Order Sum Graph of a Group

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Abstract:

The concept of the order sum graph associated with a finite group based on the order of the group and order of group elements is introduced. Some of the properties and characteristics such as size, chromatic number, domination number, diameter, circumference, independence number, clique number, vertex connectivity, spectra, and Laplacian spectra of the order sum graph are determined. Characterizations of the order sum graph to be complete, perfect, etc. are also obtained.

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MSC2010: CXCL10, CXCL16, Date.

Introduction:

Investigating graphs associated with different algebraic structures has fascinated researchers around the world. Many research articles developed new concepts by linking graph theory and algebra. The coprime graph of a finite group is a graph consisting of elements of the group as vertices and there is an edge between two vertices say u and v if $(|u|, |v|) = 1$. Some of the properties of the coprime graph were also investigated[1].

The distance coprime graph of some standard graphs such as path, star, wheel, cycle, and complete bipartite graph were obtained[2]. Some properties and characterizations for the distance coprime graph were also obtained. Recently, the notion of the *non-inverse graph* of a group (in short, i^*-graph) of G, denoted by Γ, has been introduced as a simple graph with vertex set consisting of elements of G and there is an edge between two vertices $u, v \in \Gamma$ if u and v are not inverses of one another. Certain structural properties of non-inverse graphs of groups have been discussed.

Motivated by the studies mentioned above, in this paper, a graph called order sum graph based on the properties of groups is introduced. The definition of the order sum graph is given in the next section. Different types of domination are studied in the recent past such as tadpole domination[3], independent domination[4], H_n-

domination[5,] etc. The variants of domination are investigated in this paper for order sum graphs.

Order Sum Graph of a Group

Recall that the order of a group $(G, *)$ is the number of elements of its underlying set G and the order of an element, say a, of G is the least positive integer k, such that $a^k = e$, the identity element of the group. In this paper, n denotes the order of the group G. The notion of the order sum graph of a given finite group is given below:

Definition 1: The *order sum graph,* $\Gamma_{os}(G)$ associated with a group G, is a graph whose vertices are the elements of G and two vertices say $a, b \in \Gamma_{os}(G)$ are adjacent if $o(a) + o(b) > o(G)$.

For example, consider the group G of fourth roots of unity under multiplication, that is, $G = \{1, -1, i, -i\}$ and $o(G) = 4$. Here, the identity element of G is 1 and hence $o(-1) = 2, o(1) = 1, o(i) = 4, o(-i) = 4$. The order sum graph associated with G is given in Fig. 1.
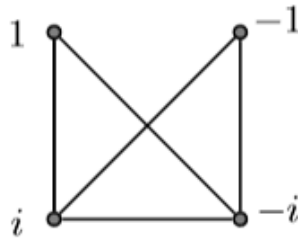
Figure 1. The order sum graph of the group of 4-th roots of unity.

The finite group is represented by G and its order sum graph by Γ_{os}. In this context, the questions about the structural properties of the order sum graphs of groups arise much interest. The following establish a necessary and sufficient for $\Gamma_{os}(G)$ to be connected.

Theorem 1: The order sum graph of a group G is connected if and only if G is a cyclic group.

Proof. Let G be a group of order n. Now, let the order sum graph associated with G be connected. Then in Γ_{os}, there is a path connecting any two vertices. Assume G is not a cyclic group now. Then, there is no element in G whose order is the same as n. Let a, b $\in G$. Since the order of G is n, $o(\text{a}) \neq n$, $o(\text{a})|n$. Therefore, $o(\text{a}) \leq \frac{n}{2}$. Similarly, $o(\text{b})|n$, $o(\text{b}) \neq n$ and hence $o(\text{b}) \leq \frac{n}{2}$. Thus, for any two elements in G, $o(\text{a}) + o(\text{b}) \leq n$. Therefore, by Definition 1, there is no path between any two vertices in $\Gamma_{os}(G)$. This is in direct opposition to the hypothesis that $\Gamma_{os}(G)$ is connected. Therefore, there exists at least one element in G whose order is equal to $o(G)$. Thus, G is a cyclic group. □

Conversely, let G be a cyclic group. Then, there is at least one element, say a in G which generates G. Here, $o(\text{a}) = o(G) = n$. Then, $o(\text{a}) + o(a_i) > n$, where a_i, $i = 2,3, \dots, n$ are the remaining elements of G. Therefore, by Definition-1, a is adjacent to all a_i's, $i = 2,3, \dots, n$ in $\Gamma_{os}(G)$. Therefore, any two vertices of $\Gamma_{os}(G)$ have a path connecting them through a. Hence, Γ_{os} is connected.

For example, consider the group of integers modulo 6 under the usual addition, that is $Z_6 = \{0,1,2,3,4,5\}$. Here, $o(Z_6) = 6$, $o(0) = 1, o(1) = 6, o(2) = 3, o(3) = 2, o(4) = 3, o(5) = 6$. The order sum graph associated with $(Z_6, +)$ is given in Fig. 2.

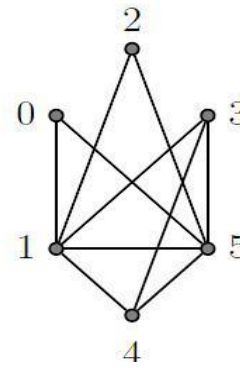**Figure 2. The order sum graph of the group $(Z_6, +)$**

Now, recall the following two important properties of cyclic groups.

Theorem 2: The order of a cyclic group is the same as the order of any generator of group [7,8].

Theorem 3: A cyclic group with exactly one generator can have a maximum of two elements [7,8].

Theorem 4: For a group G consisting of exactly one element a_1 such that $o(a_1) = o(G)$, then $\Gamma_{os}(G)$ is a path of order 2.

Proof. Let $G = \{a_1, a_2, \dots, a_n\}$. Without loss of generality, let a_1 be the only element in G such that $o(a_1) = o(G) = n$. Then, G must be a cyclic group with exactly one generator and $o(a_1) + o(a_i) > n$, where $i = 2,3, \dots, n$. By Definition 1, a_1 will be adjacent to all a_i, $i = 2,3, \dots, n$ in $\Gamma_{os}(G)$. Since a_1 is the only element in G such that $o(a_1) = n$, a_i's, $i = 2,3, \dots, n$ are not adjacent to each other in $\Gamma_{os}(G)$. Therefore, $\Gamma_{os}(G)$ is a star graph $K_{1,n-1}$. But, by Theorem 3, such a group can have at most two elements. Hence, $\Gamma_{os}(G) \cong K_2$. □

In this context, it is interesting to check the case when there exists no element in G with the order $o(G)$. Consider the following example:

Let U_8 be a group of elements less than 8 and relatively prime to 8 under the operation of multiplication modulo 8. That is, $U_8 = \{1,3,5,7\}$. Here, $o(U(8)) = 4, o(3) = 2, o(5) = 2, o(1) = 1, o(7) = 2$. The order sum graph associated with the group U_8 is given in Fig. 3:
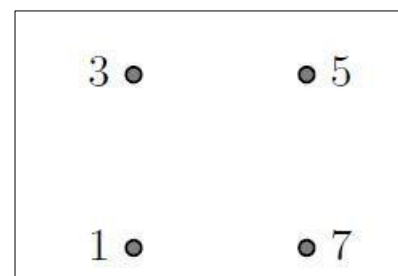

Figure 3. Order sum graph of the group $(U_8, \times)$.

At this point, it is interesting to investigate the conditions under which the order sum graphs of groups are null graphs and hence the following result:

Proposition 5: The $\Gamma_{os}(G)$ of order n is a null graph if and only if G is not a cyclic group.

Proof. Consider a cyclic group G with n elements then, there is no element in G whose order is equal to n. Let $a, b \in G$, then, $o(a)|n$, $o(a) \neq n$ and $o(b)|n$, $o(b) \neq n$. This implies that $o(a) \leq \frac{n}{2}$ and $o(b) \leq \frac{n}{2}$. Hence, $o(a) + o(b) \leq n$ for any two elements in G, then, by Definition 1, any two pairs of vertices a and b in $\Gamma_{os}(G)$ have no edge. Therefore, $\Gamma_{os}(G)$ is a null graph of n vertices.
On the contrary, let $\Gamma_{os}(G)$ be a null graph. Assuming G is a cyclic group, there is at least one element, say a in G whose order is equal to n. Then, by Definition 1, there is an edge between a and all the other vertices in $\Gamma_{os}(G)$. Thus, there is a path connecting any two vertices of $\Gamma_{os}(G)$ through a. Therefore, $\Gamma_{os}(G)$ is connected which is a contradiction to the fact that it is a null graph. Hence, G is not a cyclic group.
The following theorem discusses the case where the order sum graph of a group becomes a complete graph.

Theorem 6: The order sum graph associated with a group G of order $n \geq 3$, is complete if and only if all the elements of G except the identity element have order equal to n.

Proof. Consider a group G with $n \geq 3$ elements. Let $\Gamma_{os}(G)$ be a complete graph. Assume that there are $n - 2$ elements, say a_i, $i = 1,2, \ldots, n - 2$ in G whose order is equal to n. Out of the remaining two elements of G, one is the identity element a_n whose order is equal to 1 and the other element say a_{n-1} has to order less than n. Hence, $o(a_{n-1}) + o(a_n) \leq n$. Therefore, by Definition 1, there is no edge between a_n and a_{n-1}, which is a contradiction that $\Gamma_{os}(G)$ is complete.
Now, let all the elements of G say a_i, $i = 1,2, \ldots, n - 1$ other than the identity element a_n have order equal to n. Thus, $o(a_i) + o(a_j) > n$, $i = 1,2, \ldots, n$, $j = 1,2, \ldots, n$; $i \neq j$. Therefore, there is an edge between any two pair of vertices in $\Gamma_{os}(G)$. Hence, $\Gamma_{os}(G)$ is complete. □

As an illustration of the above result, consider the group $(Z_5, +)$ of integers modulo 5 under the operation addition. Since $Z_5 = \{0,1,2,3,4\}$, $o(Z_5) = 5$, $o(0) = 1, o(1) = 5, o(2) = 5, o(3) = 5, o(4) = 5$. The order sum graph of Z_5 is given by Fig. 4.
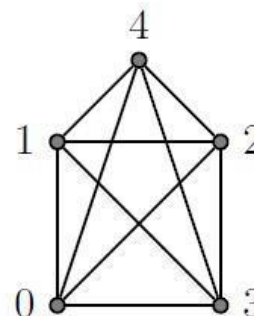

Figure 4. The order sum graph of the group $(\mathbf{Z_5}, +)$

Consider the following result on a group of prime order.

Theorem 7: Let G be a group with a prime number of elements say p. Then, G is cyclic and has $p - 1$ generators[6].

In view of Theorem 7, the following theorem examines the structural properties of the order sum graph of a group of prime order.

Theorem 8: The order sum graph associated with a group of prime order p is complete.

Proof. Consider a group G with p elements, where p is prime. Then, by Theorem 7, G is cyclic and has $p - 1$ generators. By Theorem 2, the order of all the $p - 1$ generators of G is equal to p and the order of the identity element is 1. Thus, all the elements of G except the identity element have order equal to p. Therefore, by Theorem 6, $\Gamma_{os}(G)$ is complete. □

The following theorem discusses the diameter of an order sum graph of a group.

Proposition 9: Let G be a cyclic group with finite order, then $diam(\Gamma_{os}(G)) \leq 2$.

Proof. Consider a cyclic group G with finite order then, there is at least one element whose order is equal to $o(G)$. By Theorem 1, $\Gamma_{os}(G)$ is connected. If $\Gamma_{os}(G)$ is complete, then $diam(\Gamma_{os}(G))=1$. If $\Gamma_{os}(G)$ is not complete, the maximum eccentricity of the vertices in $\Gamma_{os}(G)$ is 2, since G is connected. Therefore, $diam(\Gamma_{os}(G)) \leq 2$. □

Proposition 10 The chromatic number of the order sum graph associated with G is $\vartheta + 1$, where $\vartheta < n$ is the number of generators of the group G.

Proof. Consider a cyclic group G of order $n \geq 2$. $\vartheta < n$ be the number of generators in G. When $\vartheta = 1$, $\Gamma_{os}(G) = K_2$ and $\chi(K_2) = 2$. Therefore, $\chi(\Gamma_{os}(G)) = 2 = \vartheta + 1$. □
In general, ϑ colors are required to color the vertices $v_1, v_2, \ldots, v_\vartheta$, since there are ϑ vertices with degree $n - 1$ in $\Gamma_{os}(G)$. The remaining $n - \vartheta$ vertices with degree ϑ say $v_{\vartheta+1}, v_{\vartheta+2}, \ldots, v_{\vartheta+(n-\vartheta)}$ can be colored with one color. Hence, $\chi(\Gamma_{os}(G)) = \vartheta + 1$. □

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The size of the order sum graph of a finite group is investigated below:

Theorem 11: The size of the order sum graph associated with a cyclic group of order n is $\frac{\vartheta(2n-\vartheta-1)}{2}$, where $\vartheta < n$ is the number of generators of G.

Proof. Consider a cyclic group G of order $n \geq 2$ and let $\vartheta < n$ be the number of generators of G, then there are ϑ vertices with degree $n-1$ and $n-\vartheta$ vertices with degree ϑ in $\Gamma_{os}(G)$. Therefore, the sum of degrees of the vertices in $\Gamma_{os}(G)$ is $\vartheta(n-1) + (n-\vartheta)(\vartheta) = \vartheta(2n-\vartheta-1)$. Hence, by the handshaking lemma, the size of $\Gamma_{os}(G)$ is $\frac{\vartheta(2n-\vartheta-1)}{2}$.   □

The following theorem establishes the isomorphism between the order sum graphs of two isomorphic groups.

Theorem 12: The order sum graphs associated with two cyclic groups with the same order are isomorphic.

Proof. Let G_1 and G_2 be cyclic groups of the same order. Then, both G_1 and G_2 are isomorphic groups[9]. That is, the number of generators of G_1 is the same as that of G_2. Note that a vertex in the order sum graph, corresponding to a generator of the group G will have an edge with all the other vertices of the graph. Also, it is observed that no two vertices of the order sum graph corresponding to the non-generators of the group will be adjacent to each other (see Definition 1). Therefore, both the order sum graphs $\Gamma_{os}(G_1)$ and $\Gamma_{os}(G_2)$ have the same number of edges from the corresponding vertices. It is clear from the above-mentioned facts that the adjacency will also be preserved in the graphs $\Gamma_{os}(G_1)$ and $\Gamma_{os}(G_2)$. Hence, $\Gamma_{os}(G_1)$ and $\Gamma_{os}(G_2)$ are also isomorphic. □

A *perfect graph* is a graph G in which for every induced subgraph of G, the clique number is equal to its chromatic number.[9] Recall the following theorem which characterizes the perfect graphs.

Theorem 13: A graph G is perfect if and only if neither G nor its complement contains an induced odd cycle of length greater than or equal to five.[10]

It is interesting to check when the order sum graph of a group is perfect.

Theorem 14 Let G be a cyclic group with ϑ generators. Then, the order sum graph associated with G is perfect if and only if any one of the following is true:

    i. $\vartheta = 1$ and $n < 6$;
    ii. $\vartheta = 2$ and $n \leq 6$;
    iii. $\vartheta = 3$ and $n = 4$.

Proof. Let G be a cyclic group with ϑ generators. Let $\Gamma_{os}(G)$ be perfect. Now, if $\vartheta \geq 4$, then n must be at least $\vartheta + 1$, including the ϑ generators and identity element of G. Therefore, there is an induced odd cycle of length 5 consisting of four generator vertices and one non-generator vertex in the order sum graph with at least 4 generators. By Theorem 13, this is a contradiction to the hypothesis that $\Gamma_{os}(G)$ is perfect. Hence, $\vartheta < 4$.

There exist three cases:

Case 1: Let $\vartheta = 1$ and let $\Gamma_{os}(G)$ be perfect. Assume that $n \geq 6$. Then, the complement of $\Gamma_{os}(G)$ contains an induced odd cycle of length 5 consisting of 5 non-generator vertices (Ref. Definition 1. By Theorem 13, this is a contradiction to the hypothesis that $\Gamma_{os}(G)$ is perfect. Hence, $n < 6$.

Case 2: Let $\vartheta = 2$ and let $\Gamma_{os}(G)$ be perfect. Assume that $n > 6$. Then, the complement of $\Gamma_{os}(G)$ contains an induced odd cycle of length 5 consisting of 5 non-generator vertices (Ref. Definition 1). By Theorem 13, this is a contradiction to the hypothesis that $\Gamma_{os}(G)$ is perfect. Hence, $n \leq 6$.

Case 3: Let $\vartheta = 3$ and let $\Gamma_{os}(G)$ be perfect. Assume that $n \geq 5$. Then, $\Gamma_{os}(G)$ contains an induced odd cycle of length 5 consisting of 3 generator vertices and 2 non-generator vertices (Ref. Definition 1). By Theorem 13, this is a contradiction to the hypothesis that $\Gamma_{os}(G)$ is perfect. Hence, $n < 5$ but since there are three generators, n can be a minimum 4 including the identity element of G. Therefore, $n = 4$.

Hence, the order sum graph associated with G is perfect if and only if either $\vartheta = 1$ and $n < 6$ or $\vartheta = 2$ and $n \leq 6$ or $\vartheta = 3$ and $n = 4$.   □

Some Parameters of Order Sum Graphs
Naturally, investigation on different graph parameters of the order sum graphs sounds interesting at this point. In this context, the following result discusses the domination in order sum graph of groups.

Proposition 15: The domination number of the order sum graph associated with a group G of order n is either 1 or n.

Proof. By Theorem 1 and Proposition 5, $\Gamma_{os}(G)$ is either connected or a null graph. Hence, either there is a minimum one vertex in $\Gamma_{os}(G)$ which is adjacent to all other vertices in Γ_{os} or between any two pairs of vertices in $\Gamma_{os}(G)$, there is no edge. Therefore, $\gamma(\Gamma_{os}(G)) = 1$ or n. □

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It is observed that the order sum graph is connected if and only if the group is cyclic. Otherwise, the graph will be a null graph. Hence, in this section, some parameters of the order sum graphs associated with cyclic groups are discussed.

An *independent set* of a graph G is the set of vertices that are not adjacent to each pair. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of a maximal independent set of G. The following theorem discusses the independence number of the order sum graph of a cyclic group.

Theorem 16: The independence number of the order sum graph is equal to $n - \vartheta$, where ϑ is the number of generators of the group G.

Proof. Let G be a cyclic group with ϑ generators. By Definition 1, two vertices $a, b \in \Gamma_{os}(G)$ are adjacent if $o(a) + o(b) > n$. This is possible only when either $o(a)$ or $o(b)$ or both are equal to n. That is, either a or b or both are generators of G. Therefore, if a vertex in $\Gamma_{os}(G)$ is corresponding to a generator, then it is adjacent to all the other vertices of $\Gamma_{os}(G)$. Similarly, a vertex in $\Gamma_{os}(G)$ corresponding to a non-generator is adjacent only to all the vertices corresponding to generators. Hence, the set of vertices corresponding to all non-generators is the largest independent set of $\Gamma_{os}(G)$. Therefore, the independent number of $\Gamma_{os}(G)$ is $n - \vartheta$.

The *vertex cover* of a graph G is the vertex set S of G so that each edge of G has at least one end vertex in S. The *covering number* ν of G, is the minimum cardinality of a minimal vertex cover of G.

The covering number of the order sum graph of a group is discussed in the following result.

Corollary 17: The vertex covering the number of $\Gamma_{os}(G)$ is equal to the number of generators of the group.

Proof. The proof follows immediately from the fact that $\alpha(G) + \nu(G) = n$, the order of the graph. □

Recall that the clique number of G is the order of a maximum clique in G. The clique number of the order sum graph is determined in the theorem given below:

Theorem 18: The clique number of the order sum graph associated with a cyclic group G of order n is equal to $\vartheta + 1$, where ϑ is the number of generators of G.

Proof. Let G be a cyclic group with ϑ generators. By Definition 1, it is clear that a generator is adjacent to all the other vertices in $\Gamma_{os}(G)$ and a non-generator is adjacent only to generators. Now, if G has exactly one generator, the clique number of $\Gamma_{os}(G)$ is 2. If G has two generators, then maximum clique in $\Gamma_{os}(G)$ should be K_3 whose vertex set consists of two generators and one non-generator.

Therefore, the clique number of $\Gamma_{os}(G)$ is 3. If proceeded similarly, when there are ϑ generators, a maximum clique in $\Gamma_{os}(G)$ by joining the vertices corresponding to ϑ generators and one non-generator is obtained. Therefore, the clique number of $\Gamma_{os}(G)$ is $\vartheta + 1$. □

If G is a graph whose vertex set is given by $V(G)$. Then, the set $S \subseteq V(G)$ is an independent dominating set if S is a dominating set and an independent set. The minimum cardinality of such a set is the independent domination number and is represented by $i(G)$. The following theorem determines the independent domination number of an order sum graph of a cyclic group.

Proposition 19 The independent domination number of $\Gamma_{os}(G)$ is equal to 1.

Proof. Consider G to be a cyclic group. Therefore, G has at least one generator, say a. By Definition 1, a is adjacent to all the other vertices in $\Gamma_{os}(G)$. Therefore, the singleton set $\{a\}$ is a minimal dominating set as well as an independent set with minimum cardinality. Hence, $i(\Gamma_{os}(G)) = 1$. □

The following theorem determines the upper domination number of the order sum graph of a cyclic group.

Theorem 20 The upper domination number of the order sum graph of a cyclic group with ϑ generators is $n - \vartheta$. □

Proof. Consider a cyclic group G with ϑ generators. Then, the minimal dominating sets possible in the order sum graph of G are the singleton sets each consisting of a generator and the set containing all the non-generators of G. It is known that the set of all non-generators of G is a minimal dominating set with maximum cardinality $n - \vartheta$. Therefore, the upper domination number of $\Gamma_{os}(G)$ is $n - \vartheta$.

Distance Related Parameters of Order Sum Graphs

A vertex cutset of a connected graph G is a set of its vertices whose deletion makes the graph disconnected. Similarly, a cutset or an edge cutset is a set of edges whose deletion makes the graph disconnected. The connectivity of a graph, denoted by $\kappa(G)$ is the minimum cardinality of a minimal vertex-cut of G, whereas the edge connectivity $\kappa'(G)$ is the minimum cardinality of a minimal cutset. The following result discusses the relation between vertex connectivity and the edge connectivity of the order sum graph of a group.

Theorem 21: For the order sum graph Γ_{os} associated with a cyclic group G, $\kappa(\Gamma_{os}(G)) = \kappa'(\Gamma_{os}(G))$.

Proof. By Definition 1, a vertex corresponding to a generator of G is adjacent to all the other vertices in $\Gamma_{os}(G)$. Since there are ϑ generators, a minimum

number of vertices whose removal makes the order sum graph disconnected or trivial is ϑ. Therefore, the vertex connectivity $\kappa(\Gamma_{os}(G)) = \vartheta$. □

Furthermore, by Definition-1, a vertex corresponding to a non-generator is adjacent only to all generators. Therefore, minimum all the edges which are incident on a fixed non-generator vertex have to be removed to make $\Gamma_{os}(G)$ disconnected or trivial. Since these edges are all coming from different generators and the no. of generators is equal to ϑ, the edge connectivity $\kappa'(\Gamma_{os}(G)) = \vartheta$. Hence, $\kappa(\Gamma_{os}(G)) = \kappa'(\Gamma_{os}(G)) = \vartheta$, completing the proof.
□

The *center* of a graph is the set of all vertices whose eccentricity is equal to the radius of the graph. The following theorem discusses the center of the order sum graph of a given finite group.

Theorem 22: The center of the order sum graph of a group G is the set of its vertices corresponding to the generators of G.

Proof. Consider the order sum graph $\Gamma_{os}(G)$. A vertex corresponding to a generator of G is adjacent to all other vertices in $\Gamma_{os}(G)$. Therefore, the eccentricity of each of the vertices corresponding to generators of G is 1. Further, a vertex corresponding to a non-generator of G is adjacent to all vertices corresponding to generators and is at a distance 2 from other vertices corresponding to non-generators. Therefore, the eccentricity of each of the vertices corresponding to non-generators of G is 2. Since the eccentricity of vertices corresponding to generators of G is minimum, the set of vertices in $\Gamma_{os}(G)$ corresponding to all the generators of the group G forms the center of $\Gamma_{os}(G)$. Hence the result. □

The *periphery* of a graph is the set of all vertices whose eccentricity is equal to the diameter of the graph. Since the vertices of $\Gamma_{os}(G)$ corresponding to the non-generators of a finite cyclic group G are not adjacent to each other, the distance between such vertices will be 2, which is equal to the diameter of the graph.

Theorem 23 The periphery of the order sum graph of a (cyclic) group G is the set of its vertices corresponding to the non-generators of G.

Theorem 24 Let G be a cyclic group with ϑ generators. Then, the circumference of the order sum graph associated with G is equal to

$$\begin{cases} 2\vartheta; & \vartheta < \left\lceil \dfrac{n}{2} \right\rceil \\ n; & \vartheta \geq \left\lceil \dfrac{n}{2} \right\rceil \end{cases}$$

Proof. To find the circumference of the order sum graph associated with G, the following steps are undertaken:

Step 1: Visit a generator vertex say v.
Step 2: Traverse through a non-generator vertex and then to a generator vertex that is not visited.
Step 3: Repeat step 2 till all the generator vertices are visited.
Step 4: Traverse to a non-generator vertex that is not visited and then go back to initial vertex v.

Now, the following cases arise:

Case 1: Let $\vartheta < \left\lceil \dfrac{n}{2} \right\rceil$. Then, the number of generators is less than the number of non-generators in G. The circumference of $\Gamma_{os}(G)$ is 2ϑ since there are ϑ generators and the equal number of non-generators required to form the largest cycle are obtained using the above steps.

Case 2: Let $\vartheta \geq \left\lceil \dfrac{n}{2} \right\rceil$. Also, let us first consider the case when $\vartheta = \left\lceil \dfrac{n}{2} \right\rceil$. Then, $\vartheta = \dfrac{n}{2}$ when n is even or $\vartheta = \dfrac{n+1}{2}$ when n is odd. In both cases, either number of generators is greater than or equal to the number of non-generators in G. Therefore, the circumference of Γ_{os} obtained by following the above steps is equal to n as all the vertices are traversed. Since there are n vertices in G, for $\vartheta \geq \left\lceil \dfrac{n}{2} \right\rceil$, the circumference is equal to n. □

Spectra of Order Sum Graphs
The spectral values of the order sum graphs are studied in the following theorem.

Theorem 25: Let G be a cyclic group of order n, ϑ be the number of generators of G, then the spectra of Γ_{os} associated with G is

$$\begin{pmatrix} \lfloor\frac{\vartheta}{2}\rfloor + \sqrt{\vartheta n - \frac{(\vartheta+1)}{4}(3\vartheta-1)} & \lfloor\frac{\vartheta}{2}\rfloor - \sqrt{\vartheta n - \frac{(\vartheta+1)}{4}(3\vartheta-1)} & 0 & -1 \\ 1 & 1 & n-(\vartheta+1) & \vartheta-1 \end{pmatrix},$$

when ϑ is odd and

$$\begin{pmatrix} \frac{(\vartheta-1)+\sqrt{4\vartheta n-(3\vartheta^2+2\vartheta-1)}}{2} & \frac{(\vartheta-1)-\sqrt{4\vartheta n-(3\vartheta^2+2\vartheta-1)}}{2} & 0 & -1 \\ 1 & 1 & n-(\vartheta+1) & \vartheta-1 \end{pmatrix}$$
,

when ϑ is even.

Proof. The adjacency matrix $A(\Gamma_{os}(G))$ of $\Gamma_{os}(G)$ is given by

$$A(\Gamma_{os}(G)) = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1(n-1)} & a_{1n} \\ a_{12} & 0 & a_{23} & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{(n-1)n} & 0 \end{bmatrix}_{n\times n}$$

For $1 \le k \le n-1$ and $\vartheta = k$, $a_{ij} = \begin{cases} 1, \text{if } 1 \le i \le k, i < j \le n \\ 0, \text{otherwise} \end{cases}$

Consider $\det(\lambda I - A(\Gamma_{os}(G)))$. Here, the following steps are performed:

Step 1: Let $R_i \to R_i - R_\vartheta$, for $i = 1,2,\dots,\vartheta-1$. Then, $\det(\lambda I - A(\Gamma_{os}(G)))$ is of the form $[\lambda + 1]^{\vartheta-1}\det(A)$.

Step 2: In $\det(A)$, let $R_i \to R_i - R_{\vartheta+1}$, where $i = \vartheta+2, \vartheta+3, \dots, \vartheta+(n-\vartheta)$ to obtain $\det(A)$ of the form $[\lambda]^{n-(\vartheta+1)}\det(B)$. Here, consider the following cases.

Case 1: Let ϑ be even. Then, on expansion and simplification of $det(B)$,

$$det(B) = [\lambda - (\frac{(\vartheta-1)+\sqrt{4\vartheta n-(3\vartheta^2+2\vartheta-1)}}{2})][\lambda - (\frac{(\vartheta-1)-\sqrt{4\vartheta n-(3\vartheta^2+2\vartheta-1)}}{2})].$$

Case 2: Let ϑ be odd. Then, on expansion and simplification of $\det(B)$, $\det(B) = [\lambda - (\lfloor\frac{\vartheta}{2}\rfloor + \sqrt{\vartheta n - \frac{(\vartheta+1)}{4}(3\vartheta-1)})][\lambda - (\lfloor\frac{\vartheta}{2}\rfloor - \sqrt{\vartheta n - \frac{(\vartheta+1)}{4}(3\vartheta-1)})].$

Therefore, the characteristic polynomial of $A(\Gamma_{os}(G))$ is given as

$$\phi\left(A(\Gamma_{os}(G))\right) = \begin{cases} (\lambda+1)^{\vartheta-1}(\lambda)^{n-(\vartheta+1)}\left[\lambda - \left(\frac{(\vartheta-1)+\eta}{2}\right)\right] \\ \left[\lambda - \left(\frac{(\vartheta-1)-\eta}{2}\right)\right]; & \text{when } \vartheta \text{ is even}; \\ (\lambda+1)^{\vartheta-1}(\lambda)^{n-(\vartheta+1)}\left[\lambda - \left(\lfloor\frac{\vartheta}{2}\rfloor + \sqrt{\vartheta n - \frac{\zeta}{4}}\right)\right] \\ \left[\lambda - \left(\lfloor\frac{\vartheta}{2}\rfloor - \sqrt{\vartheta n - \frac{\zeta}{4}}\right)\right]; & \text{when } \vartheta \text{ is odd}. \end{cases}$$

where, $\eta = \sqrt{4\vartheta n - (3\vartheta^2 + 2\vartheta - 1)}$ and $\zeta = (\vartheta+1)(3\vartheta-1))$. This completes the proof. □

The following theorem determines the spectra of the Laplacian matrix of the order sum graph of a cyclic group G.

Theorem 26: Let G be a cyclic group of order n and ϑ be the number of generators of G. Then, the Laplacian spectra of Γ_{os} associated with G is

$$\begin{pmatrix} n & \vartheta & 0 \\ \vartheta & n-\vartheta-1 & 1 \end{pmatrix}$$

Proof. Let G be cyclic group with ϑ generators. The Laplacian matrix $L(\Gamma_{os}(G))$ of the graph $\Gamma_{os}(G)$ is given by $L(\Gamma_{os}(G)) = [a_{ij}]_{n\times n}$, where

$$a_{ij} = \begin{cases} -1, & \text{if } 1 \le i \le k, \quad i < j \le n; \\ n-1, & \text{if } i = j \ \forall i = 1, \quad 2, \dots, \quad k; \\ \vartheta, & \text{if } i = j \ \forall \ i = k+1, \quad k+2, \quad \dots k+(n-k); \\ 0, & \text{otherwise}. \end{cases}$$

and $1 \le k \le n-1$ and $\vartheta = k$.

Consider $\det(\lambda I - L(\Gamma_{os}(G)))$. Then, the following steps are performed:

Step 1: Let $R_i \to R_i - R_1$, for $i = 2,3,\dots,\vartheta$. Then, $\det(\lambda I - L(\Gamma_{os}(G)))$ is of the form $[\lambda - n]^{\vartheta-1}\det(A)$.

Step 2: In det(A), let $R_i \to R_i - R_n$, where $i = \vartheta + 1, \vartheta + 2, \ldots, \vartheta + (n - \vartheta - 1)$ to obtain det(A) of the form $[\lambda - \vartheta]^{(n-\vartheta-1)}\det(B)$.

Step 3: In det(B), let $R_1 \to R_1 - R_{\vartheta+1} - R_{\vartheta+2} - \cdots - R_{\vartheta+(n-\vartheta)}$ to obtain det(B) of the form $[\lambda - n]\det(C)$.

Step 4: In det(C), let $C_n \to C_1 + C_2 + C_3 + \cdots + C_n$ to obtain $\det(C) = \lambda\det(D)$, where $\det(D) = 1$.

Therefore, the characteristic polynomial of $L(\Gamma_{os}(G))$ is given by

$$\phi(L(\Gamma_{os}(G)), \lambda) = \lambda[\lambda - n]^\vartheta[\lambda - \vartheta]^{(n-\vartheta-1)}.$$

This completes the proof. □

Conclusion:

In this paper, some properties and characteristics of a new algebraic graph, called an order sum graph associated with a group have been discussed. Some of the parameters such as size, diameter, chromatic number, clique number, independence number, vertex connectivity, edge connectivity, domination number, spectra, and Laplacian spectra of the order sum graph have been obtained. Some characterizations of the order sum graph have also been obtained. As a newly introduced notion, many structural characteristics of order sum graphs are yet to be investigated. Moreover, determining the order sum graphs corresponding to different standard groups are also much promising for future studies.

Authors' declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in CHRIST (Deemed to be University).

Authors' contributions statement:

The first author conceived of the presented idea and developed the theory through discussions with the second author. The second author supervised the study and verified the analytical methods. The statement of contributions is prepared and approved by both authors.

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رتبة مجموع الرسم البياني للزمرة

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الخلاصة:

في هذا البحث: تم تقديم مفهوم رتبة مجموع الرسم البياني بالنسبة الى الزمرة المنتهية بالاعتماد على رتبة الزمرة ورتبة عناصر الزمرة, كما تم اعطاء بعض الخواص لرتبة مجموع الرسم البياني مثل الحجم ، العدد اللوني ، العدد المهيمن ، القطر ، المحيط ، العدد المستقل، العدد الزمري، الارتباط الرأسي ، الأطياف ، وأطياف لابلاس. كذلك تم اعطاء توصيفات لرتبة مجموع الرسم البياني للحصول على مفهوم تام وشامل.

الكلمات المفتاحية: الرسوم البيانية الجبرية ، المركز ، الهيمنة، أطياف الرسم البياني، رتبة مجموع الرسم البياني.

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