Some properties of the Oscillatory and Nonoscillatory Solutions Of Second Order Nonlinear Neutral Differential Equation

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Abstract

In this paper sufficient conditions for oscillation of all solutions of nonlinear second order neutral differential equation and sufficient conditions for nonoscillatory solutions to converge to zero are obtained.

Introduction

Consider the second order nonlinear neutral differential equation

(1.1) $[x(t)+p(t)x(\tau(t))]''+q(t)f(x(\sigma(t)))=0, \quad t\ge t_0$

under the standing hypotheses:

(1) $p\in C[(t_0,\infty),R]$;

(2) $\tau,\sigma\in C[(t_0,\infty),R]$, τ,σ are strictly increasing and $\lim_{t\to\infty}\tau(t)=\infty$, $\lim_{t\to\infty}\sigma(t)=\infty$

(3) $q\in C[(t_0,\infty),R]$, $q(t)$ not equivalent to zero.

Our aim is to obtain new sufficient conditions for the oscillation of all solutions of equation (1.1) and sufficient conditions for nonoscillatory solutions to converge to zero . By a solution of equation (1.1) we means a continuous function $x:[t_x,\infty)\to R$ such that $x(t)+p(\tau)x(t))$ is two times continuously differentiable , and $x(t)$ satisfies equation (1.1) for all sufficiently large $t\ge t_x$. A solution of (1.1) is said to be oscillatory if it has an infinite sequence of zero tending to infinity , otherwise a solution is said to be nonoscillatory. The problem of oscillation for neutral differential equations has received considerable attention in recent years, see e.g. [1-6] and the references cited therein, however many of these papers discuss the cases when the coefficients and the arguments are constants and a few of them investigate the cases of variable coefficients and arguments. In this paper we improve some results of [3], [5] , and give some other new theorems .

Main Results

In this sections we studied the oscillation of all solutions of equation (1.1), and obtained some new sufficient conditions to the bounded and all solutions of (1.1). Let $u(t)=x(t)+p(t)x(\tau(t))$

So, equation (1.1) reduce to

(2.1) $u''(t)=-q(t)f(x(\sigma(t)))$

The next theorem concerns bounded oscillatory solutions of equation (1.1).

Theorem (2.1) : Suppose that $p(t)\ge 0$ is bounded . $q(t)\le 0$ for $t\ge t_0$, f is an increasing function, such that $u\,f(u)>0$ for $u\ne 0$ and

(2.2) $\int^{\infty} q(s)\,ds=-\infty$

Then all bounded solutions of (1.1) are oscillatory.

Proof. Without loss of generality we may assume that $x(t)$ is an eventually

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positive and bounded solution of equation (1.1), and $x(\sigma(t)) > 0$ for $t \geq t_0$, then $u''(t) \geq 0$ for all large t. For sufficiently large t_0 we have two cases :

1. $u'(t) > 0, \quad t \geq t_1 \geq t_0$
2. $u'(t) < 0, \quad t \geq t_1 \geq t_0$

Case (1) : Since $x(t)$ and $p(t)$ are bounded, then also $u(t)$ is bounded, this is impossible according to $u''(t) \geq 0$ and $u'(t) > 0$ for all large t.

Case (2) : We can consider one possibility $u(t) > 0$, for $t \geq t_1 \geq t_0$ where t_1 is sufficiently large , integrating the equation $u''(t) = -q(t)f(x(\sigma(t)))$ from t_1 to t we ge

$$u'(t) - u'(t_1) = \int_{t_1}^{t} q(s) f(x(\sigma(s)))ds \geq$$

and so

$$f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

$$-u'(t_1) \geq f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

as $t \to \infty$ We get a contradiction. The proof is complete.

Example (2.2): Consider the nonlinear neutral differential equation

$$(2.3) \qquad \frac{d^2}{dt^2}[x(t) + 4x(t+\pi)] - \frac{3}{2} f(\sin(t+2\pi)) = 0$$

Which satisfied all conditions of theorem 2.1 so all solutions of equation (2.3) are oscillatory for instance $x(t) = \sin t$, is such solution.

Remark (2.3): We can replace the conditions of theorem 2.1 by the conditions: $p(t) \geq 0$ is bounded, $q(t) \geq 0$, f is decreasing function such that $u f(u) < 0$ for $u \neq 0$ and

$$\int^{\infty} q(s)\, ds = \infty$$

Then every bounded solution of equation (1.1) is oscillatory.

Theorem (2.4) : Suppose that $p(t) \geq 0$, $q(t) \geq 0$, and f is an increasing function such that $u f(u) > 0$, for $u \neq 0$, and (2.4) $\int^{\infty} q(s)\, ds = \infty$

Then every solution of (1.1) is oscillatory.

Proof : Assume that $x(t)$ is an eventually positive solution of (1.1), and $x(\sigma(t)) > 0$ for $t \geq t_0$. Then $u''(t) \leq 0$ for all large t. We consider two cases : 1. $u'(t) < 0$, $t \geq t_0$; 2. $u'(t) > 0$, $t \geq t_0$.

Cases (1) : One can find that $u(t) < 0$ for $t \geq t_1 \geq t_0$, and $\lim_{t\to\infty} u(t) = -\infty$, which is Impossible ,since $u(t) > 0$.

Cases (2) :- We have only the possibility $u(t) > 0$ for $t \geq t_1 \geq t_0$, and we have, $u''(t) = -q(t) f(x(\sigma(t)))$, by integrating this equation from t_1 to t we get

$$u'(t) - u'(t_1) = \int_{t_1}^{t} -q(s) f(x(\sigma(s)))\, ds$$

$$\leq f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

Then

$$-u'(t_1) \leq f(x(\sigma(t_1))) \int_{t_1}^{t} -q(s)\, ds$$

as $t \to \infty$ We get a contradiction . The proof is complete.

Example (2.5) : Consider the nonlinear neutered differential equation

(2.5) $\frac{d^2}{dt^2}[x(t)+2x(t+4\pi)]+\frac{3}{2}f(\sin(t+2\pi))=0, \quad t\geq 0$

Which satisfied all conditions of theorem 2.4, so all solution of equation (2.5) are oscillatory, for instance $x(t)=\sin t$, is such oscillatory solution.

Remark(2.6): We can replace the conditions in Theorem 2.4 by the conditions $p(t)\geq 0,\ q(t)\leq 0,\ f$ is decreasing function such that $u\,f(u)<0$ for $u\neq 0$, and

$$\int_1^\infty q(s)\,ds=-\infty$$

Then all solutions of (1.1) are oscillatory.

3- Asymptotic Behavior of equation (1.1):

In this section we investigate the converges to zero of the solution of equation (1.1). In addition to $u\,f(u)>0$ for $u\neq 0$ we will assume that $f(u)$ is bounded away from zero if u is bounded away from zero .

Theorem (3.1): Suppose that $-1<p_1\leq p(t)\leq 0,\ q(t)\geq 0,\ \tau(t)<t$ and

(3.1) $$\int^\infty q(s)\,ds=\infty$$

Then every nonoscillatory solution of (1.1) satisfy $x(t)\to 0$ as $t\to\infty$

proof : Let $x(t)$ be nonoscillatory solution of (1.1), without loss of generality let $x(t)>0$, and $x(\tau(t))>0$ for all $t\geq t_0$, from (1.1) we get $u''(t)\leq 0$, so we have two cases to investigate.

case (1) : $u'(t)>0$, we have $u'(t)$ is decreasing and bounded then $\lim_{t\to\infty}u(t)=l\geq 0$, by integrating equation (2.1) from t_1 to t and let $t\to\infty$ we ge

$$u'(t_1)=l+\int_1^\infty q(s)\,f(x(\sigma(s)))\,ds$$

Which implies that $\liminf_{t\to\infty}x(t)=0$, otherwise $\liminf_{t\to\infty}x(t)=b>0$ so there exists t_2 large enough such that $x(t)\geq b>0$ for all $t\geq t_2\geq t_1$ which implies that $f(x(\sigma(t)))\geq b_1>0$ for $t\geq t_3\geq t_2$.

Then $$u'(t_3)\geq l+b_1\int_3^\infty q(s)\,ds$$

which leads to a contradiction, so $\liminf_{t\to\infty}x(t)=0$.

We claim that $u(t)<0$ for $t\geq t_3$, otherwise $u(t)>0$, by the mean value theorem

$$u(t)=u(t_3)+(t-t_3)\,u'(s), \qquad s\in(t_3,t)$$

$$\geq u(t_3)+(t-t_3)\,u'(t_4), \qquad t_4\in(s,t)$$

Which implies that $\lim_{t\to\infty}u(t)=\infty$

And this is impossible since $\liminf_{t\to\infty}x(t)=0$ so $u(t)<0$ and we have

$$x(t)<-p(t)x(\tau(t))<x(\tau(t))$$

Then $x(t)$ is bounded. Suppose that $\limsup_{t\to\infty}x(t)=b>0$, so there exist increasing sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$ such that

$$\lim_{t\to\infty}x(t_n)=b$$

$$0>u(t_n)>x(t_n)+p_1\,x(\tau(t_n))$$

then $$x(\tau(t_n))\geq\frac{x(t_n)}{-p_1}$$

and so $$\lim_{t\to\infty}x(\tau(t_n))\geq\frac{b}{-p_1}>b$$

which is a contradiction then $b=0$.

Case(2) : $u'(t)<0$ implies that $u(t)<0$ for $t \geq t_1 \geq t_0$ and the prove is
similar to that in case (1) .

Example(3.2): Consider the second order nonlinear neutral differential Equation

$$[x(t)-e^{-2}x(t-2)]'' + 4(2e^{4}-1)e^{4(t-1)}x^{3}(t-1)=0, \qquad t>0 \tag{3.2}$$

All the conditions of theorem 3.1 are hold so all nonoscillatory solutions of the above equation tends to zero as $t \to \infty$, for instance $x(t)=e^{-2t}$ is such nonoscillatory solution.

Corollary(3.3): Suppose that all conditions of theorem 3.1 are held , then all unbounded solution of equation (1.1) are oscillatory .

Proof : By theorem 3.1 all nonoscillatory solution of (1.1) are bounded.

Example(3.4): Consider the nonoscillatory neutral differential equation

$$\left[x(t)-\frac{e^{\pi}}{24}x(t-\pi)\right]'' + \frac{9}{4}e^{\pi}x\left(t-\frac{\pi}{2}\right)=0, \qquad t>0 \tag{3.3}$$

Then all conditions of corollary 3.3 are held, then every unbounded solutions of 3.3 are oscillatory , for instance $x(t)=e^{t}\sin t$ is such unbounded oscillatory solution .

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بعض الخصائص لتذبذب وعدم تذبذب حلول المعادلات التفاضلية المحايدة غيرالخطية من الرتبة الثانية

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الخلاصة

في هذا البحث تمت دراسة سلوك حلول المعادلة التفاضلية المحايدة غير الخطية من الرتبة الثانية حيث اعطيت في البند الاول بعض الشروط الكافية لضمان تذبذب حلول المعادلة

$$[x(t)+p(t)x(\tau(t))]''+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0 \qquad (1.1)$$

تضمن هذا البند نظريتين ونتيجة مع الامثلة ، اما في البند الثاني فقد تمت مناقشة تقارب حلول هذه المعادلة وتباعدها ، فقد اعطيت شروط كافية اخرى لضمان تقارب الحلول غير المتذبذبة للمعادلة (1.1) نحو الصفر أوأي عدد او تباعد هذه الحلول ، تضمن هذا البند نظرية مع نتيجه كذالك اعطيت بعض الامثلة لتأكيد وجود مثل هذه الحلول.