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Efficient Approach for Solving (2+1) D- Differential Equations

Noor A. Hussein[1] ***Luma N. M. Tawfiq***[2*]

[1]Department of Mathematics, College of Education, University of Al-Qadisiyah,Al-Diwaniyah , Iraq
[2]Department of Mathematics, College of Education for Pure Science Ibn Al-Haitham, University of Baghdad, Iraq
*Corresponding author: luma.n.m@ihcoedu.uobaghdad.edu.iq
E-mail addresses: noor.alli@qu.edu.iq

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Abstract:

In this article, a new efficient approach is presented to solve a type of partial differential equations, such (2+1)-dimensional differential equations non-linear, and nonhomogeneous. The procedure of the new approach is suggested to solve important types of differential equations and get accurate analytic solutions i.e., exact solutions. The effectiveness of the suggested approach based on its properties compared with other approaches has been used to solve this type of differential equations such as the Adomain decomposition method, homotopy perturbation method, homotopy analysis method, and variation iteration method. The advantage of the present method has been illustrated by some examples.

Keywords: Boussinesq equations, Cubic Klein-Gordon equations, Decomposition method, (2+1)-dimensional PDEs, Kadomtsev-Petviashvili equation.

Introduction:

Differential equations especially partial differential equations (PDEs) play an important role in everyday life, they have become a part of modern life [1]. Therefore, it has become necessary to have many and varied ways to solve such equations, which in turn solve life problems associated with them [2].

They are used to describe many life models such as exponential growth, population growth of species or the change in investment return over time[3], cooling and heating problems, bank interest, radioactive decay problems even flow problems in solving continuous compound interest problems, orthogonal trajectories [4] and also involving fluid mechanics problems, population or conservation biology [5], circuit design, heat transfer, seismic waves [6]. They are used in specific fields such as, in the field of medicine, where modeling cancer growth or the spread of disease may be described as differential equations [7].

The (1+1)-dimensional PDEs is applied to simulate the propagation of waves in a line. Actual atmospheric and oceanic motions do not occur on lines but planes. Accordingly, it is necessary to study higher-dimensional PDEs to describe the propagation of Rossby solitary waves. Gottwald first derived the (2+1) dimensional Zakharov kuznetsov (ZK) equation for nonlinear Rossby solitary waves in barotropic fluids[8]. In recent years, numerous scholars have obtained higher-dimensional PDEs for Rossby solitary waves to explain the wave phenomenon in large-scale atmospheres and oceans. Yang et al [9] obtained three –dimensional ZK-Burgers equation in barotropic fluids. Zhang et al [10] derived (2+1)–dimensional generalized fZK equation and ZK equation with complete Coriolis force. Yin et al[11] obtained two-dimensional nonlinear Rossby waves with the dissipation and external source under complete Coriolis force effects and discussed the effects of these factors on the Rossby waves fluctuations.

Many methods for solving (2+1)D- PDEs such as variable separation approach [12], hyperbola function method [13], expanded $\left(G/G^2\right)$ expansion method [14], extended F-expansion method [15], and complex method [16, 17], a Darboux Transformation [18, 19]. In this paper, the researchers will use a stunner method to solve partial differential equations with (2+1)-dimension and obtain distinct and accurate analytical results. The next section explains the steps of the proposed method.

This paper has been arranged as follows: In section 2, the basic ideas of the suggested method will be given. In section 3, solving some examples of (2+1)D, such as cubic Klein-Gordon equation, Kadomtsev-Petviashvili equation, and Boussinesq equations by using the suggested method will be given. The convergence of the suggested techniques will be illustrated in section 4. Finally, the conclusion is given in section 5.

Suggested Method

Consider the (2+1) D-PDE as follows

$$L(u(x,y,t)) + R(u(x,y,t)) + N(u(x,y,t)) = g(x,y,t) \quad \dots \quad 1$$

With initial conditions: $\left.\frac{\partial^k u(x,y,t)}{\partial t^k}\right|_{t=0} = f_k(x,y), \quad k = 0,1,\dots,n-1 \quad \dots \quad 2$

Where $L(.) = \frac{\partial^n(.)}{\partial t^n}$, $n = 1,2,3,\dots$ is a linear operator of the partial derivation with respect to t, $g(x,y,t)$ is the nonhomogeneous part, $N(.)$is a nonlinear term $R(.)$ is the remainder of the linear term, and x and y are space independent variables. $R(.)$ and $N(.)$ are free orders of partial derivation with respect to t.

In the suggested method the unknown dependent function $u(x,y,t)$ can be construed as infinite series of the form:

$$u(x,y,t) = u_0(x,y) + u_1(x,y)t + u_2(x,y)t^2 + \cdots = \sum_{k=0}^{\infty} u_k(x,y)t^k \quad \dots \quad 3$$

Where $u_k(x,y) = \frac{1}{k!}\left.\frac{\partial^k u(x,y,t)}{\partial t^k}\right|_{t=0} \quad \dots \quad 4$

In the next step calculate the terms u_n $(n = 0,1,2 \ \dots)$.

Rewrite Eq.1 as follow:

$$L(u(x,y,t)) = -R(u(x,y,t)) - N(u(x,y,t)) + g(x,y,t) \quad \dots \quad 5$$

Taking L^{-1} (inverse of the linear operator L) to both sides of the Eq.5 to get:

$$L^{-1}\left(L(\mathcal{U}(x,y,t))\right) = -L^{-1}[R(\mathcal{U}) + N(\mathcal{U})] + L^{-1}[g(x,y,t)] \quad \dots \quad 6$$

$$u(x,y,t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \left.\frac{\partial^k u(x,y,t)}{\partial t^k}\right|_{t=0} = -L^{-1}[R(u) + N(u)] + L^{-1}[g(x,y,t)] \quad \dots \quad 7$$

From Eq.2 , obtain that:

$$u(x,y,t) = \sum_{k=0}^{n-1} t^k f_k(x,y) - L^{-1}[R(u) + N(u)] + L^{-1}[g(x,y,t)] \quad \dots \quad 8$$

Now substitute Eq.3 in Eq.8, to get:

$$L^{-1}(R(u)) = L^{-1}\left(R\left(\sum_{k=0}^{\infty} u_k(x,y)t^k\right)\right) = \sum_{k=0}^{\infty} R(u_k(x,y)) \frac{k!}{(n+k)!} t^{n+k} \quad \dots \quad 9$$

In Eq.8 the nonlinear part $N(u)$, can be written as follows:

$$N(u) = \sum_{k=0}^{\infty} N_k t^k \quad \dots \quad 10$$

Such that $N_k = \frac{1}{k!}\left.\frac{\partial^k N(u(x,y,t))}{\partial t^k}\right|_{t=0} \quad \dots \quad 11$

Thus

$$L^{-1}(N(u)) = L^{-1}\left[\sum_{k=0}^{\infty} N_k t^k\right] = \sum_{k=0}^{\infty} N_k L^{-1}(t^k) = \sum_{k=0}^{\infty} N_k \frac{k!}{(n+k)!} t^{n+k} \quad \dots \quad 12$$

Also, the nonhomogeneous term will be written as:

$$G(x,y,t) = L^{-1}[g(x,y,t)] = \sum_{k=0}^{\infty} g_k \frac{t^k}{k!} \quad \dots \quad 13$$

Where

$$g_k = \frac{1}{k!}\left.\frac{\partial^k G(x,y,t)}{\partial t^k}\right|_{t=0} \quad \dots \quad 14$$

Substituting Eq. 9, 12, and 13 in Eq. 8 to have:

$$u(x,y,t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} f_k(x,y) - \sum_{k=0}^{\infty} R(u_k(x,y)) \frac{\mathcal{K}!}{(n+\mathcal{K})!} t^{n+\mathcal{K}} - \sum_{k=0}^{\infty} \mathcal{N}_k \frac{\mathcal{K}!}{(n+\mathcal{K})!} t^{n+k} + \sum_{k=0}^{\infty} g_k \frac{t^k}{k!} \quad ..15$$

substituting Eq. 15 in 14 to get:

$$u_j(x,y) = \frac{1}{j!}\left.\frac{\partial^j u(x,y,t)}{\partial t^j}\right|_{t=0} = \frac{1}{j!}\frac{\partial^j}{\partial t^j}\left[\sum_{k=0}^{n-1} f_k(x,y)\frac{t^k}{k!} - \sum_{k=0}^{\infty} \frac{k!}{(n+k)!}(R(u_k(x,y)) + N_k)t^{n+k} + \sum_{k=0}^{\infty} g_k t^k\right]_{t=0}, \ \forall\ j \ \geq \ n, \quad \dots \quad 16$$

then $\frac{\partial^j}{\partial t^j}\left[\sum_{k=0}^{n-1} f_k(x,y)\frac{t^k}{k!}\right] = 0$ and

$$\frac{\partial^j}{\partial t^j} t^{\mathcal{K}} = \begin{cases} 0 & \mathcal{K} < j \\ \frac{\mathcal{K}!}{(\mathcal{K}-j)!} t^{\mathcal{K}-j} & \mathcal{K} \geq j \end{cases}$$

Thus Eq. 16 becomes:

$$u_j(x,y) = \frac{1}{j!}\left[-\sum_{k=j-n}^{\infty} \frac{k!}{(n+k)!}\left(R\left(u_k(x,y)\right) + N_k\right)\frac{(n+k)!}{(n+k-j)!} t^{n+\mathcal{K}-j} + \sum_{k=0}^{\infty} g_k \frac{k!}{(k-j)!} t^{k-j}\right]_{t=0} \quad \dots \quad 17$$

$$u_j(x,y) = \frac{1}{j!}\left[-\frac{(j-n)!}{0!}\left(R\left(u_{j-n}\right) + N_{j-n}\right) + g_j \frac{j!}{0!} t^{k-j}\right] \quad \dots \quad 18$$

Hence $u_j(x,y) = g_j - \frac{(j-n)!}{j!}\left(R\left(u_{j-n}(x,y)\right) + N_{j-n}\right), \quad j \geq n \quad \dots \quad 19$

Finally, substitute Eq. 19 in 3 to get $u(x,y,t)$.

Convergence Analysis for Series Solution

The analysis of convergence for the series solution of the (2+1) D-PDEs is discussed. The sufficient requirement for convergence of the suggested approach is addressed. That is the series solution for (2+1) D-PDEs will appear to be close to the exact solution.

Theorem 1. Let A_n presented as $u_0 + \ldots + u_n$ be an operator from a Hilbert space H to H. The series solution

$$u = \sum_{k=0}^{\infty} u_k(x,y)t^k$$

is convergent if $\exists\ 0<\lambda<1$ when $\|A_{n+1}\| \leq \lambda\|A_n\|$ (such that $\|u_{n+1}\| \leq \lambda\|u_n\|$) $\forall n = 0,1,\ldots$.

Theorem 1, is a specific case from *the Banach's fixed point theorem* which is a sufficient condition to study the convergence of the proposed method.

Theorem 2. If the series solution $u = \sum_{k=0}^{\infty} u_k(x,y)t^k$ convergent, then this series will consider the exact solution of the present non-linear problem.

Now the following theorem shows the series solution $u = \sum_{k=0}^{\infty} u_k(x,y)t^k$ is convergent

Theorem 3 (Sufficient Condition for Convergence)

"If χ and Υ are Banach spaces and $\aleph: \chi \to \Upsilon$ is a contractive nonlinear mapping, that is

$$\forall\ \omega, \omega^* \in \chi;\ \|\aleph(\omega) - \aleph(\omega^*)\| \leq \gamma \|\omega - \omega^*\|, 0 < \gamma < 1$$

Then according to Banach's fixed point theorem, $\aleph$ has a unique fixed point ω, consider the exact solution of the present non-linear problem.

Proof

Assume that the sequence generated by the suggested method can be written as:

$$\omega_n = \aleph(\omega_{n-1}), \omega_{n-1} = \sum_{i=0}^{n-1} \omega_i, n = 1,2,3, \ldots$$

Suppose that $\omega_0 \in B_r(\omega)$ where $B_r(\omega) = \{\omega^* \in \chi; \|\omega^* - \omega\| < r\}$. Then:

i. $\omega_n \in B_r(\omega)$

ii. $\lim_{n\to\infty} \omega_n = \omega$

(i) From the inductive approach, for $n = 1$, one can get:

$$\|\omega_1 - \omega\| = \|\aleph(\omega_0) - \aleph(\omega)\| \leq \gamma \|\omega_0 - \omega\|$$

Assume that $\|\omega_{n-1} - \omega\| \leq \gamma \|\omega_{n-2} - \omega\| \leq \gamma^2 \|\omega_{n-3} - \omega\|$

$$\leq \gamma^3 \|\omega_{n-4} - \omega\|$$
$$\leq \gamma^{n-1} \|\omega_0 - \omega\|$$

As induction hypothesis, then

$$\|\omega_n - \omega\| = \|\aleph(\omega_{n-1}) - \aleph(\omega)\| \leq \gamma \|\omega_{n-1} - \omega\| \leq \gamma^n \|\omega_0 - \omega\|$$

Using (i), to get

$$\|\omega_n - \omega\| \leq \gamma^n \|\omega_0 - \omega\| \leq \gamma^n r < r \Rightarrow \omega_n \in B_r(\omega)$$

Because of $0 < \gamma < 1$, so

$$\lim_{n\to\infty} \gamma^n = 0,\ \lim_{n\to\infty} \|\omega_n - \omega\| \leq \lim_{n\to\infty} \gamma^n r = 0$$

that is: $\lim_{n\to\infty} \omega_n = \omega$

Theorem's 1, 2 and 3 show that the achieved solution from the suggested method is convergent to the exact solution under the given condition, $\exists\ 0<\lambda<1$, such that $\|u_{n+1}\| \leq \lambda\|u_n\|$, $\forall n = 0,1,\ldots$.

Illustrative Examples

In this section, some illustrative examples for solving (2+1) D-PDEs by using the suggested method are presented.

Example1

The suggested method is used to solve the (2+1)-dimensional cubic Klein-Gordon equation. This equation prescribes many problems in classical (quantum) mechanics, solitons, and condensed matter physics. For example, it models the dislocations in crystals and the motion of rigid pendula attached to a stretched wire.[20]

Consider (2+1) D- cubic Klein-Gordon equation

$$u_{xx} + u_{yy} - u_{tt} - u + 2u^3 = 0,$$ with initial conditions

$$u(x,y,0) = \operatorname{sech}(x+y),\ u_t(x,y,0) = \operatorname{sech}(x+y)\tanh(x+y)$$

$$\Longrightarrow u_{tt} = u_{xx} + u_{yy} - u + 2u^3$$

It is clear that $L(u) = \frac{\partial^2}{\partial t^2}$, $R(u) = u_{xx} + u_{yy} - u$, $(u) = 2u^3$, $g(x,y,t) = 0$

From ICs: $u_0 = \operatorname{sech}(x+y)$, $u_1 = \operatorname{sech}(x+y)\tanh(x+y)$

So, from Eq. 11, it follows that:

$N_0 = 2{u_0}^3 = 2sech^3(x+y)$, and $N_1 = \frac{\partial}{\partial t}(2u^3) = 6u^2u_t = 6(u_0)^2u_1 = 6sech^3(x+y)\tanh(x+y)$

Also, $(u_0) = u_{0xx} + u_{0yy} - u_0$,

$$u_{0x} = -\operatorname{sech}(x+y)\tanh(x+y)$$

"$u_{0xx} = u_{0yy} = -sech^3(x+y) + \operatorname{sech}(x+y)\tanh^2(x+y)$

$$R(u_0) = -2sech^3(x+y) + 2\operatorname{sech}(x+y)\tanh^2(x+y) - \operatorname{sech}(x+y)$$

$$u_{1x} = sech^3(x+y) - \operatorname{sech}(x+y)\tanh^2(x+y)$$

$$u_{1xx} = u_{1yy} = -5sech^3(x+y)\tanh(x+y) + \operatorname{sech}(x+y)\tanh^3(x+y)$$

$$\Longrightarrow R(u_1) = -10sech^3(x+y)\tanh(x+y) + 2\operatorname{sech}(x+y)\tanh^3(x+y) - \operatorname{sech}(x+y)\tanh(x+y)$$

By Eq. 19,

$$u_2 = -\frac{1}{2!}[R(u_0) + N_0]$$

$$u_2 = \frac{1}{2!}[-2sech^3(x+y) + 2\operatorname{sech}(x+y)\tanh^2(x+y) - \operatorname{sech}(x+y) + 2sech^3(x+y)]$$

$u_2 = \frac{1}{2!}[\text{sech}(x+y)\tanh^2(x+y) + \text{sech}(x+y) - sech^3(x+y) - \text{sech}(x+y)]$

$u_2 = \frac{1}{2!}[\text{sech}(x+y)\tanh^2(x+y) - sech^3(x+y)]$ "(6)

Also, $u_3 = -\frac{1}{3!}[R(u_1) + N_1]$

$u_3 = \frac{1}{3!}[-10sech^3(x+y)\tanh(x+y) + 2\,\text{sech}(x+y)\tanh^3(x+y) - \text{sech}(x+y)\tanh(x+y) + 6sech^3(x+y)\tanh(x+y)]$

$u_3 = \frac{1}{3!}[-4sech^3(x+y)\tanh(x+y) + 2\,\text{sech}(x+y)\tanh^3(x+y) - \text{sech}(x+y)\tanh(x+y)]$ "(9)

$u_3 = \frac{1}{3!}[-5sech^3(x+y)\tanh(x+y) + \text{sech}(x+y)\tanh^3(x+y)]$

And so on, thus from Eq. 3, we get

"$u(x,y,t) = u_0(x,y) + u_1(x,y)t + u_2(x,y)t^2 + \cdots$

$u(x,y,t) = \text{sech}(x+y) + \text{sech}(x+y)\tanh(x+y)\,t + \frac{1}{2!}[\text{sech}(x+y)\tanh^2(x+y) - sech^3(x+y)]t^2 + \frac{1}{3!}[-5sech^3(x+y)\tanh(x+y) + \text{sech}(x+y)\tanh^3(x+y)]t^3 + \cdots$

$u(x,y,t) = \text{sech}(x+y-t) + t\left[\frac{\partial}{\partial t}\,\text{sech}(x+y-t)\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2}{\partial t^2}\,\text{sech}(x+y-t)\right]_{t=0} + \frac{t^3}{3!}\left[\frac{\partial^3}{\partial t^3}\,\text{sech}(x+y-t)\right]_{t=0} + \cdots$

$\Rightarrow u(x,y,t) = \text{sech}(x+y-t)$, this is the exact analytic solution.

Comparing the results presented in this paper with other results shows that the suggested method is powerful, efficient, and adequate.

The Riccati–Bernoulli sub-ODE method was used to construct solitary wave solutions for the (2+1)-dimensional cubic nonlinear Klein–Gordon (cKG) equation and obtain a new infinite sequence of solutions by using a Bäcklund transformation. The Riccati–Bernoulli sub-ODE gives infinite solutions. Indeed, all presented solutions have so important contributions for the explanation of some practical physical phenomena and further nonlinear problems[20].

Wang et al.[21] have presented only five solutions for the cKG equation, using the multi-function expansion method. Whereas Khan et al. [22] gave eight solutions, using the modified simple equation (MSE) method. Comparing these results with the presented result in this paper, one can deduce that the suggested method gives a unique exact traveling wave solution. Thus, the suggested method is more effective in providing an exact solution than these two methods.

Example 2

Kadomtsev and Petviashivili in 1970 first introduced this equation to describe slowly varying nonlinear waves in a dispersive medium and study weakly nonlinear dispersive waves in plasmas and also in the modulation of weakly nonlinear long water waves which travel nearly in one dimension that is, nearly in a vertical plane. The solitons are stable[23].

Consider the 4th order nonlinear (2+1)D-Kadomtsev-Petviashvili equation

$u_{xt} - 6uu_{xx} - 6(u_x)^2 + u_{xxxx} + 3u_{yy} = 0,$

With IC: $u_x(x,y,0) = -\frac{1}{2}csc^2\left(\frac{1}{2}(\text{x}+\text{y})\right)\coth\left(\frac{1}{2}(\text{x}+\text{y})\right)$

It is clear that $L(u) = \frac{\partial}{\partial t}$, $R(u) = (u_{xxxx} + 3u_{yy}), (u) = -(6uu_{xx} + 6(u_x)^2)$, $g(x,y,t) = 0$

$\Rightarrow u_{xt} = 6uu_{xx} + 6(u_x)^2 - u_{xxxx} - 3u_{yy}$

From IC. : $u_{0x}(x,y,0) = -\frac{1}{2}csc^2\left(\frac{1}{2}(\text{x}+\text{y})\right)\coth\left(\frac{1}{2}(\text{x}+\text{y})\right) \Rightarrow u_0 = \frac{1}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)$

$u_{0xx} = u_{0yy} = \frac{1}{4}csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right) + \frac{1}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^2\left(\frac{1}{2}(\text{x}+\text{y})\right)$

$u_{0xxx} = -csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right)coth\left(\frac{1}{2}(\text{x}+\text{y})\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^3\left(\frac{1}{2}(\text{x}+\text{y})\right)$

$u_{0xxxx} = \frac{1}{2}csch^6\left(\frac{1}{2}(\text{x}+\text{y})\right) + \frac{11}{4}csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^2\left(\frac{1}{2}(\text{x}+\text{y})\right) + \frac{1}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^4\left(\frac{1}{2}(\text{x}+\text{y})\right)$

from Eq. 11, obtain that:

$-N_0 = 6(u_0u_{0xx} + (u_{0x})^2)$

$N_0 = 6\left[\frac{1}{8}csch^6\left(\frac{1}{2}(\text{x}+\text{y})\right) + \frac{1}{2}csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^2\left(\frac{1}{2}(\text{x}+\text{y})\right)\right]$

$-R(u_0) = -(u_{0xxxx} + 3u_{0yy})$

"$R(u_0) = -\frac{1}{2}csch^6\left(\frac{1}{2}(\text{x}+\text{y})\right) - \frac{11}{4}csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^2\left(\frac{1}{2}(\text{x}+\text{y})\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^4\left(\frac{1}{2}(\text{x}+\text{y})\right) - \frac{3}{4}csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right) - \frac{3}{2}csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right)coth^2\left(\frac{1}{2}(\text{x}+\text{y})\right)$ "(12)

By Eq. 19

$u_{1x}(x,y) = -\frac{0!}{1!}\left(R\left(u_0(x,y)\right) + N_0\right)$

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$$u_{1x}(x,y) = -\frac{1}{2}csch^6\left(\frac{1}{2}(x+y)\right) - \frac{11}{4}csch^4\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(x+y)\right)coth^4\left(\frac{1}{2}(x+y)\right) - \frac{3}{4}csch^4\left(\frac{1}{2}(x+y)\right) - \frac{3}{2}csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right) + \frac{3}{4}csch^6\left(\frac{1}{2}(x+y)\right) + 3csch^4\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)$$

$$u_{1x}(x,y) = \frac{1}{4}csch^6\left(\frac{1}{2}(x+y)\right) + \frac{1}{4}csch^4\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right) - \frac{1}{2}csch^2\left(\frac{1}{2}(x+y)\right)coth^4\left(\frac{1}{2}(x+y)\right) - \frac{3}{4}csch^4\left(\frac{1}{2}(x+y)\right) - \frac{3}{2}csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)$$

$$u_{1x}(x,y) = -csch^4\left(\frac{1}{2}(x+y)\right) - 2csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)$$

$$u_{1x}(x,y) = d\left[2csch^2\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right)\right]$$

$$u_1 = 2csch^2\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right)$$

$$u_{1x} = -csch^4\left(\frac{1}{2}(x+y)\right) - 2csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)$$

$$u_{1xx} = 4csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) + 2csch^2\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right)$$

$$u_{1xxx} = -2csch^6\left(\frac{1}{2}(x+y)\right) - 11csch^4\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right) - 2csch^2\left(\frac{1}{2}(x+y)\right)coth^4\left(\frac{1}{2}(x+y)\right)$$

$$u_{1xxxx} = 17csch^6\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) + 26csch^4\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right) + 2csch^2\left(\frac{1}{2}(x+y)\right)coth^5\left(\frac{1}{2}(x+y)\right)$$

$$-R(u_1) = -\left(u_{1xxxx} + 3u_{1yy}\right)$$

$$-R(u_1) = -17csch^6\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) - 26csch^4\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right) - 2csch^2\left(\frac{1}{2}(x+y)\right)coth^5\left(\frac{1}{2}(x+y)\right) - 12csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) - 6csch^2\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right)$$

Also, from Eq. 11

$$N_1 = \frac{\partial}{\partial t}\left[-6(uu_{xx} + (u_x)^2)\right] = -6(u_0u_{1xx} + u_1u_{0xx} + 2u_{0x}u_{1x})$$

$$-N_1 = 6\left[\frac{7}{2}csch^6\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) + 4csch^4\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right)\right]$$

From Eq. 19; $u_{2x}(x,y) = -\frac{1}{2!}\left(R\left(u_1(x,y)\right) + N_1\right)$

$$"u_{2x} = \frac{1}{2!}\left[4csch^6\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) - 4csch^4\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right) - 8csch^2\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right) - 12csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right)\right]$$

$$u_{2x} = \frac{1}{2!}\left[-16csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) - 8csch^2\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right)\right]$$

$$u_{2x} = \frac{1}{2!}\left[-8csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) - 8\left(csch^4\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) + csch^2\left(\frac{1}{2}(x+y)\right)coth^3\left(\frac{1}{2}(x+y)\right)\right)\right]$$

$$u_2 = \frac{1}{2!}\left[4csch^4\left(\frac{1}{2}(x+y)\right) + 8csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)\right]$$

and so on, from Eq. 3

$$u = \sum_{k=0}^{\infty} u_k(x,y)t^k$$

$$u = \frac{1}{2}csch^2\left(\frac{1}{2}(x+y)\right) + 2tcsch^2\left(\frac{1}{2}(x+y)\right)coth\left(\frac{1}{2}(x+y)\right) + \frac{t^2}{2!}\left(4csch^4\left(\frac{1}{2}(x+y)\right) + 8csch^2\left(\frac{1}{2}(x+y)\right)coth^2\left(\frac{1}{2}(x+y)\right)\right) + \cdots \text{"(13)}$$

$$u = \left[\frac{1}{2}csch^2\left(\frac{1}{2}(x+y-4t)\right)\right]_{t=0} + t\left[\frac{\partial}{\partial t}\left(\frac{1}{2}csch^2\left(\frac{1}{2}(x+y-4t)\right)\right)\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{2}csch^2\left(\frac{1}{2}(x+y-4t)\right)\right)\right]_{t=0} + \cdots$$

This is the exact solution: $u(x,y,t) = \frac{1}{2}csch^2\left(\frac{1}{2}(\mathrm{x}+\mathrm{y}-4\mathrm{t})\right)$

In[23] the exp**(-Φ(ξ))-**expansion method with the aid of Maple has been used to obtain the exact solutions of the (2+1) Kadomtsev–Petviashvili equation and get hyperbolic function solutions

Example 3

In this example, we solve the (2+1)-dimensional Boussinesq equation which contains the second-order partial derivative u_{tt} in addition to other partial derivatives. This family of nonlinear equations gained its importance because it appears in many scientific applications and physical phenomena [24]. The new family is of the form $u_{tt}-u_{xx}-u_{yy}+p(u)=0$, where u(x, y, t) is a function of space *x*, *y* and time variable *t* and the nonlinear term p(u) = $-\frac{1}{2}(u^2)_{xx}-u_{xxxx}$, with u(x,y,t) is a sufficiently often differentiable function. This is called the (2+1)-dimensional Boussinesq equation. The (2+1)-dimensional Boussinesq equation was introduced by Boussinesq to describe the propagation of long waves in shallow water under gravity propagating in both directions. The (2+1)-dimensional Boussinesq equation describes motions of long waves in shallow water under gravity and in a two-dimensional nonlinear lattice. This particular form the (2+1)-dimensional Boussinesq equation is of special interest because it is completely integrable and admits inverse scattering formalism. However, the good Boussinesq equation or the well-posed equation can be handled in a like manner [25].

Consider the nonlinear 4th order (2+1) D-Boussinesq equations

$$u_{tt}-u_{xx}-\frac{1}{2}(u^2)_{xx}-u_{yy}-u_{xxxx}=0,$$

with ICs: $u(x,y,0)=6sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)$

$$u_t(x,y,0)=24\left(\frac{1}{\sqrt{2}}\right)sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

To solve the model equation by the suggested method firstly should determine:

$$L(u)=\frac{\partial^2}{\partial t^2}\quad,\quad R(u)=-u_{xx}-u_{yy}-u_{xxxx},\ N(u)=-\frac{1}{2}(u^2)_{xx},\ g(x,y,t)=0$$

$$\Longrightarrow u_{tt}=u_{xx}+\frac{1}{2}(u^2)_{xx}+u_{yy}+u_{xxxx}$$

From ICs: $u_0=6sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)$, and $u_1=24\left(\frac{1}{\sqrt{2}}\right)sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right)$

$$\mathrm{u_{0x}}=\mathrm{u_{0y}}=\frac{-12}{\sqrt{2}}\mathrm{sech}^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)$$

$$u_{0xx}=-6sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)+12sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^2\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

$$u_{0xxx}=\frac{24}{\sqrt{2}}sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right)+\frac{24}{\sqrt{2}}\ sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right)-\frac{24}{\sqrt{2}}sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

$$\mathrm{u_{0xxxx}}=24\mathrm{sech}^6\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)-132\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)+24\mathrm{sech}^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)$$

$$-R(u_0)=u_{0xx}+u_{0yy}+u_{0xxxx}$$

$$-\mathrm{N_0}=\frac{1}{2}(\mathrm{u_0}^2)_{\mathrm{xx}}=\frac{1}{2}(2\mathrm{u_0u_{0x}})_{\mathrm{x}}=\mathrm{u_0u_{0xx}}+(\mathrm{u_{0x}})^2$$

$$-N_0=-36\mathrm{sech}^6\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)+144\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)$$

$$u_2(x,y)=-\frac{1}{2!}\left(R\left(u_0(x,y)\right)+N_0\right)$$

$$"u_2(x,y)=\frac{1}{2!}\Big[-12sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)+24sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^2\left(\frac{1}{\sqrt{2}}(x+y)\right)+24\mathrm{sech}^6\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)-132\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)+24\mathrm{sech}^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)-36\mathrm{sech}^6\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)+144\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\Big]$$

$$u_2(x,y)=\frac{1}{2!}\Big[-24sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)+48sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\Big]$$

$$\mathrm{u_{1x}}=\mathrm{u_{1y}}=12\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)-24\mathrm{sech}^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)$$

$$\mathrm{u_{1xx}}=\left(\frac{-96}{\sqrt{2}}\right)\mathrm{sech}^4\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)+\left(\frac{48}{\sqrt{2}}\right)\mathrm{sech}^2\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)\tanh^3\left(\frac{1}{\sqrt{2}}(\mathrm{x}+\mathrm{y})\right)$$

$$u_{1xxx} = -48sech^6\left(\frac{1}{\sqrt{2}}(x+y)\right) + 264sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^2\left(\frac{1}{\sqrt{2}}(x+y)\right) - 48sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^4\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

$$u_{1xxxx} = \frac{816}{\sqrt{2}}sech^6\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) - \frac{1248}{\sqrt{2}}sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{96}{\sqrt{2}}sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^5\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

Now, should be calculate $u_3(x,y)$

$$-R(u_1) = u_{1xx} + u_{1yy} + u_{1xxxx}$$

$$-N_1 = \frac{\partial}{\partial t}\left[\frac{1}{2}(u^2)_{xx}\right] = \frac{1}{2}(2uu_x)_x = \frac{\partial}{\partial t}[uu_{xx} + (u_x)^2] = uu_{xxt} + u_t u_{xx} + 2u_x u_{xt}$$

$$-N_1 = u_0 u_{1xx} + u_1 u_{0xx} + 2u_{0x}u_{1x}$$

$$-N_1 = -\frac{1008}{\sqrt{2}}\text{sech}^6\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{1152}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right)$$

$$u_3(x,y) = -\frac{1}{3!}\left(R\left(u_1(x,y)\right) + N_1\right)$$

$$u_3(x,y) = \frac{1}{3!}\left(\frac{-192}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{96}{\sqrt{2}}\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{816}{\sqrt{2}}sech^6\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) - \frac{1248}{\sqrt{2}}sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{96}{\sqrt{2}}sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^5\left(\frac{1}{\sqrt{2}}(x+y)\right) - \frac{1008}{\sqrt{2}}\text{sech}^6\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{1152}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right)\right)$$

$$u_3(x,y) = \frac{1}{3!}\left(\frac{-192}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{96}{\sqrt{2}}\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right) - \frac{192}{\sqrt{2}}sech^6\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) - \frac{96}{\sqrt{2}}sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{96}{\sqrt{2}}sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^5\left(\frac{1}{\sqrt{2}}(x+y)\right)\right)$$

$$u_3(x,y) = \frac{1}{3!}\left(\frac{-384}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{192}{\sqrt{2}}\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right)\right)$$

and so on, from Eq. 3

$$u = \sum_{k=0}^{\infty} u_k(x,y)t^k$$

$$u = 6\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{24}{\sqrt{2}}t\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{t^2}{2!}\left(-24sech^4\left(\frac{1}{\sqrt{2}}(x+y)\right) + 48sech^2\left(\frac{1}{\sqrt{2}}(x+y)\right)tanh^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\right) + \frac{t^3}{3!}\left(\frac{-384}{\sqrt{2}}\text{sech}^4\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh\left(\frac{1}{\sqrt{2}}(x+y)\right) + \frac{192}{\sqrt{2}}\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y)\right)\tanh^3\left(\frac{1}{\sqrt{2}}(x+y)\right)\right) + \cdots$$

$$u = \left[6\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y-2t)\right)\right]_{t=0} + t\left[\frac{\partial}{\partial t}\left(6\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y-2t)\right)\right)\right]_{t=0} + \frac{t^2}{2!}\left[\frac{\partial^2}{\partial t^2}\left(6\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y-2t)\right)\right)\right]_{t=0} + \frac{t^3}{3!}\left[\frac{\partial^3}{\partial t^3}\left(6\text{sech}^2\left(\frac{1}{\sqrt{2}}(x+y-2t)\right)\right)\right]_{t=0} + \cdots$$

$u = 6sech^2\left(\frac{1}{\sqrt{2}}(x+y-2t)\right)$. This is the exact solution.

The (G'/G)-expansion method is used to solve example 3, with Maple and getting solutions are in more general forms [24].

In exp(Φ(η))-expansion method is applied to find exact traveling wave solutions to the (2+1)-dimensional Boussinesq equation with the aid of Maple [25].

Zheng studied the exact traveling wave solutions of the (2+1)-dimensional Boussinesq equation by using the (G′/G)-expansion method and achieved three analytical solutions[26].

Ajeel et al [27] were discussed the related existing theorem.

Conclusion:

In this article, the new effective method for treating non-linear, (2+1)D – PDEs is implemented. A new decomposition technique has been introduced to compute exact analytic solutions for the non-linear (2+1) D- model equations such as

(2+1) D- cubic Klein-Gordon equation, (2+1) D-Kadomtsev-Petviashvili model equation, and (2+1)D- Boussinesq equations. Series formulation is used throughout the entire procedure, which leads to a series solution being made use within the new procedure. The method is generally based on the well selected base functions and produces an exact solution. Illustrated examples showed that the proposed method has better accuracy with easy implementation. Furthermore, the results showed that when the number of iterations increases, the series solution becomes closer to the exact value as well. The suggested method can be used in the future to solve (3+1)D- PDEs.

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Authors' declaration:

- Conflicts of Interest: None.
- Ethical Clearance: The project was approved by the local ethical committee in University of Baghdad.

Authors' contributions statement:

The authorship of the title above certifies that they have participated in different roles as follows:

• L. N. M. T. suggested a new efficient approach to solving a type of PDEs, that is non-linear, in-homogeneous (2+1)-D differential equations.

• N. A. H. used the procedure of the new approach to solve an important type of model equations such as cubic Klein-Gordon, Kadomtsev-Petviashvili equation, Boussinesq equations. Then three important model equations are solved with suggested method simplicity and ease implementation to get an accurate analytic solution i.e., exact solution.

• L. N. M. T. proved the convergence of series solution to exact solution analytically by using the important concept in functional analysis.

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أسلوب كفوء لحل معادلات تفاضلية ذو بعد (2+1)

نور علي حسين[1] لمى ناجي محمد توفيق[2]

[1]قسم الرياضيات, كلية التربية, جامعة القادسية, الديوانية, العراق
[2]قسم الرياضيات, كلية التربية للعلوم الصرفة - ابن الهيثم, جامعة بغداد, بغداد, العراق

الخلاصة:
في هذا البحث تم عرض اسلوب جديد كفوء لحل صنف من المعادلات التفاضلية الجزئية مثل المعادلات التفاضلية ذات البعد (2+1) خطية و غير خطية متجانسة و غير متجانسة. إجراءات الاسلوب الجديد تم اقتراحه لحل صنف مهم من المعادلات التفاضلية بأسلوب مبسط و سهل التنفيذ للحصول على حل تحليلي دقيق تحديدا الحل المضبوط . فاعلية الاسلوب المقترح مبني على اساس خواصه مقارنة بالطرق الاخرى المستخدمة لحل هذا الصنف من المعادلات التفاضلية مثل طريقة ادومين , الاضطراب الهوموتوبي, التحليل الهوموتوبي , التغاير التكرارية . ميزات الطريقة المقدمة موضحة من خلال الامثلة .

الكلمات المفتاحية: معادلات Boussinesq , معادلات Klein-Gordon التكعيبية, طريقة التفكيك, معادلات تفاضلية جزئية ذات البعد (2+1), معادلة Kadomtsev-Petviashvili