcal{A} = [\alpha_1]$ and $H_m(b)\Lambda\mathcal{A} = [\alpha_2]$ that is $[1 \quad 0 \quad \cdots \quad 0]\Lambda\mathcal{A} = [\alpha_1]$ and $[1 \quad b \quad \cdots \quad b^N]\Lambda\mathcal{A} = [\alpha_2]$, Then the last two rows of $[S; F]$ are replaced by boundary conditions. Then Eq 16 becomes as

$$[\tilde{S}; \tilde{F}] = \begin{bmatrix} s_{0,0} & s_{0,1} & s_{0,2} & \cdots & s_{0,N} & ; & f(r_0) \\ s_{1,0} & s_{1,1} & s_{1,2} & \cdots & s_{1,N} & ; & f(r_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ s_{N-2,0} & s_{N-2,1} & s_{N-2,2} & \cdots & s_{N-2,N} & ; & f(r_N) \\ 1 & 0 & 0 & \cdots & 0 & ; & \alpha_1 \\ 1 & b & b^2 & \cdots & b^N & ; & \alpha_2 \end{bmatrix}$$
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If rank $\tilde{S} = rank[\tilde{S}; \tilde{F}] = N + 1$; therefore, the coefficient matrix $C'$ can be easily computed as $C' = \tilde{S}^{-1}\tilde{F}$.
Thus, by substituting the coefficient matrix $C'$ into Eq 11, the approximate solution can be obtained as

$$x_N(r) = \sum_{l=0}^{N} c_l' \, W_i^m(r).$$

These equations generate $(m+1)$ non-linear equations, which can be handled by employing the Newton's iteration method. As a result, $x(r)$ it can be calculated.

**Error analysis and estimation of the absolute error**

The error analysis of the approach utilized is described in this section. The problem will be given a residual correction approach that can estimate the absolute inaccuracy.

Let $x_N(r)$ and $x(r)$ be the approximate solution and the exact solution of Eq 1, respectively. In the process below, for the estimation of the absolute error, the residual correction could be assigned [18].

First, the following results are obtained by removing the term from both sides of Eq 1.

$\mathfrak{R} = \varepsilon x_N''(r) + p_1(r)x_N'(r) + p_2(r)x_N(r) - f(r)$, to (1) yield the following differential equation

$$\varepsilon e_N''(r) + p_1(r)e_N'(r) + p_2(r)e_N(r) = f(r) - \mathfrak{R}$$
18

with the homogenous BCs

$$x(0) = 0, x(1) = 0 \qquad 19$$

Where $e_N = x(r) - x_N(r)$

For some choices of $M \geq N$, applying the proposed approach to problems 18 and 19 yields an approximate solution, which will be donated by $E_{N,M}^*$ .The actual error function $e_N(r)$ is estimated in this approximation solution. This estimate can be used to generate a new approximate solution, keeping in mind that $x_{exact}(r) = x_N(r) + e_N(r)$. This estimate can be utilized to compute another fresh approximate solution

$$x_{M,N}(r) = x_N(r) + E_{N,M}^*$$

of the problem 1. The error of this new solution $x_{N,M}(r)$, called the corrected solution, is directly related to the accuracy of the error estimate $E_{N,M}^*(r)$. Specifically, if the error of $x_{N,M}(r)$ has been denoted by $E_{N,M}^{\varepsilon}(r)$, it is true that

$$E_{N,M}^{\varepsilon}(r) = x_{exact}(r) - x_{N,M}(r) = E_N(r) - E_{N,M}(r)$$

As a result, the precision of the error estimate $E_{N,M}(r)$ is directly related to the success of residual correction. In the examples problems that will be presented in the following part, this scenario will become evident.

### Error bound for the solution

In this part, the error bound for the approximate solution $x_N(r)$ is related to the truncation error of the Taylor polynomial corresponding to the exact solution.

### Theorem

Let $x_N(r)$ *and* $x(r)$ denote the approximate and the exact solutions of problem 1, respectively. If

$x(r) \in C^{N+1}[0,b]$ , then

$$|x(r) - x_N(r)| \le \left|R_N^T(r)\right| + \left|x_N^T(r) - x_N(r)\right| \qquad 20$$

Where $x_N^T(r)$ denotes the $N^{th}$degree Taylor polynomial of $x(r)$ around the points $r = q \in [0,b]$ and $R_N^T(r)$ represents its reminder term.

**Proof** Since $x(r)$ is $(N+1)$-times continuously differentiable, it can be represented by its Taylor series as

$$x(r) = \sum_{k=0}^{N} \frac{(r-q)^k}{k!} x^k(q) + R_N^T(r)\,,$$

where

$$R_N^T(r) = \frac{(r-q)^{N+1}}{(N+1)!} x^{N+1}(d_r), 0 < r \le b$$

is the remaining term of the Taylor expansion $x(r)$. Thus, $x(r) - x_N^T(r) = R_N^T(r)$ by using this and the triangle inequality, the following result might be obtained

$$\begin{aligned} |x(r) - x_N(r)| &= \left|x(r) - x_N(r) + x_N^T(r) - x_N^T(r)\right| \\ &\le \left|x(r) - x_N^T(r)\right| + \left|x_N^T(r) - x_N(r)\right| \\ &= \left|R_N^T(r,q)\right| + \left|x_N^T(r) - x_N(r)\right| . \end{aligned}$$

Therefore, an upper bound of the absolute error based on the Taylor truncation error of the exact solution is found. Note that this is not an a priori error bound; it only works as a means to compare the actual error to this Taylor truncation error.

## Results and Discussion

**Problem 1**

Consider the first-order ODE with constant coefficients [4]

$$\varepsilon x''(r) + x(r) = 0 \qquad 21$$

with BCS

$$x(0) = 0, x(1) = 1 \qquad 22$$

Which has the exact solution is

$$x_{exsact}(r) = \frac{\sin(r/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})} \qquad 23$$

The problem is solved using different values of $N$ , and $\varepsilon$ as shown in Table 1. The comparison between the current method with the method reported by Yüzbaşı et al.[4] is in Table 2. As illustrated in Table 2 when higher values of N (12 and 14) are used, the absolute error for problem 1 shows better results for the present method compared to Yüzbaşı et al.[4]

**Table 1. The max absolute error for problem 1.**

| ε | N=5 | N=7 | N=9 | N=11 | N=13 | N=15 |
|---|---|---|---|---|---|---|
| $2^{-2}$ | 2.03E-4 | 6.90E-7 | 1.75E-9 | 3.17E-12 | 4.44E-15 | 8.88E-16 |
| $2^{-4}$ | 1.28E-2 | 2.13E-4 | 2.22E-6 | 1.61E-8 | 8.60E-11 | 3.53E-13 |
| $2^{-6}$ | 4.00E-1 | 3.85E-2 | 1.33E-3 | 3.56E-5 | 7.59E-7 | 1.26E-8 |

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**Table 2. Comparison of the max absolute error between the proposed methods with ref [4] for problem 1 at N=10, 12, 14.**

| $\varepsilon$ | Yüzbaşı and Karaçayır [4] | | | Present method | | |
|---|---|---|---|---|---|---|
| | N=10 | N=12 | N=14 | N=10 | N=12 | N=14 |
| $2^{-2}$ | 0.4937E-12 | 0.4076E-13 | 0.1138E-12 | 0.4936E-12 | 0.4638E-15 | 0.8882E-17 |
| $2^{-4}$ | 0.9168E-9 | 0.5626E-11 | 0.3041E-11 | 0.9179E-9 | 0.5665E-11 | 0.2633E-13 |
| $2^{-6}$ | 0.1870E-5 | 0.4735E-7 | 0.1564E-7 | 0.1871E-5 | 0.4657E-7 | 0.8771E-9 |

The exact solution and the absolute solution of different values of $N$ and $\varepsilon$ are displayed in Fig 1 (a and b). In contrast, the absolute error for various values of $N$ and $\varepsilon$ is illustrated in Fig 1(c and d).

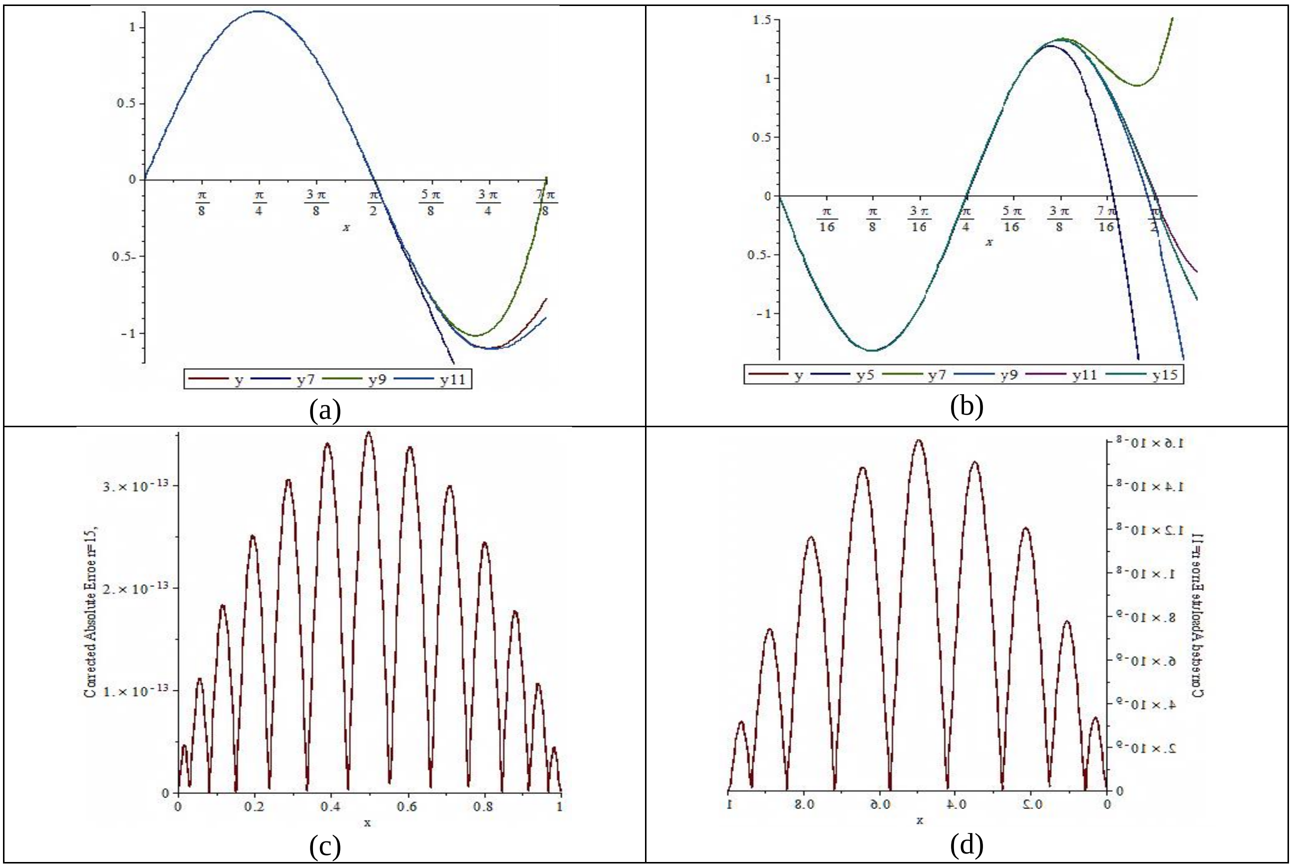


**Figure 1. The approximate solution when $N = 7, 9, 11$ at $\varepsilon = 2^{-2}$ and the exact solution, while (b) is the exact solution together with the approximate solution for, how $N = 5, 7, 9, 11$ and $15$ when $\varepsilon = 2^{-2}$, (c) and (d) are the corrected absolute error when $N = 15$ with $\varepsilon = 2^{-4}$ and $N = 11$ for $\varepsilon = 2^{-2}$**

**Problem 2.**

Consider the second-order nonhomogeneous equation [4, 15]

$$-\varepsilon x''(r) + \frac{1}{r+1}x'(r) + \frac{1}{r+1}x(r) = f(r)\,, \qquad 24$$

subject to BCs

$$x(0) = 1 + 2^{-\frac{1}{\varepsilon}}\,, x(1) = 2 + e \qquad 25$$

Where

$f(r) = \left(\frac{1}{r+1} + \frac{1}{r+2} - \varepsilon\right)e^r + \frac{2^{-\frac{1}{\varepsilon}}(r+1)^{1+\frac{1}{\varepsilon}}}{r+2}$, this problem has the exact solution given by

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$x(r) = e^r + 2^{-\frac{1}{\varepsilon}}(r+1)^{1+\frac{1}{\varepsilon}}$.

Table 3 shows the absolute error for the proposed technique compared to the published methods in refs [4] and [15] for various values of $N$. Clearly, in Table 3. It can be seen that Yüzbaşı et al. [4] reported better results for the absolute error at values of $N$ (8 and 10) compared to results reported by Lin [10]. In addition, the proposed method gave better results of the absolute error for the same $N$ values. On the other hand, Table 4 shows the actual and estimated absolute error for the suggested method. For various $N$ and $M$ values, better results of the estimated absolute errors were obtained compared to the actual absolute errors of the problem 2.

**Table 3. The comparison max absolute error for ref [4, 15] with the suggested method for problem 2.**

| $\varepsilon$ | Referance[10] | | Referance[4] | | Present method | |
|---|---|---|---|---|---|---|
| | N=8 | N=10 | N=8 | N=10 | N=8 | N=10 |
| $2^{-2}$ | 1.139E-8 | 1.824E-11 | 1.237E-10 | 1.210E-13 | 8.58E-11 | 8.53E-14 |
| $2^{-3}$ | 7.610E-6 | 1.434E-11 | 1.052E-7 | 1.323E-13 | 7.78E-8 | 6.33E-14 |
| $2^{-4}$ | 9.630E-3 | 4.495E-4 | 3.135E-4 | 3.455E-6 | 2.34E-4 | 2.91E-6 |
| $2^{-5}$ | 1.615E-1 | 5.235E-2 | 1.992E-2 | 1.566E-3 | 1.32E-2 | 1.24E-3 |
| $2^{-6}$ | 5.301E-1 | 3.410E-1 | 2.281E-1 | 5.465E-2 | 1.10E-1 | 3.34E-2 |
| $2^{-7}$ | 9.404E-1 | 7.571E-1 | 1.003E-0 | 4.390E-1 | 4.80E-1 | 1.89E-1 |

**Table 4. Actual absolute errors and Estimated absolute errors for problem 2 with N=6, 9, 12 and M=7, 10, 13 at different values of $\varepsilon$.**

| $\varepsilon$ | Actual absolute errors | | | Estimated absolute errors | | |
|---|---|---|---|---|---|---|
| | $E^{\varepsilon}_{6}$ | $E^{\varepsilon}_{9}$ | $E^{\varepsilon}_{12}$ | $E^{\varepsilon}_{6,7}$ | $E^{\varepsilon}_{9,10}$ | $E^{\varepsilon}_{12,13}$ |
| $2^{-2}$ | 0.104E-8 | 0.211E-13 | 0.113E-14 | 0.324E-10 | 0.447E-15 | 0.888E-17 |
| $2^{-3}$ | 0.119E-5 | 0.223E-13 | 0.677E-15 | 0.449E-7 | 0.486E-15 | 0.666E-17 |
| $2^{-4}$ | 0.858E-4 | 0.293E-6 | 0.137E-9 | 0.159E-4 | 0.291E-7 | 0.627E-11 |
| $2^{-5}$ | 0.850E-3 | 0.429E-4 | 0.723E-6 | 0.363E-3 | 0.124E-4 | 0.148E-6 |
| $2^{-6}$ | 0.381E-2 | 0.634E-3 | 0.758E-4 | 0.196E-2 | 0.334E-3 | 0.325E-4 |
| $2^{-7}$ | 0.115E-1 | 0.296E-2 | 0.852E-3 | 0.670E-2 | 0.189E-2 | 0.566E-3 |

**Problem 3.**

Thirdly, consider the singularly perturbed two-point boundary value problem[19, 20]

$$-\varepsilon x''(r) + x(r) = -\cos^2(\pi r) - 2\varepsilon\pi^2\cos(2\pi r)\,, \quad 26$$

with BCs

$$x(0) = x(1) = 0 \quad 27$$

The exact solution is given by

$$x(r) = \frac{\exp\left(-\frac{1-r}{\sqrt{\varepsilon}}\right) + \exp\left(-\frac{r}{\sqrt{\varepsilon}}\right)}{1+\exp\left(-\frac{1}{\sqrt{\varepsilon}}\right)} - \cos^2(\pi r)$$

Fig. 2a displays the exact and approximate solutions for various values of $N$ and $\varepsilon = 16$, whereas Fig 2d shows the approximate solution for various values of. However, for problem 3, Fig 2 (b and c) shows the absolute error for a variety of $N$ and $\varepsilon = 16$ values.

In Table 5, it is easy to see that there were slit differences in the findings of Aziz and Khan [19,20] when quintic spline and a spline method were used for larger N values. However, by applying the present method for lower values of (N, M) = (12, 15), (14, 17), the max absolute error results are better compared to the findings of Aziz and Khan [19,20].

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**Table 5. Comparison of the max absolute error for our method with reported work [19, 20] for problem 3**

| $\varepsilon$ | Reference [19] | | Reference [20] | | Present method | | | |
|---|---|---|---|---|---|---|---|---|
| | N=128 | N=256 | N=128 | N=256 | N=12 | N=12,M=15 | N=14 | N=14,M=17 |
| 1/16 | 0.330E-10 | 0.205E-11 | 0.988E-10 | 0.6172E-11 | 0.151E-11 | 0.125E-15 | 0.161E-15 | 0.945E-18 |
| 1/32 | 0.162E-10 | 0.100E-11 | 0.484E-10 | 0.3032E-11 | 0.137E-11 | 0.131E-15 | 0.161E-15 | 0.945E-18 |
| 1/64 | 0.439E-10 | 0.278E-11 | 0.134E-9 | 0.8397E-11 | 0.316E-11 | 0.732E-15 | 0.907E-15 | 0.672E-13 |
| 1/128 | 0.145E-9 | 0.944E-11 | 0.481E-9 | 0.3011E-10 | 0.136E-7 | 0.368E-11 | 0.453E-11 | 0.913E-15 |

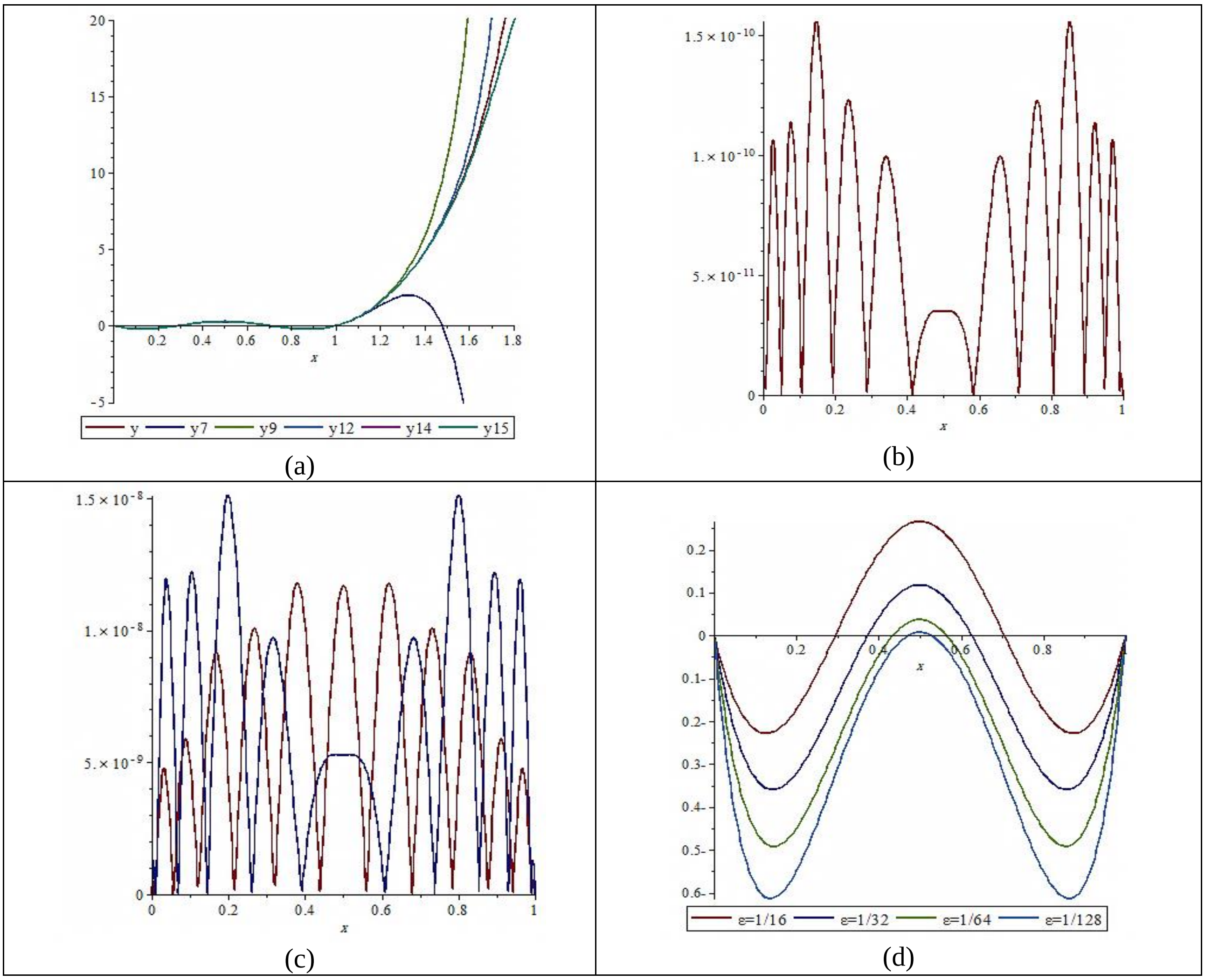


**Figure 2. (a) The exact solution and the approximate solution when $N = 7, 9, 12, 14, 15$ and $\varepsilon = 16$, (b) The absolute error were $N = 15$ and $\varepsilon = 16$ (c) The absolute error where $N = 12$ and $N = 13$, and (d) The approximate solutions for different values of $\varepsilon$ .**

**Problem 4.**

Consider a singular perturbation two-point boundary value problem is [19]

$$-\varepsilon x''(r) + 4x(r) = 4 + 2\sqrt{\varepsilon}\left(e^{-\frac{r}{\sqrt{\varepsilon}}} + e^{\frac{r-1}{\sqrt{\varepsilon}}}\right) - 3(1-r)e^{-\frac{r}{\sqrt{\varepsilon}}} - 3r\left(e^{\frac{r-1}{\sqrt{\varepsilon}}}\right), \quad 28$$

Subject to BCs
$x(0) = x(1) = 0.$
The exact solution is
$$x(r) = 1 - (1-r)e^{-\frac{r}{\sqrt{\varepsilon}}} - r\left(e^{\frac{r-1}{\sqrt{\varepsilon}}}\right).$$

Table 6 displays the absolute error and the correct absolute error for different values of M, N, and $\varepsilon$ when the present Wang-Ball method was applied. Also, table 7 gives the values of the absolute error and the correct absolute error for different values of M, N, and $\varepsilon$ for the suggested method against the one reported by Aziz and Khan[19]. The results show that in spite of the lower values of N Wang –Ball method gives better result comparing to Aziz and Khan[19] method.

**Table 6. Maximum absolute errors, problem 4 present method.**

| $\varepsilon$ | N=16 | N=16,M=19 | N=16,M=22 | N=15 | N=15,M=20 | N=15,M=22 |
|---|---|---|---|---|---|---|
| $2^{-1}$ | 2.90E-12 | 8.88E-16 | 1.23E-15 | 1.61E-12 | 1.11E-15 | 1.01E-15 |
| $2^{-2}$ | 3.48E-12 | 1.89E-15 | 1.84E-15 | 1.85E-12 | 1.55E-15 | 2.04E-15 |
| $2^{-3}$ | 4.31E-12 | 2.23E-9 | 2.12E-15 | 2.08E-12 | 2.89E-15 | 1.03E-15 |
| $2^{-4}$ | 5.55E-12 | 6.88E-15 | 5.78E-15 | 2.43E-12 | 2.70E-8 | 4.98E-15 |
| $2^{-5}$ | 7.22E-12 | 3.48E-8 | 3.16E-14 | 2.31E-11 | 3.03E-14 | 1.97E-14 |
| $2^{-6}$ | 2.72E-11 | 4.37E-13 | 3.55E-13 | 1.54E-9 | 5.65E-7 | 2.84E-13 |
| $2^{-7}$ | 2.25E-9 | 4.30E-11 | 6.82E-12 | 7.24E-8 | 5.02E-12 | 5.91E-12 |

**Table 7. Comparison of the max absolute error for problem 4 with ref [19]**

| $\varepsilon$ | Reference [19] | | | | Present Method | |
|---|---|---|---|---|---|---|
| | N=32 | N=64 | N=128 | N=256 | N=15 | N=17, M=22 |
| $2^{-4}$ | 0.657E-09 | 0.438E-10 | 0.279E-11 | 0.175E-12 | 0.39146E-14 | 0.52535E-16 |
| $2^{-5}$ | 0.182E-08 | 0.130E-09 | 0.854E-11 | 0.540E-12 | 0.56366E-14 | 0.30243E-15 |
| $2^{-6}$ | 0.535E-08 | 0.420E-09 | 0.283E-10 | 0.182E-11 | 0.21544E-12 | 0.18119E-14 |
| $2^{-7}$ | 0.299E-07 | 0.135E-08 | 0.977E-10 | 0.637E-11 | 0.19633E-10 | 0.77307E-13 |

## Conclusion

The subject of this article is the numerical solution of singularly perturbed second-order differential equations with boundary conditions. The Wang-Ball operational matrix, which was developed to generalize the ordinary Ball polynomial, is the method presented. The novel approach converts the (SPSODEs) into a set of linear and non-linear algebraic equations with respect to the DE's property for each numerical issue considered. Therefore, the DEs are easier to solve while still yielding precise results. The Wang-Ball operational matrix has shown impressive performance when compared to current literature, in addition to recovering the exact solution of specific DEs. Consequently, the method provided in this article may be used to solve any real-life scenario model in the form of either first or second-order DEs.

## Authors' Declaration

- Conflicts of Interest: None.
- I/We hereby confirm that all the Figures and Tables in the manuscript are mine/ours. Furthermore, any Figures and images, that are not mine/ours, have been included with the necessary permission for re-publication, which is attached to the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in Hadhramout University, Mukalla, Yemen.

## Authors' Contribution Statement

A.K. contributed to the design and implementation of the research, the analysis of the results, and the writing of the manuscript. H.A.A. and S.F.B. interpretation, drafting, revision, proofreading, and verifying the analytic approximate methods of the manuscript. The authors discussed the results and contributed to the final manuscript.

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# مخطط رقمي لحل مسائل القيمة الحدية التي تتضمن اضطراب مفرد

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## الخلاصة

نستخدم المصفوفات العملياتية لمشتقات وانج-بول متعددة الحدود في هذه الدراسة لحل المعادلات التفاضلية الشاذه المضطربة من الدرجة الثانية (WPSODEs) ذات الشروط الحدية. باستخدام مصفوفة كثيرات حدود وانج-بول، يمكن تحويل مشكلة الاضطراب الرئيسية الشاذ إلى أنظمة معادلات جبرية خطية. كما يمكن الحصول على معاملات الحل التقريبي المطلوبة عن طريق حل نظام المعادلات المذكور. وتم استخدام أسلوب الخطاء المتبقي أيضًا لتحسين الخطأ، كما تمت مقارنة النتائج بالطرق المنشورة في عدد من المقالات العلمية. استُخدِمت العديد من الأمثلة لتوضيح موثوقية وفائدة مصفوفات وانج بول العملياتية. طريقة وانج بول لديها القدرة على تحسين النتائج عن طريق تقليل درجة الخطأ بين الحلول التقريبية والدقيقة. أظهرت سلسلة وانج-بول فائدتها في حل أي نموذج واقعي كمعادلات تفاضلية من الدرجة الأولى أو الثانية

**الكلمات المفتاحية:** شروط حدية، معادلات متكاملة، حلول عددية، مضطرب بشكل فردي، متعدد حدود وانغ بول.