An Asymptotic Analysis of the Gradient Remediability Problem for Disturbed Distributed Linear Systems

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Abstract:
The goal of this work is demonstrating, through the gradient observation of a disturbed distributed parameter systems of type linear (DDPL-systems), the possibility for reducing the effect of any disturbances (pollution, radiation, infection, etc.) asymptotically, by a suitable choice of related actuators of these systems. Thus, a class of asymptotically gradient remediable system (AGR-system) was developed based on finite time gradient remediable system (GR-system). Furthermore, definitions and some properties of this concept AGR-system and asymptotically gradient controllable system (AGC-controllable) were stated and studied. More precisely, asymptotically gradient efficient actuators ensuring the weak asymptotically gradient compensation system (WAGC-system) of known or unknown disturbances are examined. Consequently, under convenient hypothesis, the existence and the uniqueness of the control of type optimal, guaranteeing the asymptotically gradient compensation system (AGC-system), are shown and proven. Finally, an approach that leads to a Mathematical approximation algorithm is explored.

Keywords: Asymptotic analysis, Controllability, Disturbance, Optimal control, Remediability.

Introduction:
Driven by environmental, pollution1, radiation and infection problems 2,3, the authors have studied the problem with regard to the gradient observation of a class of DDPL-systems considering the possibility of lessening or compensating asymptotically the effect of any disturbances. Thus, the study constitutes a development to the case of asymptotic type for the previous investigates to the remediability linear parabolic problem of different systems, introduced in the finite time case 4,7 and asymptotic case 4,8,9.

One can note that studying compensation problem with respect to the gradient observation and the so-called gradient remediability, is of considerable interest 10. Thus, it was shown that there exists a system that is not remediable, however may be gradient remediable.

Gradient remediability concept in usual and regional case is considered and studied for DPL-systems 10,12. Regarding the asymptotic case aspect 13, the great importance of the asymptotic analysis in systems theory 14,15, takes into consideration the problem of AGC-systems and studies a prospective extension of the development methods, in addition to analyzing the results in finite time. Hereafter, through likeness the relationship among the remediability and controllability of the gradient case has been inspected and studied in a considerable time.

Also, the link among remediability and controllability in asymptotic gradient case has been studied and analyzed.

This paper is structured as follows: Section 2, is devoted to the introduction of the gradient remediability concepts of type exact and weak under convenient hypothesis.

Section 3 relates to the asymptotic form in various cases in connection with suitable actuators and sensors. Also, an asymptotically gradient efficient actuators enable the guaranteeing an asymptotic gradient compensation of weak type is presented.

In section 4, weakly and exactly a asymptotically gradient controllable system
Formulation of the Considered Problem:

Assume that $\Omega$ stands as an open and bounded set in $\mathbb{R}^n$, with a boundary of smooth type $\partial \Omega$. Considering a class of DDPL-system defined by the form:

$$(S) \begin{cases} \dot{y}(t) = A y(t) + B u(t) + f(t) ; 0 < t < T \\ y(0) = y_0 \end{cases}$$

where $A$ generates a strongly continuous semi-group $S(t)_{t \geq 0}$; $B \in L(\Omega, \mathbb{U})$, $u \in L^2(0, T; \mathbb{U})$, $\mathbb{U}$ is a space of Hilbert type is denoted the input space and $X = H_0^1(\Omega)$, the space of state.

The system $(S)$ admits a unique solution $y \in C\left(0, T; H_0^1(\Omega) \right) \cap C^1\left(0, T; L^2(\Omega) \right)$ given by:

$$y(t) = S(t) y_0 + \int_0^t S(t - s) B u(s) ds + \int_0^t S(t - s) f(s) ds$$

The system $(S)$ is augmented by the following output (gradient observation) equation:

$$(O) z_{u,f}(t) = CV y(t) ; 0 < t < T$$

where $C \in L\left(L^2(\Omega)^n, Y \right)$, $Y$ is a Hilbert space (gradient observation space) and $V$ is the operator defined by:

$$\nabla : H_0^1(\Omega) \rightarrow \left(L^2(\Omega)^n \right)$$

$$y \rightarrow \nabla y = \left( \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial y}{\partial x_n} \right)$$

while $\nabla^*$ its adjoint operator. Then, the gradient observation at the final time $T$ is given by:

$$z_{u,f}(T) = CVS(T) y_0 + CVH_T u + CVF_T f$$

where $H_T$ and $F_T$ are operators formulated by

$$H_T : L^2(0, T; U) \rightarrow X$$

$$u \rightarrow H_T u = \int_0^T S(T - s) B u(s) ds$$

and

$$F_T : L^2(0, T; \mathbb{X}) \rightarrow X$$

and

$$f \rightarrow F_T f = \int_0^T S(T - s) f(s) ds$$

In the autonomous case, that is to say, deprived of disturbance ($f = 0$) and control ($u = 0$) the observation of gradient, $z_{0,0}(\cdot) = CVS(\cdot) y_0$, is then normal. But if the system is disturbed by a term $f$, the gradient observation becomes

$$z_{0,f}(T) = CVS(T) y_0 + CVF_T f$$

Generally $z_{0,f}(\cdot) \neq CVS(\cdot) y_0$. Then a control term $Bu$ is introduced in order to reduce, in finite time, the effect of this disturbance according to the gradient observation, such that: For any $f \in L^2(0, T; \mathbb{X})$, there exists $u \in L^2(0, T; \mathbb{U})$ satisfying

$$CVH_T u + CVF_T f = 0$$

The next definition characterizes the gradient remediable notion of type exactly and weakly in finite time as follows:

**Definition 1**

1. System $(S)$ augmented by $(O)$, (or $(S) + (O)$) is called exactly gradient remediable (EGR-system) on $[0, T]$, if for every $f \in L^2(0, T; X)$, there exists a control $u \in L^2(0, T; U)$ such that $CVH_T u + CVF_T f = 0$.

2. $(S) + (O)$ is called weakly gradient remediable (WGR-system) on $[0, T]$, if for every $f \in L^2(0, T; X)$ and for every $\varepsilon > 0$, there exists a control $u \in L^2(0, T; U)$ such that $\|CVH_T u + CVF_T f\|_Y < \varepsilon$.

**Remark 1**

The finite time gradient compensation problem is equivalent to:

For any $f \in L^2(0, T; X)$, does there exists a control $u \in L^2(0, T; U)$ such that

$$\int_0^T CVS(T - s) B u(s) ds + \int_0^T CVS(T - s) f(s) ds = 0$$

or equivalently

$$\int_0^T CVS(t) B v(t) dt + \int_0^T CVS(t) g(t) dt = 0$$

where $g(t) = f(T - t)$ and $v(t) = u(T - t)$.

Consequently, the finite time gradient remediability of $(S) + (E)$ can be also formulated as follows:

For any $g \in L^2(0, T; X)$, there exists a control $v \in L^2(0, T; U)$ satisfying Eq.1.

The characterizations consequences on the WEGR-systems and in limited time have been
established by Rekkab and Benhadid, and they have shown that the remediability concept of type gradient is a weaker than controllability of type gradient\(^\text{10}\).

**Asymptotic Gradient Compensation Problem:**

**Formalism statement:**

An asymptotic analysis of the problem is given by considering the system:

\[
(S_\infty) \begin{cases} 
\dot{y}(t) = A y(t) + B u(t) + f(t); t > 0 \\
y(0) = y_0 
\end{cases}
\]

augmented by the output (gradient observation) equation:

\[
(O_\infty) z_{uf}(t) = \nabla y(t); t > 0
\]

with \( f \in L^2(0, +\infty; X) \) and \( u \in L^2(0, +\infty; U) \).

Let us consider the following operators

\[
H_\infty: L^2(0, +\infty; U) \to X
\]

\[
u \mapsto H_\infty \nu = \int_0^{+\infty} S(s)B\nu(s)ds
\]

and

\[
F_\infty: L^2(0, +\infty; X) \to X
\]

\[
f \mapsto F_\infty f = \int_0^{+\infty} S(s)f(s)ds
\]

The asymptotic gradient remediability problem was studied to consist an investigation regarding the output operator \( \mathcal{C} \), the existence of an input one \( B \) confirming the gradient compensation asymptotically of any disturbance, that is: For any \( f \in L^2(0, +\infty; X) \), there exists \( u \in L^2(0, +\infty; U) \) such that

\[
\nabla H_\infty \nu + \nabla F_\infty f = 0
\]

Note that the operators \( H_\infty \) and \( F_\infty \) are not generally well defined. They are, if and only if the following condition is verified:\(^{14}\):

\[
\exists k \in L^2(0, +\infty; \mathbb{R}^+) \text{ such that }
\|
\nabla S(t)\| \leq k(t); \forall t \geq 0
\]

**Remark 2**

\begin{itemize}
  \item If \( \{S(t)\}_{t\geq0} \) is exponentially stable, that is to say, if \( \exists \beta > 0 \) and \( \exists \alpha > 0 \) such that
  \[
  \|S(t)\| \leq \beta e^{-\alpha t}; \forall t \geq 0
  \]
  then Eq.3, is satisfied with \( k(t) = \beta e^{-\alpha t} \in L^2(0, +\infty; \mathbb{R}^+) \), consequently \( H_\infty \) and \( F_\infty \) are well defined. This hypothesis concern the choice of the dynamics \( \mathcal{A} \) of the system through the semi-group \( \{S(t)\}_{t\geq0} \) and also the input operator \( B \).
  \item Actually, this work is concerned with the operators \( K_\infty^C \) and \( R_\infty^C \) which are defined by
  \[
  K_\infty^C: L^2(0, +\infty; U) \to Y
  \]
  \[
  u \mapsto K_\infty^C u = \int_0^{+\infty} \nabla S(t)B\nu(t)dt
  \]
  and
  \[
  R_\infty^C: L^2(0, +\infty; X) \to Y
  \]
  \[
  f \mapsto R_\infty^C f = \int_0^{+\infty} \nabla S(t)f(t)dt
  \]
  Then some weaker hypotheses are needed than Eq.3. Certainly, it is supposed that \( \exists k \in L^2(0, +\infty; \mathbb{R}^+) \) satisfied
  \[
  \|\nabla S(t)\| \leq k(t); \forall t \geq 0
  \]
  In this case, \( K_\infty^C \) and \( R_\infty^C \) are well defined and Eq.2 becomes:
  \[
  K_\infty^C u + R_\infty^C f = 0
  \]
  Under hypothesis Eq.4, therefore, the WEAGR-system can be expressed in the next manner:
  \end{itemize}

**Definition 2**

(i) \( (S_\infty) + (O_\infty) \) is called EAGR-system, if \( \forall f \in L^2(0, +\infty; X) \), there exists a control \( u \in L^2(0, +\infty; U) \) such that \( K_\infty^C u + R_\infty^C f = 0 \).

(ii) \( (S_\infty) + (O_\infty) \) is called WAGR-system, if \( \forall f \in L^2(0, +\infty; X) \) and every \( \varepsilon > 0 \) there exists a control \( u \in L^2(0, +\infty; U) \) such that \( \|K_\infty^C u + R_\infty^C f\|_{IR^2} < \varepsilon \).

Let us note that for \( T > 0; f \in L^2(0, +\infty; X) \) and \( u \in L^2(0, +\infty; U) \) and under hypothesis Eq.4, it follows that:

\[
K_\infty^C u + R_\infty^C f = \int_0^T \nabla S(t)B\nu(t)dt + \int_0^{+\infty} \nabla S(t)f(t)dt
\]

\[
= \int_0^T \nabla S(t)B\nu(t)dt + \int_0^T \nabla S(t)f(t)dt + [e_1(T) + e_2(T)]
\]

where \( e_1(T) = \int_T^{+\infty} \nabla S(t)B\nu(t)dt \) and
\[
\varepsilon_2(T) = \int_T^{+\infty} CVS(t)f(t)\,dt, \text{ with } \\
\varepsilon_1(T) + \varepsilon_2(T) \to 0 \text{ when } T \to +\infty, \text{ then for any } \\
f \in L^2(0, +\infty; X) \text{ and } u \in L^2(0, +\infty; U), \text{ it follows that } \\
\lim_{T \to +\infty} \left( \int_0^T CVS(t)Bu(t)\,dt + \int_0^T CVS(t)f(t)\,dt \right) \\
= K_C^{\infty}u + R_C^{\infty}f
\]

Characterization:

For the following results, let \( B^* \) and \( C^* \) be the adjoint operators of \( B \) and \( C \) respectively and \( (S^*(t))_{t \geq 0} \) is considered for the semigroup of \( (S(t))_{t \geq 0} \) of type adjoint. Let also \( X', U' \) and \( Y' \) be the dual space of \( X, U \) and \( Y \). Under hypothesis Eq.4, the following general characterization results are obtained:

Proposition 1

The following properties are equivalent

(i) \( (S_{\alpha}) + (E_{\alpha}) \) is EAGR-system.

(ii) \( \text{Im}(R^{\infty}_C) = \text{Im}(K^{\infty}_C) \).

(iii) \( \exists y > 0 \text{ such that } \forall \theta \in Y', \text{ it follows that } \\
\|S^{\infty}(\cdot)V^{\infty}C^{\ast}\theta\|_{L^2(0, +\infty; \mathbb{X})} \\
\leq y\|B^{\ast}S^{\infty}(\cdot)V^{\infty}C^{\ast}\theta\|_{L^2(0, +\infty; \mathbb{U})}
\]

Proof

(i) \( \iff \) (ii) Derives from Definition 1. Indeed, it is assumed that \( (S_{\alpha}) + (E_{\alpha}) \) is EAGR-system.

Let \( y \in \text{Im}(R^{\infty}_C) \), then there exists \( f \in L^2(0, +\infty; \mathbb{X}) \) such that \( y = R^{\infty}_Cf \).

From the property of exact asymptotic gradient remediability for the considered system, there exists \( u \in L^2(0, +\infty; U) \) such that \( K^{\infty}_Cu + R^{\infty}_Cu = 0 \Rightarrow R^{\infty}_Cu = -K^{\infty}_Cu \).

By the linearity of the operator \( K^{\infty}_C \), it follows that \( y = R^{\infty}_Cf = K^{\infty}_C(-u) \), then \( y \in \text{Im}(K^{\infty}_C) \).

The other inclusion is obtained as the previous one. Then, it follows that \( \text{Im}(R^{\infty}_C) = \text{Im}(K^{\infty}_C) \).

- Conversely, it is assumed that \( \text{Im}(R^{\infty}_C) = \text{Im}(K^{\infty}_C) \) and one can show that \( (S_{\alpha}) + (E_{\alpha}) \) is EAGR-system.

Let \( f \in L^2(0, +\infty; \mathbb{X}) \), then \( R^{\infty}_Cf \in \text{Im}(R^{\infty}_C) \). Since \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \), it follows that \( R^{\infty}_Cf \in \text{Im}(K^{\infty}_C) \) then there exists \( u \in L^2(0, +\infty; U) \) such that \( R^{\infty}_Cf = K^{\infty}_Cu \), this gives \( R^{\infty}_Cf - K^{\infty}_Cu = 0 \) and by putting \( u_1 = -u \in L^2(0, +\infty; U) \). Thus \( R^{\infty}_Cf + K^{\infty}_Cu_1 = 0 \) where \( (S) + (E) \) is EAGR-system.

(ii) \( \iff \) (iii) Derives from the fact that the adjoint operators \( (R^{\infty}_C)^{\ast} \) and \( (K^{\infty}_C)^{\ast} \) of \( (R^{\infty}_C) \) and \( (K^{\infty}_C) \) respectively, are defined by

\[(R^{\infty}_C)^{\ast}: Y' \to L^2(0, +\infty; X') \\
\theta \to (R^{\infty}_C)^{\ast}\theta = S^{\ast}(\cdot)V^{\ast}C^{\ast}\theta \]

and

\[(K^{\infty}_C)^{\ast}: Y' \to L^2(0, +\infty; U') \\
\theta \to (K^{\infty}_C)^{\ast}\theta = B^{\ast}(R^{\infty}_C)^{\ast}\theta = B^{\ast}S^{\ast}(\cdot)V^{\ast}C^{\ast}\theta \]

Set \( P = (R^{\infty}_C)^{\ast} \), \( Q = (K^{\infty}_C)^{\ast} \) and use the following lemma.

Lemma 1

Let \( X, Y, Z \) be spaces of Banach reflexive type, \( P \in \mathcal{U}(X, Z) \) and \( Q \in \mathcal{U}(Y, Z) \). There is an equivalence between:

\[\text{Im}(P) \subset \text{Im}(Q)\]

and

\[\exists y > 0 \text{ such that } \|P^\ast z\|^2 \leq y\|Q^\ast z\|^2, \forall z \in Z'.\]

\[\square\]

The following proposition 2 is proved with regard of the weak asymptotic gradient remediability characterization.

Proposition 2

There is equivalence between

(i) \( (S_{\alpha}) + (O_{\alpha}) \) is WAGR-system.

(ii) \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \).

(iii) \( \ker(B^{\ast}(R^{\infty}_C)^{\ast}) = \ker((R^{\infty}_C)^{\ast}) \).

Proof

(i) \( \iff \) (ii) Derives from Definition 1. Indeed, it is assumed that \( (S_{\alpha}) + (O_{\alpha}) \) is WAGR-system.

Let \( f \in L^2(0, +\infty; \mathbb{X}) \), then \( \forall \varepsilon > 0, \exists u \in L^2(0, +\infty; U) \) such that

\[\|K^{\infty}_Cu + R^{\infty}_Cf\|_Y < \varepsilon, \text{ is that to say }\]

\[\|K^{\infty}_Cu + R^{\infty}_Cf\|_Y < \varepsilon.\]

Set \( u = -u \in L^2(0, +\infty; U) \), then \( \forall \varepsilon > 0, \exists u_1 \in L^2(0, +\infty; U) \) such that \( \|R^{\infty}_Cf - K^{\infty}_Cu_1\|_Y < \varepsilon, \) this gives \( R^{\infty}_Cf \in \text{Im}(K^{\infty}_C) \), where \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \).

- Conversely, assume that \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \) and let \( f \in L^2(0, +\infty; \mathbb{X}) \), then \( R^{\infty}_Cf \in \text{Im}(K^{\infty}_C) \), then \( \forall \varepsilon > 0, \exists u_1 \in L^2(0, +\infty; U) \) such that \( \|R^{\infty}_Cf - K^{\infty}_Cu_1\|_Y < \epsilon.\)

Put \( u_1 = -u \in L^2(0, +\infty; U) \), then

\[\forall \varepsilon > 0, \exists u \in L^2(0, +\infty; U) \text{ such that } \|R^{\infty}_Cf + K^{\infty}_Cu\|_Y < \varepsilon, \text{ where } (S_{\alpha}) + (E_{\alpha}) \text{ is WAGR-system. }\]

(ii) \( \iff \) (iii) by considering orthogonal. Indeed, it is assumed that \( (S_{\alpha}) + (O_{\alpha}) \) is WAGR-system. So, one can show that \( \ker(B^{\ast}(R^{\infty}_C)^{\ast}) = \ker((R^{\infty}_C)^{\ast}) \).

\[\theta \in IR^q \text{ such that } B^{\ast}(R^{\infty}_C)^{\ast}\theta = 0.\]

In addition, \( (K^{\infty}_C)^{\ast} = B^{\ast}(R^{\infty}_C)^{\ast} \), this gives \( \theta \in \ker((K^{\infty}_C)^{\ast}) \). Thus \( \text{Im}(K^{\infty}_C) \) = \( \overline{\text{ker}((K^{\infty}_C)^{\ast})} \).

By Proposition 3.5, if follows that \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \). Then, \( \text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \).

\[\forall f \in L^2(0, +\infty; \mathbb{X}) \text{ or } R^{\infty}_Cf \in \text{ker}((K^{\infty}_C)^{\ast}), \text{ we have }\]

\[\text{Im}(R^{\infty}_C) \subset \text{Im}(K^{\infty}_C) \text{ and } \ker(B^{\ast}(R^{\infty}_C)^{\ast}) \subset \ker((R^{\infty}_C)^{\ast}) \text{ where } \ker(B^{\ast}(R^{\infty}_C)^{\ast}) \subset \ker((R^{\infty}_C)^{\ast}).\]
Conversely, assume that $\text{Ker}(B^* (R_{C}^\infty)^*) = \text{Ker}(R_{C}^\infty)$ and one can show that $\text{Im}(R_{C}^\infty) \subset \text{Im}(K_{C}^\infty)$. Let $f \in L^2(0, +\infty; \chi)$ such that $f \in \text{Im}(R_{C}^\infty)$, it follows that $\text{Im}(K_{C}^\infty) = (\text{Ker}(K_{C}^\infty))^\perp$. For every $\theta \in IR^q$ such that $(R_{C}^\infty)^* \theta = 0$ that is, $B^* (R_{C}^\infty)^* \theta = 0$, it follows that $(R_{C}^\infty)^* \theta = 0$ because $\text{Ker}(B^* (R_{C}^\infty)^*) = \text{Ker} ((R_{C}^\infty)^*)$, then $(R_{C}^\infty f, \theta) = 0$. □

Asymptotic Gradient Remediability via Actuators and Sensors:

In connection with the system $(S_{\infty})$ is motivated by $(\Omega_k, g_k)_{1 \leq k \leq p}$, actuators suite of type zone with $g_i \in L^2 (\Omega_k)$ and $\Omega_k = \text{Supp} (g_k) \subset \Omega, \forall k = 1, \ldots, p$, with control space $U = \mathbb{R}^p$ and $B$ is specified by

$$B : \mathbb{R}^p \to X$$

$$u(t) \to Bu(t) = \sum_{k=1}^{p} \chi_{\Omega_k} g_k u_k (t)$$

and where $u = (u_1, \ldots, u_p ) \in L^2 (0, +\infty; \mathbb{R}^p)$. Its adjoint is given by

$$B^* z = \left( \langle g_1, z \rangle_{L^2(\Omega_1)}, \ldots, \langle g_p, z \rangle_{L^2(\Omega_p)} \right) \in \mathbb{R}^p$$

then, the following result is obtained:

**Corollary 1**

$(S_{\infty}) + (O_{\infty})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied the next inequality

$$\int_0^{+\infty} \| S^* (t) V^C \theta \|_{L^2(0, +\infty; \mathcal{X})}^2 dt \leq \gamma \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^* (t) V^C \theta) \rangle^2 dt$$

for every $\theta \in Y$.

**Proof**

Since Proposition 1, $(S_{\infty}) + (O_{\infty})$ is EAGR-systems $\iff \exists \gamma > 0$ with

$$\| S^* (\cdot) V^C \theta \|_{L^2(0, +\infty; \mathcal{X})} \leq \gamma \| B^* S^* (\cdot) V^C \theta \|_{L^2(0, +\infty; \mathcal{U})}$$

for every $\theta \in Y$.

By using Eq.5, the formula of the operator $B^*$, yields that

$$\int_0^{+\infty} \| S^* (t) V^C \theta \|_{L^2(0, +\infty; \mathcal{X})}^2 dt \leq \gamma \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^* (t) V^C \theta) \rangle^2 dt$$

By supposing the output function $(S_{\infty})$ is specified via suite of sensor of type zones of $(D_i, h_i)_{1 \leq i \leq q}$, $h_i \in L^2 (D_i)$, represent the distribution zone sensor, $D_i = \text{Supp} h_i \subset \Omega$, intended for $i = 1, \ldots, q$ as well as $D_i \cap D_j = \emptyset$ for $i \neq j, Y = \mathbb{R}^q$ and the operator $C$ is formed by

$$C : (L^2 (\Omega))^q \to \mathbb{R}^q$$

$$y(t) \to Cy(t) = \left( \sum_{i=1}^{n} \langle h_i, y_i(t) \rangle_{D_i}, \ldots, \sum_{i=1}^{n} \langle h_q, y_q(t) \rangle_{D_q} \right)$$

its adjoint is given by $C^*$ with for $\theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$

$$C^* \theta = \left( \sum_{i=1}^{q} \chi_{D_i} \theta_i, \ldots, \sum_{i=1}^{q} \chi_{D_i} \theta_i \right) \in (L^2 (\Omega))^n$$

Without loss of generality, consider the system $(S_{\infty})$ with a dynamics $A$ having the form

$$A y = \sum_{m=1}^{+\infty} \lambda_m \sum_{j=1}^{r_m} \langle y, \varphi_{mj} \rangle_{L^2(\Omega)} \varphi_{mj}, \forall y \in D(A)$$

where $\lambda_1, \lambda_2, \ldots$ are real parameters such that $\lambda_1 > \lambda_2 > \lambda_3 > \ldots, (\varphi_{mj})_{1 \leq j \leq r_m}$ is an orthogonal basis in $H_0^1 (\Omega)$ of eigenvectors for $A$ which is orthonormal in $L^2 (\Omega)$, related to eigenvalues $\lambda_n$ with a multiplicity $\gamma_n$. It is well known that $A$ produces a semi – group $(S(t))_{t \geq 0}$ of type strongly continuous given by [14, 15]:

$$S(t)y = \sum_{m=1}^{+\infty} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle y, \varphi_{mj} \rangle_{L^2(\Omega)} \varphi_{mj}$$

Obviously, if $\sup \lambda_m = \lambda_1 < 0$, $(S(t))_{t \geq 0}$ is exponentially stable. □

The following characterization results have obtained

**Corollary 2**

$(S_{\infty}) + (E_{\infty})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied the next inequality

$$\sum_{m=1}^{+\infty} \frac{1}{2\lambda_m} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla \varphi_{mj} \rangle^2_{(L^2(\Omega))^n} \leq \gamma \sum_{k=1}^{p} \sum_{m=1}^{+\infty} \frac{1}{2\lambda_m} \sum_{j=1}^{r_m} \langle C^* \theta, \nabla \varphi_{mj} \rangle^2_{(L^2(\Omega))^n}$$

for every $\theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$.

**Proof**

Since Corollary 1, $(S_{\infty}) + (E_{\infty})$ is EAGR-systems $\iff \exists \gamma > 0$ satisfied that $\forall \theta = (\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$, yields that

$$\int_0^{+\infty} \| S^* (t) V^C \theta \|_{L^2(\Omega)}^2 dt \leq \gamma \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^* (t) V^C \theta) \rangle^2 dt$$
Since
\[ S(t) y = \sum_{m \geq 1} e^{\lambda_m t} \sum_{j=1}^{r_m} \langle y, \varphi_{mj} \rangle_{L^2(\Omega)} \varphi_{mj} \]
it follows that
\[ \int_0^{+\infty} \| S'(t) \mathbf{P}^{*} C^* \theta \|_{L^2}^2 dt \leq \int_0^{+\infty} \| S'(t) \mathbf{P} C^* \theta \|_{L^2(\Omega)}^2 dt \]
\[ = \int_0^{+\infty} \sum_{m \geq 1} e^{2\lambda_m t} \sum_{j=1}^{r_m} \langle (\mathbf{P}^{*} C^* \theta, \varphi_{mj}) \rangle_{L^2(\Omega)}^2 dt \]
\[ = \sum_{m \geq 1} \frac{-1}{2\lambda_m} \sum_{j=1}^{r_m} \langle (C^* \theta, \nabla \varphi_{mj}) \rangle_{L^2(\Omega)}^2 \]
and
\[ \int_0^{+\infty} \sum_{k=1}^{p} \langle (g_k, S^{*}(t) \mathbf{V}^* C^* \theta) \rangle dt = \sum_{k=1}^{p} \int_0^{+\infty} \sum_{m \geq 1} e^{2\lambda_m t} \sum_{j=1}^{r_m} \langle (\mathbf{V}^* C^* \theta, \varphi_{mj}) \rangle_{L^2(\Omega)}^2 dt \]
\[ = \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2\lambda_m} \sum_{j=1}^{r_m} \langle (C^* \theta, \nabla \varphi_{mj}) \rangle_{L^2(\Omega)}^2 \]
\[ \Rightarrow \exists \gamma > 0 \text{satisfied that} \quad \forall \theta = (\theta_1, ..., \theta_q) \in \mathbb{R}^q. \]

**Proof**
Since Corollary 2, \((S_{\infty}) + (E_{\infty})\) is \(EAGR\)-systems \(\Leftrightarrow \exists \gamma > 0\) satisfied that \(\forall \theta = (\theta_1, ..., \theta_q) \in \mathbb{R}^q\), then
\[ \sum_{m \geq 1} \frac{-1}{2\lambda_m} \sum_{j=1}^{r_m} \langle (C^* \theta, \nabla \varphi_{mj}) \rangle_{L^2(\Omega)}^2 \]
\[ \leq \gamma \sum_{k=1}^{p} \sum_{m \geq 1} \frac{-1}{2\lambda_m} \sum_{j=1}^{r_m} \langle (g_k, \varphi_{mj}) \rangle_{L^2(\Omega)}^2 \langle (g_k, \varphi_{mj}) \rangle_{\Omega_k} \]
\[ = \sum_{k=1}^{p} \sum_{l=1}^{q} \langle \partial \theta_l, \partial \varphi_{ml} \rangle_{\Omega_l} \]
\[ = \sum_{l=1}^{q} \langle \theta_l, \partial \varphi_{ml} \rangle_{\Omega_l} \]
\[ = \sum_{l=1}^{q} \langle \theta_l, \partial \varphi_{ml} \rangle_{L^2(\Omega_l)} \]
\[ = \sum_{l=1}^{q} \langle \theta_l, \partial \varphi_{ml} \rangle_{L^2(\Omega_l)} \]

**Asymptotic Gradient Efficient Actuators and Sensors:**
The notion of asymptotic gradient efficient actuator have been presented analogy to the concept of gradient efficient actuator in finite time given as follows:

**Definition 3**
The suite \((\Omega_k, g_k)_{1 \leq k \leq p}\) is called asymptotic gradient efficient actuators \((AGE\)-actuators\) if, \((S_{\infty}) + (E_{\infty})\) is \(WAGR\)-systems.

**Proposition 3**
The suite \((\Omega_k, g_k)_{1 \leq k \leq p}\), \(AGE\)-actuators if and only if
\[ \bigcap_{m \geq 1} \text{Ker} (M_m f_m) = \text{Ker} (B^*(R_{\infty}^0)^*) \]
anywhere, for \(m \geq 1\), \(M_m\) is the matrix of order \((p \times r_m)\) defined by
\[ M_m = ((g_k, \varphi_{mj})_{L^2(\Omega_k)})_{k \leq p} \]
and

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\( f_m : \theta \in \mathbb{R}^d \rightarrow f_m(\theta) = \left( (\nabla C^* \theta, \varphi_{m1}), (\nabla C^* \theta, \varphi_{m2}), \ldots, (\nabla C^* \theta, \varphi_{mr_m}) \right) \in \mathbb{R}^{rm} \)

**Proof**

Since Proposition 3, \((S_\infty) + (O_\infty)\) is WAG\-R systems if and only if \(Ker (B^*(R_C^{\infty} )) = Ker ((R_C^{\infty} ))\)

Let \( \theta \in \mathbb{R}^d \), then

\[
B^*(R_C^{\infty} ) \theta = B^* S^* (\cdots) \nabla C^* \theta = \begin{pmatrix}
(g_1, S^* (\cdots) \nabla C^* \theta)_{L^2(\Omega_1)} \\
(g_2, S^* (\cdots) \nabla C^* \theta)_{L^2(\Omega_2)} \\
\vdots \\
(g_p, S^* (\cdots) \nabla C^* \theta)_{L^2(\Omega_p)}
\end{pmatrix}
\]

\[
= \left( \sum_{m \geq 1}^{rm} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_1, \varphi_{mj})_{L^2(\Omega_1)} \\
\sum_{m \geq 1}^{rm} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_2, \varphi_{mj})_{L^2(\Omega_2)} \\
\vdots \\
\sum_{m \geq 1}^{rm} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_p, \varphi_{mj})_{L^2(\Omega_p)}
\right)
\]

and then, for \( m \geq 1 \),

\[
M_{mf}(\theta) = \left( \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_1, \varphi_{mj})_{L^2(\Omega_1)} \\
\sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_2, \varphi_{mj})_{L^2(\Omega_2)} \\
\vdots \\
\sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_p, \varphi_{mj})_{L^2(\Omega_p)}
\right)
\]

Assume that \( \theta \in \bigcap_{m \geq 1} Ker (M_{mf}) \), this gives \( \theta \in Ker (M_{mf}) \), \( \forall m \geq 1 \) \( \Rightarrow \)

\[
\forall k \in \{ 1, 2, \ldots, p \}, \forall m \geq 1 \Rightarrow \sum_{m \geq 1} e^{\lambda_m} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_k, \varphi_{mj})_{L^2(\Omega_k)} = 0, \forall k \in \{ 1, 2, \ldots, p \}, \forall m \geq 1 \Rightarrow \sum_{m \geq 1} e^{\lambda_m} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} (g_k, \varphi_{mj})_{L^2(\Omega)} = 0, \forall k \in \{ 1, 2, \ldots, p \}, \forall m \geq 1 \Rightarrow B^*(R_C^{\infty} ) \theta = 0 \Rightarrow \theta \in Ker (B^*(R_C^{\infty} )).
\]

Where

\[
\bigcap_{m \geq 1} Ker (M_{mf}) \subset Ker (B^*(R_C^{\infty} ))
\]

that is

\[
\bigcap_{m \geq 1} Ker (M_{mf}) = Ker (B^*(R_C^{\infty} ))
\]

\( \Box \)

**Proposition 4**

The suite \((Q_k, g_k)_{1 \leq k \leq p}\), is AG\-E actuators if and only if

\[
Ker (\nabla C^* ) = \bigcap_{m \geq 1} Ker (M_{mf})
\]

**Proof**

Suppose that the suite \((Q_k, g_k)_{1 \leq k \leq p}\), is AG\-E actuators to prove

\[
Ker (\nabla C^* ) = \bigcap_{m \geq 1} Ker (M_{mf})
\]

Since Proposition 2 and Proposition 3, \((S_\infty) + (E_\infty)\) is WAG\-R system if and only if

\[
\bigcap_{m \geq 1} Ker (M_{mf}) = Ker (B^*(R_C^{\infty} ))
\]

it follows that for every \( \theta \in \mathbb{R}^d \),

\[
(R_C^{\infty} )^* \theta = S^* (\cdots) \nabla C^* \theta = \sum_{m \geq 1} e^{\lambda_m} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} \varphi_{mj}
\]

Assuming that \( \theta \in Ker ((R_C^{\infty} )) \), then

\[
(R_C^{\infty} )^* \theta = 0.
\]

Then,

\[
(R_C^{\infty} )^* \theta = S^* (\cdots) \nabla C^* \theta = \sum_{m \geq 1} e^{\lambda_m} \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} \varphi_{mj} = 0
\]

\( \forall m \geq 1 \),

\[
\forall m \geq 1 \Rightarrow \sum_{j=1}^{r_m} (\nabla C^* \theta, \varphi_{mj})_{L^2(\Omega_1)} \varphi_{mj} = 0
\]

\( \Rightarrow \nabla C^* \theta = 0 \Rightarrow \theta \in Ker (\nabla C^* ) \).

Hence,

\[
Ker ((R_C^{\infty} )) \subset Ker (\nabla C^* )
\]

On the other hand, if it is assumed that \( \theta \in Ker (\nabla C^* ) \), then \( \nabla C^* \theta = 0 \) that is to say,

\[
(R_C^{\infty} )^* \theta = S^* (\cdots) \nabla C^* \theta = 0
\]

\( \Rightarrow \theta \in Ker ((R_C^{\infty} )) \).

That is to say, \( Ker (\nabla C^* ) \subset Ker ((R_C^{\infty} )) \), then \( Ker (\nabla C^* ) = ker ((R_C^{\infty} )) \).

\( \Box \)

By analogy with the finite time case and under a condition given by: If there exists \( m_0 \geq 1 \) such that \( \text{rank } G_m = q \)

where, for \( m \geq 1 \), \( G_m \) is the matrix of order \((q \times r_m)\) defined by

\[
G_m = \left( \sum_{i=1}^{n} (h_i, \frac{\partial \varphi_{mj}}{\partial x_i})_{L^2(\Omega)} \right)_{ij},
\]

\( 1 \leq i \leq q \) and \( 1 \leq j \leq r_m \)

and \( G_m^T \) is the transpositional matrix of \( G_m \), the two following corollary’s are obtained, where the demonstrations are similar to the finite time case given in 12.
Corollary 5
The suite \((Ω_k, g_k)_{1≤k≤p}\) is \(AGE\)-actuators if and only if
\[\bigcap_{m=1}^{∞} Ker (M_mG_m^T) = \{0\}\]

Corollary 5
If \(\text{rank} (M_mG_m^T) = q\) or \(\text{rank} M_m = r_m\)
Then, the suite \((Ω_k, g_k)_{1≤k≤p}\) is \(AGE\)-actuators

Asymptotic Gradient Remediability and Asymptotic Gradient Controllability:
The case of asymptotic relation is difficult and requires more conditions.

Asymptotic Gradient Controllability:
Assuming the system that is described by the following equation:
\[(S_0) \begin{cases} \dot{y}(t) = Ay(t) + Bu(t); t > 0 \\ y(0) = y \end{cases}\]
and \(A\) is supposed generates a strongly continuous semi-group \((S(t))_{t≥0}\) such that
\[\|\nabla S(t)\| ≤ k(t); \forall t ≥ 0\]
Next, some sufficient conditions to characterize the \(AGC\)-system are given in the following results.

Definition 4
System \((S_0)\) is called
\(\ast\) \(EAGC\)-system if for every \(y ∈ \mathcal{E} = (L^2(Ω))^n\), there exists \(u ∈ L^2(0, +∞; U)\)
such that \(\nabla y_0 + \nabla H_\omega u = y\), or equivalently \(\text{Im} \nabla H_\omega = (L^2(Ω))^n\).
\(\ast\) \(WAGC\)-system if for every \(y ∈ \mathcal{E} = (L^2(Ω))^n\), and every \(ε > 0\), there exists \(u ∈ L^2(0, +∞; U)\)
such that \(\|\nabla y_0 + \nabla H_\omega u - y\| < ε\), or equivalently \(\text{Im} \nabla H_\omega = (L^2(Ω))^n\).

Let \(\mathcal{E}', U'\) be the dual spaces of \(\mathcal{E}\) and \(U\) respectively, then using Lemma 1, it is easy to show the following results the following proposition 5 characterizes the \(EAGC\)-systems, and \(WAGC\)-systems.

Proposition 5
The system \((S_0)\) is
(i) \(EAGC\)-systems if and only if
\[\exists γ > 0 \text{ such that } \forall z^* ∈ \mathcal{E}', \|z^*\|_{\mathcal{E}'} ≤ γ\|\nabla (VH_\omega)^*z^*\|_{L^2(0, +∞; U)}\]
Or equivalently
\[\exists γ > 0 \text{ such that } \forall z^* ∈ \mathcal{E}', \|z^*\|_{\mathcal{E}'} ≤ γ\|B^*S'\cdot(\nabla V)^*z^*\|_{L^2(0, +∞; U)}\]
(ii) \(WAGC\)-systems if and only if
\[\text{Ker} (\nabla (VH_\omega)^*) = \{0\}\]

The following results in proposition 6 demonstrate that the asymptotic controllability concept of type gradient is strongest than the asymptotic remediability of type gradient in various situations.

Proposition 6
If \((S_0)\) is \(EAGC\)-system (resp. \(WAGC\)-system), then, \((S_∞) + (E_∞)\), it is \(EAGR\)-system (resp. \(WAGR\)-system).

Proof
\(\ast\) By hypothesis Eq.8, if follows that, for \(θ ∈ Y'\),
\[\|S'(. )\nabla V' C' θ\|_{L^2(0, +∞; X')}' = \left(\int_0^{+∞} \|S'(t)\nabla V' C' θ\|_{L^2(X')}^2 dt\right)^{\frac{1}{2}}\]
\[≤ \left(\int_0^{+∞} \|S'(t)\nabla V'\|_{L^2(0, +∞; X')}^2 \|C' θ\|_{L^2(0, +∞; U)}^2 dt\right)^{\frac{1}{2}}\]
\[= \left(\int_0^{+∞} \|\nabla S(t)\|_{L^2(0, +∞; X')}^2 \|C' θ\|_{L^2(0, +∞; U)}^2 dt\right)^{\frac{1}{2}} ≤ k\|C' θ\|_{L^2(0, +∞; X')}\] with \(k > 0\).

from Proposition 5, and since \((S_0)\) is \(EAGC\)-system, \(∃ γ > 0\), with
\[\|C' θ\|_{L^2(0, +∞; X')} ≤ γ\|B^*S'(. )\nabla V' C' θ\|_{L^2(0, +∞; U)}\]
then,
\[\|S'(. )\nabla V' C' θ\|_{L^2(0, +∞; X')} ≤ M\|B^*S'(. )\nabla V' C' θ\|_{L^2(0, +∞; U)}\] with \(M = kγ > 0\).
By using the equivalence of part (i) and part (ii) in proposition 1, \((S_∞) + (E_∞)\) is \(EAGR\)-systems.

\(\ast\) From Proposition 5, \((S_∞) + (E_∞)\) is \(WAGC\)-system and remains equivalent to,
\[\text{Ker} (B^* (R_{c_∞}' )) = Ker ( (R_{c_∞}' ))\]
that is to say
\[\text{Ker} (B^* (R_{c_∞}' )) ⊂ Ker ( (R_{c_∞}' ))\]
This is equivalent to \(Ker ( (K_{c_∞}' )) ⊂ Ker ( (R_{c_∞}' ))\), because \((K_{c_∞}' ) = B^* (R_{c_∞}' )\).
For \(θ ∈ Ker ( (K_{c_∞}' ))\), it follows that
\[(K_{c_∞}' ) θ = B^*S' (\nabla V' C' θ = (\nabla VH_ω)^*C' θ = 0,\]
then \(C' θ = 0\) because \(\text{Ker} ( (\nabla VH_ω)^* ) = \{0\}\),
and then \(θ ∈ Ker ( C' ) ⊂ Ker ( (R_{c_∞}' ))\).

Remark 3
The opposite of Proposition 6 is not correct; this case may be exemplified via the following.

Example 1
Reflect the subsequent one dimensional system of type diffusion.
boosted via observation function allows by q sensors of type zone

\[ (O_1) \sum_{k=1}^{q} \left( h_i \frac{\partial y}{\partial x_i} \right)_{t=0} = C \nabla y(t) \]

\[ \left( \sum_{i=1}^{n} \left( h_i \frac{\partial y}{\partial x_i} \right)_{t=0} \right) \]

So, \( \Omega = [0, 1] \) gives the corresponding operator \( \Delta \) of type Laplace that confuses an appropriate basis of eigenfunctions via next form

\[ \varphi_m(x) = \sqrt{2} \sin \left( m \pi x \right); m \geq 1 \]

The corresponding eigenvalues are specified through \( \lambda_m = -m^2 \pi^2; m \geq 1 \). The operator \( \Delta \) generates a self adjoint strongly continuous semi group \( (S(t))_{t \geq 0} \) defined by

\[ S(t)y = \sum_{m=1}^{\infty} e^{-m^2 \pi^2 t} (y, \varphi_m) \varphi_m \]

is exponentially stable \(^{14}\) with the transformations

\[ H_\gamma u = \sum_{k=1}^{p} \sum_{m=1}^{\infty} \int_{0}^{\infty} e^{-m^2 \pi^2 t} u_k(t) dt \langle g_k, \varphi_m \rangle \varphi_m \]

and

\[ F_\gamma f = \sum_{m=1}^{\infty} \int_{0}^{\infty} e^{-m^2 \pi^2 t} \langle f(\cdot, t), \varphi_m \rangle \varphi_m dt \]

are well defined and since Corollary 3, \((S_1) + (O_1)\) is EAGR-systems if and only if \( \exists \gamma > 0 \) such that

\[ \sum_{m=1}^{\infty} \frac{1}{2m^2 \pi^2} \sum_{i=1}^{q} \left( \theta_i h_i \right) \frac{\partial \varphi_m}{\partial x_i} L^2(\Omega) \]

\[ \leq \gamma \sum_{m=1}^{\infty} \frac{1}{2m^2 \pi^2} \left( g \varphi_m + \varphi_m \right) L^2(\Omega) \]

for every \( \theta = (\theta_1, ..., \theta_q) \in \mathbb{R}^q \)

If a unique actuator (sensor) represents the input (output) of system \( (S_1) + (O_1) \), then the last inequality becomes as follows:

\[ \sum_{m=1}^{\infty} \frac{1}{2m^2 \pi^2} \left( \theta h_i \frac{\partial \varphi_m}{\partial x_i} L^2(\Omega) \right) \]

\[ \leq \gamma \sum_{m=1}^{\infty} \frac{1}{2m^2 \pi^2} \left( g \varphi_m + \varphi_m \right) L^2(\Omega) \]

for \( g = \varphi_{m_0} \) with \( m_0 \geq 1 \), it is obtained that

\[ \frac{1}{2m_0^2 \pi^2} \left( h \frac{\partial \varphi_{m_0}}{\partial x} L^2(\Omega) \right) \leq \gamma \frac{1}{2m_0^2 \pi^2} \left( h \frac{\partial \varphi_{m_0}}{\partial x} L^2(\Omega) \right) \]

\[ \text{this is verified for } \gamma \geq 1 \]

The considered system \( (S_1) \) is not EAG-System because it is not WAGC-system. Indeed, let \( y \in L^2(\Omega) \)

\[ (\nabla H_\gamma)^* y = (H_\gamma)^* \nabla^* y = B^* S^*(\cdot) \nabla y \]

Putting \( y(x) = \sin(m_0 \pi x) \), yields that

\[ (\nabla H_\gamma)^* y = e^{-m_0^2 \pi^2}(y, \nabla \varphi_{m_0}) \]

\[ = m_0 \pi e^{-m_0^2 \pi^2} \sqrt{\int_{0}^{1} y(x) \cos(m_0 \pi x) dx} \]

Putting \( y(x) = \sin(m_0 \pi x) \)

\[ e^{-m_0^2 \pi^2} \int_{0}^{1} m_0 \pi \sin(m_0 \pi x) \cos(m_0 \pi x) dx \]

\[ = \int_{0}^{1} e^{-m_0^2 \pi^2} [\sin^2 (m_0 \pi x)] dx = 0 \]

then \( Ker [(\nabla H_\gamma)^*] \neq \{0\} \) and by proposition 5, the result is proven.

\[ \square \]

Asymptotic Gradient Remediability with Minimum Energy:

Theorem 5. Under the condition Eq.7, and the hypothesis of WAGC-system, then in the present section the problem of WAGC-system with Minimal Energy is studied. Thus, through \( f \in L^2(0, +\infty; \mathbb{R}) \), exists a control of type optimal \( u \in L^2(0, +\infty; \mathbb{R}^p) \) ensuring, asymptotically, the gradient remediability of the disturbance \( f \) such that \( K^\infty_C u + R^\infty_C f = 0 \), are studied. That is the set defined by

\[ D = \{ u \in L^2(0, +\infty; \mathbb{R}^p): K^\infty_C u + R^\infty_C f = 0 \} \]

is non empty. Next, the following function is considered

\[ J(u) = \|K^\infty_C u + R^\infty_C f\|_{\mathbb{R}^q} + \|u\|_{L^2(0, +\infty; \mathbb{R}^p)} \]

The considered problem becomes : \( \min_{u \in D} J(u) \).

For its resolution, one can use a modification of \( (H, U, M) \).

For \( \theta \in \mathbb{R}^q \), it is noted that

\[ \|\theta\|_* = \left( \int_{0}^{+\infty} \|B^* S^*(\cdot) \nabla C^* \theta \|_{\mathbb{R}^p} dt \right) ^{\frac{1}{2}} \]

The conforming inner product is specified by

\[ (\theta, \sigma)_* = \int_{0}^{+\infty} (B^* S^*(\cdot) \nabla C^* \theta, B^* S^*(\cdot) \nabla C^* \sigma) dt \]
and the operator \(\Lambda^\infty_C : \mathbb{R}^q \to \mathbb{R}^q\) defined by
\[
\Lambda^\infty_C \theta = K_C^\infty (K_C^\infty)^* \theta
\]
Then, the following proposition have obtained.

**Proposition 7**

If the condition Eq.7 is verified, then \(\|\cdot\|_\ast\) is a norm on \(\mathbb{R}^q\) if and only if \((E_\infty) + (E_\infty)\) is WAGR-system and the operator \(\Lambda^\infty_C\) is invertible.

**Proof**

Since,
\[
\|\theta\|_\ast = \left( \int_0^{+\infty} \|B^\ast S(t) \nabla C^* \theta\|_{L_2(0, +\infty; \mathbb{R}^q)}^2 dt \right)^{\frac{1}{2}} = 0
\]
\[
\Rightarrow \|B^\ast S(\cdot) \nabla C^* \theta\|_{L_2(0, +\infty; \mathbb{R}^q)}^2 = 0
\]
\[
\Rightarrow B^\ast S(\cdot) \nabla C^* \theta = 0
\]
\[
\Rightarrow \theta \in \text{Ker} (B^\ast S(\cdot) \nabla C^*) = \text{Ker} B^\ast (K_C^\infty)^*\]

However, from Proposition 3, it follows that
\[
\cap_{m \geq 1} \text{Ker} (M_m f_m) = \text{Ker} (B^\ast (R_C^\infty)^*)
\]
and the control \(u_{\theta_f}\) is optimal with
\[
\|u_{\theta_f}\|_{L^2(0, +\infty; \mathbb{R}^p)} = \|\theta_f\|_2.
\]

Moreover, it is optimal and
\[
\|u_{\theta_f}\|_{L^2(0, +\infty; \mathbb{R}^p)} = \|\theta_f\|_2.
\]

**Proof**

By utilizing Proposition 7, the mapping \(\Lambda^\infty_C\) has inverse, now, \(f \in L^2(0, +\infty; X)\), then there exists a unique \(\theta_f \in \mathbb{R}^q\) such that \(\Lambda^\infty_C \theta_f = -R_C^\infty f\) and by putting \(u_{\theta_f} = (K_C^\infty)^* \theta_f\), yields that
\[
\Lambda^\infty_C \theta_f = K_C^\infty (K_C^\infty)^* \theta_f = \int_0^{+\infty} C \nabla S(t) B^\ast S^\ast(t) \nabla C^* \theta_f dt = K_C^\infty u_{\theta_f} = -R_C^\infty f \Rightarrow K_C^\infty u_{\theta_f} + R_C^\infty f = 0.
\]
The set \(D\) defined by Eq.9, is closed, convex and not empty.

For \(u \in D\), \(J(u) = \|u\|_{L^2(0, +\infty; \mathbb{R}^p)}^2\). So, \(J\) is convex mapping of type strictly in \(D\), and hence ensures a unique minimum at \(u^* \in D\), characterized by \(\langle u^*, v - u^* \rangle_{L^2(0, +\infty; \mathbb{R}^p)} \geq 0\) \(\forall v \in D\).

For \(v \in D\),
\[
\langle u_{\theta_f}, v - u_{\theta_f} \rangle_{L^2(0, +\infty; \mathbb{R}^p)} = \langle (K_C^\infty)^* \theta_f, v - (K_C^\infty)^* \theta_f \rangle_{L^2(0, +\infty; \mathbb{R}^p)} = \langle (\theta_f, K_C^\infty v - (K_C^\infty)^* \theta_f \rangle_{L^2(0, +\infty; \mathbb{R}^p)} = 0
\]
Since \(u^*\) is unique, then \(u^* = u_{\theta_f}\) and \(u_{\theta_f}\) is optimal with
\[
\|u_{\theta_f}\|_{L^2(0, +\infty; \mathbb{R}^p)}^2 = \|\theta_f\|_2.
\]

**Mathematical Approximations**

The current part of this paper, presents important approximations augmented with an approximation approach for AGR-system. First we give an approximation of \(\theta_f\) as a solution of a finite dimension linear system \(A \theta_f = b\) and then the optimal control \(u_{\theta_f}\), with a comparison between the corresponding observation noted \(z_{u_{\theta,f}}\), and the normal case.

**The Approximations Approach:**

- **System coefficients components:**
  For \(i,j \geq 1\), consider \(a_{ij} = \langle \Lambda^\infty_C e_i, e_j \rangle_{\mathbb{R}^q}\) such that \((e_i)_{1 \leq i \leq q}\) is the canonical basis of \(\mathbb{R}^q\), it follows that
  \[
  \Lambda^\infty_C e_i = \int_0^{+\infty} C \nabla S(t) B^\ast S^\ast(t) \nabla C^* e_i dt
  \]
and since \(N\) and \(M\) represent the number of eigenfunctions of the dynamic operator A. Thus, sufficiently large because the space have an infinite dimension.

Then, \(M, N\) be sufficiently large:
and \( b_j = -\langle R_C^0 f, e_j \rangle_{\mathbb{R}^q} \).

Because \( N \) represent the number of eigenvectors \( (\varphi_m)_{m=1}^{q} \) and really it is infinite.

\[
b_j \equiv - \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} \left( \partial \varphi_{m'} h, h_j \right)_{L^2(D_j)} \int_0^{+\infty} e^{\lambda m t} (f(t), \varphi_{m'} h)_{L^2(\Omega)} \, dt
\]

- **The optimal control:**
  In this part, an approximation of the optimal control \( u_{\theta f}(\cdot) \) is given, which is defined by:
  
  \[
  u_{\theta f}(s) = B^* S^* (\cdot) \nabla^* C^* \theta_f
  \]

  \[
  \equiv \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} \sum_{i=1}^{q} \theta_{i,f} e^{\lambda m t} (g_j, \varphi_{m'} h)_{L^2(\Omega)} \left( \partial \varphi_{m'} h, h_i \right)_{L^2(D_i)}
  \]

- **Cost:**
  The minimum energy (cost), for \( N \) sufficiently large, is defined by
  
  \[
  \|u_{\theta f}\|_{L^2(0, +\infty; {\mathbb{R}^q})} = \left( \int_0^{+\infty} ||B^* S^* (\cdot) \nabla^* C^* \theta_f||^2_{\mathbb{R}^q} dt \right)^{\frac{1}{2}}
  \]

  \[
  \equiv \left( \sum_{j=1}^{p} \int_0^{+\infty} \left( \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} \sum_{i=1}^{q} \theta_{i,f} e^{\lambda m t} (g_j, \varphi_{m'} h)_{L^2(\Omega)} \left( \partial \varphi_{m'} h, h_i \right)_{L^2(D_i)} \right)^2 dt \right)^{\frac{1}{2}}
  \]

- **The related observation:**
  The measurement information related to a given control is described by
  
  \[
  z_{u_{\theta f}}(s) = C \nabla S(t) y_0 + C \nabla \int_0^{s} S(\tau) B u_{\theta f}(\tau) \, d\tau
  \]

  \[
  + C \nabla \int_0^{s} S(\tau) f(\tau) \, d\tau
  \]

  Its coordinates \( (z_{j u_{\theta f}}(\cdot))_{1 \leq j \leq q} \) are achieved for a specific integer \( N \), given by:

  \[
  z_{j,u_{\theta f}}(t) \equiv \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} \sum_{i=1}^{q} e^{\lambda m t} (y_0, \varphi_{m'} h)_{L^2(\Omega)} \left( \partial \varphi_{m'} h, h_i \right)_{L^2(D_i)}
  \]

  \[
  + \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} (g_j, \varphi_{m'} h)_{L^2(\Omega)} \left( \partial \varphi_{m'} h, h_j \right)_{L^2(D_j)} \int_0^{t} e^{\lambda m t} u_{j,\theta f}(\tau) \, d\tau
  \]

  \[
  + \sum_{m'=1}^{N} \sum_{h=1}^{m'} \sum_{k=1}^{n} \left( \partial \varphi_{m'} h, h_j \right)_{L^2(D_j)} \int_0^{t} e^{\lambda m t} f(\tau), \varphi_{m'} h)_{L^2(\Omega)} \, d\tau
  \]
Moreover, the issue of how to discover an asymptotically gradient remediable system, that is to say, of gradient type, remains stronger than the remediability concept proved that the controllability concept of gradient results. Indeed, in the asymptotic case, it has been shown demonstrated in different important ways.

Algorithm

First Step: Data: domain $\Omega$, initial state $\psi^0$, disturbance function $f$, sensors $(D, h)$, gradient of efficient actuators $(\sigma, g)$ and precision threshold $\varepsilon$. Select a truncation low of order $M = N$.

Third Step: Calculate $z_{0,0}$: output with $f = 0$ and $u = 0$.

Fourth Step: Calculate $z_{0,f}$: output with $f \neq 0$ and $u = 0$.

Fifth Step: Resolve a finite system $A\theta = b$ such that the parameters are represented by Eq.10 and Eq.11.

Sixth Step: Calculate $u$ given by Eq.12.

Seventh Step: Compute $z_{u,f}$: output where $f \neq 0$ and $u \neq 0$.

Eighth Step: If $\|z_{u,f} - z_{0,0}\|_{L^2(\Omega)} \leq \varepsilon$, then stop.

Otherwise,第十步: $M \leftarrow M + 1$ and $N \leftarrow N + 1$ and return to third step.

Ten Step: Control $u$ of type optimal links to $u^*$ the solution of $(\mathbb{P})$.

Conclusion:

In this paper, the problem of AGC analysis has been presented. Certainly, it is based on suitable hypothesis and an appropriate choice of operators and spaces. Furthermore, WEAGR-system and AGER-actuators have been presented firstly. Also the problem of WEAGC-system has been examined under a suitable hypothesis with appropriate choice of spaces and operators. More precisely, the relationship between WEAGC-system and AGR-system has been demonstrated in different important results. Indeed, in the asymptotic case, it has been proved that the controllability concept of gradient type remains stronger than the remediability concept of gradient type, that is to say, AGR-system can be asymptotically gradient remediable but, it is not AGC-system.

Thus, through the choice of sensors and hypothesis of WAGR-system, the problem of EAGR-system with minimum energy has been studied. Moreover, the issue of how to discover an optimal control has been examined in a way compensating for the influence of the disturbances about the observation of gradient via the use of $\mathbb{H} \mathbb{U} \mathbb{M}$ modified.

Regarding the digital processing, some mathematical approximations are proposed, using a multi-step algorithm.

Later, the obtained outcomes have been introduced for class DDPL-systems and may be interesting to expand this work to regional or regional bounded case with other classes under the suitable different select of spaces, for example, the possibility to replace the observability concept in this paper by an asymptotic observer.

Authors’ declaration:

- Conflicts of Interest: None.
- We hereby confirm that all the Figures and Tables in the manuscript are mine ours. Besides, the Figures and images, which are not mine ours, have been given the permission for re-publication attached with the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee in University of Montouri.

Authors’ contributions statement:

S. R. and S. B. conceived of the presented idea and developed the theory. R. AL, verified the analytical methods and contributed to the analysis of the results and to the writing of the manuscript. All authors discussed the results and contributed to the final manuscript.

References:

تحليل مقارب لمسائل قابلية معالجة التدرج للأنظمة الخطيّة التوزيعية المصيرة

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الخلاصة:
الهدف من هذا العمل، برناج إمكانية التقليل من تأثير أي اضطرابات (تلتوث، إشعاع، عدوى، الخ) بشكل تقاربي، من خلال مراقبة تدرج نوع من الأنظمة الخطية المصيرة ذات المعاملات التوزيعية (أنظمة DDPL). بواسطة اختبار مناسب للمحفزات ذات العلاقة بتلك الأنظمة (المنظمة AGR)، بالإضافة إلى منظومة قابلية معالجة التدرج (منظمة GR). وهذا، تم تطوير منظومة قابلية معالجة التدرج (منظمة WAGC) في ظل فرضية التهيئة المتقدمة، وتم تقدير بعض خصائصها، وتم تنوع منظومة قابلية السيطرة في تدرج المنظومة (ungan AGR)، وذلك باستخدام خوارزمية قابلية معالجة التدرج (منظمة WAGC). وبالتالي، فإن المنظومة لتحديد مقرر (منظمة AGC). وبالتالي، فإن منظومة تطوير مقرر (منظمة WAGC).

الكلمات المفتاحية: تحليل مقارب، قابلية المعالجة، ضغوط، التحكم الأمثل، قابلية السيطرة، قابلية المعالجة.